In-Class Problems Week 11, Fri.

Problem 1. Define the function $f : \mathbb{N} \to \mathbb{N}$ recursively by the rules

$$\begin{aligned} f(0) &= 1, \\ f(1) &= 6, \\ f(n) &= 2f(n-1) + 3f(n-2) + 4 \qquad \text{for } n \geq 2. \end{aligned}$$

(a) Find a closed form for the generating function

$$G(x) ::= f(0) + f(1)x + f(2)x^2 + \cdots + f(n)x^n + \cdots .$$

(b) Find a closed form for $f(n)$. *Hint:* Find numbers a, b, c, d, e, g such that

$$G(x) = \frac{a}{1+dx} + \frac{b}{1+ex} + \frac{c}{1+gx}.$$

Problem 2. (Carried over from Wednesday, April 25)

(a) Let

$$S(x) ::= \frac{x^2+x}{(1-x)^3}.$$

What is the coefficient of x^n in the generating function series for $S(x)$?

Hint: A formula for the coefficient of x^n in $1/(1-x)^k$ follows from the Convolution Counting Principle and is given in the Appendix (and in part (d) below).

(b) Explain why $S(x)/(1-x)$ is the generating function for the sums of squares. That is, the coefficient of x^n in the series for $S(x)/(1-x)$ is $\sum_{k=1}^{n} k^2$.

(c) Use the previous parts to prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

(d) Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Then it's easy to check that

$$a_n = \frac{A^{(n)}(0)}{n!},$$

where $A^{(n)}$ is the nth derivative of A. Use this fact (which you may assume) instead of the Convolution Counting Principle, to prove that

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n.$$

Appendix

Products of Series

Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad B(x) = \sum_{n=0}^{\infty} b_n x^n, \qquad C(x) = A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Then

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0.$$

By the Convolution Counting Property, or by Problem 2.(d),

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n.$$

Finding a Generating Function for Fibonacci Numbers

The Fibonacci numbers are defined by:

$$\begin{aligned} f_0 &::= 0 \\ f_1 &::= 1 \\ f_n &::= f_{n-1} + f_{n-2} \qquad \text{(for } n \geq 2\text{)} \end{aligned}$$

Let F be the generating function for the Fibonacci numbers, that is,

$$F(x) ::= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots$$

So we need to derive a generating function whose series has coefficients:

$$\langle 0,\ 1,\ f_1 + f_0,\ f_2 + f_1,\ f_3 + f_2,\ \ldots \rangle$$

Now we observe that

$$\begin{array}{rlllllllll} & \langle\ 0, & 1, & 0, & 0, & 0, & \ldots & \rangle & \longleftrightarrow & x \\ & \langle\ 0, & f_0, & f_1, & f_2, & f_3, & \ldots & \rangle & \longleftrightarrow & xF(x) \\ + & \langle\ 0, & 0, & f_0, & f_1, & f_2, & \ldots & \rangle & \longleftrightarrow & x^2F(x) \\ \hline & \langle\ 0, & 1 + f_0, & f_1 + f_0, & f_2 + f_1, & f_3 + f_2, & \ldots & \rangle & \longleftrightarrow & x + xF(x) + x^2F(x) \end{array}$$

This sequence is almost identical to the right sides of the Fibonacci equations. The one blemish is that the second term is $1 + f_0$ instead of simply 1. But since $f_0 = 0$, the second term is ok.

So we have

$$F(x) = x + xF(x) + x^2 F(x).$$
$$F(x) = \frac{x}{1 - x - x^2}. \tag{1}$$

Finding a Closed Form for the Coefficients

Now we expand the righthand side of (1) into partial fractions. To do this, we first factor the denominator

$$1 - x - x^2 = (1 - \alpha_1 x)(1 - \alpha_2 x)$$

where $\alpha_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\alpha_2 = \frac{1}{2}(1 - \sqrt{5})$ by the quadratic formula. Next, we find A_1 and A_2 which satisfy:

$$F(x) = \frac{x}{1 - x - x^2} = \frac{A_1}{1 - \alpha_1 x} + \frac{A_2}{1 - \alpha_2 x} \tag{2}$$

Now the coefficient of x^n in $F(x)$ will be A_1 times the coefficient of x^n in $1/(1 - \alpha_1 x)$ plus A_2 times the coefficient of x^n in $1/(1 - \alpha_2 x)$. The coefficients of these fractions will simply be the terms α_1^n and α_2^n because

$$\frac{1}{1 - \alpha_1 x} = 1 + \alpha_1 x + \alpha_1^2 x^2 + \cdots$$
$$\frac{1}{1 - \alpha_2 x} = 1 + \alpha_2 x + \alpha_2^2 x^2 + \cdots$$

by the formula for geometric series.

So we just need to find find A_1 and A_2. We do this by plugging values of x into (2) to generate linear equations in A_1 and A_2. It helps to note that from (2), we have

$$x = A_1(1 - \alpha_2 x) + A_2(1 - \alpha_1 x),$$

so simple values to use are $x = 0$ and $x = 1/\alpha_2$. We can then find A_1 and A_2 by solving the linear equations. This gives:

$$A_1 = \frac{1}{\alpha_1 - \alpha_2} = \frac{1}{\sqrt{5}}$$
$$A_2 = -A_1 = -\frac{1}{\sqrt{5}}$$

Substituting into (2) gives the partial fractions expansion of $F(x)$:

$$F(x) = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \alpha_1 x} - \frac{1}{1 - \alpha_2 x} \right).$$

So we conclude that the coefficient, f_n, of x^n in the series for $F(x)$ is

$$\begin{aligned} f_n &= \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right) \end{aligned}$$