In-Class Problems Week 11, Mon.

Problem 1. Find the coefficients of

(a) x^5 in $(1 + x)^{11}$

(b) x^8y^9 in $(3x + 2y)^{17}$

(c) a^6b^6 in $(a^2 + b^3)^5$

Problem 2. According to the Multinomial theorem, $(w + x + y + z)^n$ can be expressed as a sum of terms of the form

$$\binom{n}{r_1, r_2, r_3, r_4} w^{r_1}x^{r_2}y^{r_3}z^{r_4}.$$

How many terms are there in the sum?

Combinatorial proof

Combinatorial proofs of identities

Recall the basic plan for a combinatorial proof of an identity $x = y$:

1. Define a set S.
2. Show that $|S| = x$ by counting one way.
3. Show that $|S| = y$ by counting another way.
4. Conclude that $x = y$.

Problem 3. You want to choose a team of m people from a pool of n people for your startup company, and from these m people you want to choose k to be the team managers. You took 6.042, so you know you can do this in

$$\binom{n}{m}\binom{m}{k}$$

ways. But your CFO, who went to Harvard Business School, comes up with the formula

$$\binom{n}{k}\binom{n-k}{m-k}.$$

Before doing the reasonable thing —dump on your CFO or Harvard Business School— you decide to check his answer against yours.

(a) Start by giving an *algebraic proof* that your CFO's formula agrees with yours.

(b) Now give a *combinatorial argument* proving this same fact.

Problem 4. Now give a combinatorial proof of the following, more interesting theorem:

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$$

Hint: Let S be the set of all length-n sequences of 0's, 1's and a single *.

Problem 5. What do the following expressions equal? Give both algebraic and combinatorial proofs for your answers.

(a)

$$\sum_{i=0}^{n} \binom{n}{i}$$

(b)

$$\sum_{i=0}^{n} \binom{n}{i}(-1)^i$$

Hint: Consider the bit strings with an even number of ones and an odd number of ones.