In-Class Problems Week 10, Fri.

Problem 1. **(a)** Show that the Magician could not pull off the trick with a deck larger than 124 cards.

Hint: Compare the number of 5-card hands in an n-card deck with the number of 4-card sequences.

(b) Show that, in principle, the Magician could pull off the Card Trick with a deck of 124 cards.

Hint: Hall's Theorem and degree-constrained graphs.

Problem 2. The Tao of BOOKKEEPER: we seek enlightenment through contemplation of the word $BOOKKEEPER$.

(a) In how many ways can you arrange the letters in the word $POKE$?

(b) In how many ways can you arrange the letters in the word BO_1O_2K? Observe that we have subscripted the O's to make them distinct symbols.

(c) Suppose we map arrangements of the letters in BO_1O_2K to arrangements of the letters in $BOOK$ by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

O_2BO_1K	
KO_2BO_1	$BOOK$
O_1BO_2K	$OBOK$
KO_1BO_2	$KOBO$
BO_1O_2K	...
BO_2O_1K	
...	

(d) What kind of mapping is this, young grasshopper?

(e) In light of the Division Rule, how many arrangements are there of $BOOK$?

(f) Very good, young master! How many arrangements are there of the letters in $KE_1E_2PE_3R$?

(g) Suppose we map each arrangement of $KE_1E_2PE_3R$ to an arrangement of $KEEPER$ by erasing subscripts. List all the different arrangements of $KE_1E_2PE_3R$ that are mapped to $REPEEK$ in this way.

(h) What kind of mapping is this?

(i) So how many arrangements are there of the letters in $KEEPER$?

(j) *Now you are ready to face the BOOKKEEPER!*
How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?

(k) How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?

(l) How many arrangements of $BOOKKE_1E_2PE_3R$ are there?

(m) How many arrangements of $BOOKKEEPER$ are there?

(n) How many arrangements of $VOODOODOLL$ are there?

(o) **(IMPORTANT)** How many n-bit sequences contain k zeros and $(n-k)$ ones?

Remember well what you have learned: subscripts on, subscripts off.

This is the Tao of Bookkeeper.

Problem 3. Solve the following counting problems. Define an appropriate mapping (bijective or k-to-1) between a set whose size you know and the set in question.

(a) How many different ways are there to select a dozen donuts if four varieties are available?

(b) How many paths are there from $(0,0)$ to $(10,20)$ consisting of right-steps (which increment the first coordinate) and up-steps (which increment the second coordinate)?

(c) An independent living group is hosting nine new candidates for membership. Each candidate must be assigned a task: 1 must wash pots, 2 must clean the kitchen, 3 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. In how many ways can this be done?

(d) In how many ways can Mr. and Mrs. Grumperson distribute 13 identical pieces of coal to their two— no, three!— children for Christmas?

(e) How many solutions over the natural numbers are there to the equation:

$$x_1 + x_2 + \ldots + x_{10} \quad \leq \quad 100$$

(f) Suppose that two identical 52-card decks are mixed together. In how many ways can the cards in this double-size deck be arranged?