Bases for Non-Homogeneous Polynomial C_k Splines on the Sphere

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Abstract. We investigate the use of non-homogeneous spherical polynomials for the approximation of functions defined on the sphere $\mathbf{S}^2$. A *spherical polynomial* is the restriction to $\mathbf{S}^2$ of a polynomial in the three coordinates x, y, z of $\mathbf{R}^3$. Let $\mathcal{P}^d$ be the space of spherical polynomials with degree $\leq d$. We show that $\mathcal{P}^d$ is the direct sum of $\mathcal{H}^d$ and $\mathcal{H}^{d-1}$, where $\mathcal{H}^d$ denotes the space of *homogeneous* degree-d polynomials in x, y, z.

We also generalize this result to splines defined on a geodesic triangulation T of the sphere. Let $\mathcal{P}^d_k[T]$ denote the space of all functions f from $\mathbf{S}^2$ to $\mathbf{R}$ such that (1) the restriction of f to each triangle of T belongs to $\mathcal{P}^d$; and (2) the function f has order-k continuity across the edges of T. Analogously, let $\mathcal{H}^d_k[T]$ denote the subspace of $\mathcal{P}^d_k[T]$ consisting of those functions that are $\mathcal{H}^d$ within each triangle of T. We show that $\mathcal{P}^d_k[T] = \mathcal{H}^d_k[T] \oplus \mathcal{H}^{d-1}_k[T]$. Combined with results of Alfeld, Neamtu and Schumaker on bases of $\mathcal{H}^d_k[T]$ this decomposition provides an effective construction for a basis of $\mathcal{P}^d_k[T]$.

There has been considerable interest recently in the use of the homogeneous spherical splines $\mathcal{H}^d_k[T]$ as approximations for functions defined on $\mathbf{S}^2$. We argue that the non-homogeneous splines $\mathcal{P}^d_k[T]$ would be a more natural choice for that purpose.

1 Introduction

Spherical functions—real functions defined on the sphere $\mathbf{S}^2$—arise in many applications: geophysics, meteorology, computer graphics, robotics, lighting, acoustics, etc.. In such applications, one usually represents a point of the sphere by its spherical coordinates, the longitude ϕ and the latitude θ. The function is typically modeled as (or approximated by) a polynomial in ϕ and θ, or a sum of spherical harmonics up to a certain order.

However, the latitude-longitude representation has some drawbacks. Spherical coordinates have an essential discontinuity at the poles; the geodesic lines have complicated descriptions in terms of ϕ and θ; a regular (ϕ, θ) grid covers the sphere with non-uniform resolution; and so on. These problems are particularly annoying when implementing advanced numerical methods, such as multiscale

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integration and approximation by irregular adaptive meshes. As a consequence, there has been a lot of interest recently on the modeling of spherical functions *in situ*, that is, as functions defined in terms of the three Cartesian coordinates (x, y, z) of $\mathbf{R}^3$, which are then restricted to the sphere.

Recently, Alfeld, Neamtu and Schumaker [1–3] have proposed the use of a special class of functions, the *homogeneous spherical splines* $\mathcal{H}_k^d[T]$, as an approximation space for functions defined on $\mathbf{S}^2$. Here we define an alternative class, the *polynomial spherical splines* $\mathcal{P}_k^d[T]$. We show that $\mathcal{P}_k^d[T] = \mathcal{H}_k^d[T] \oplus \mathcal{H}_k^{d-1}[T]$, which allows us to obtain a characterization for the bases of the space $\mathcal{P}_k^d[T]$. We argue that $\mathcal{P}_k^d[T]$ is a more natural choice for approximating spherical functions.

2 Polynomial Functions on $\mathbf{R}^n$

Let $\mathcal{P}^{d,n}$ be the space of polynomials on n variables of degree $\leq d$, viewed as functions from $\mathbf{R}^n$ to $\mathbf{R}$. A function p belongs to $\mathcal{P}^{d,n}$ if and only if it can be written in the form

$$p(x) = \sum_{0 \leq i_1 + i_2 + \ldots + i_n \leq d} c_{i_1 i_2 \ldots i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$$

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$, where $c_{i_1 i_2 \ldots i_n}$ are real coefficients. (All indices in this paper are non-negative integers). The set $\mathcal{P}^{d,n}$ is obviously a vector space, of dimension

$$\dim \mathcal{P}^{d,n} = \binom{d+n}{n}$$

We say that a function f defined on $\mathbf{R}^n$ is *homogeneous of degree d* if $f(ax) = a^d f(x)$, for all $a \in \mathbf{R}$ and all $x \in \mathbf{R}^n$. Let $\mathcal{H}^{d,n}$ the space of the polynomials on the $\mathbf{R}^n$ of degree $\leq d$ which are homogeneous of degree d. Obviously $\mathcal{H}^{d,n}$ is a subspace of $\mathcal{P}^{d,n}$. A function h belongs to $\mathcal{H}^{d,n}$ if and only if it can be written in the form

$$h(x) = \sum_{i_1 + i_2 + \ldots + i_n = d} c_{i_1 i_2 \ldots i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$$

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$, where $c_{i_1 i_2 \ldots i_n}$ are real coefficients. It follows that

$$\dim \mathcal{H}^{d,n} = \binom{d+n-1}{n-1}$$

It easy to see that, if $d \neq d'$, the spaces $\mathcal{H}^{d,n}$ and $\mathcal{H}^{d',n}$ are linearly independent; that is, $\mathcal{H}^{d,n} \cap \mathcal{H}^{d',n} = \{0\}$.

3 Polynomial Functions on $\mathbf{S}^2$

If a function f is defined on $\mathbf{R}^n$, and $X \subseteq \mathbf{R}^n$, we denote by f/X the restriction of f to the set X. By extension, we define the restriction of a function space $\mathcal{F}$ to the set X as

$$\mathcal{F}/X = \{\, f/X : f \in \mathcal{F} \,\}$$

We will use the following notation: if $f/X = g/X$, we write $f \equiv g \pmod{X}$; or just $f \equiv g$, when X is implicit in the context. It is obvious that '$\equiv$' is an equivalence relation.

We are interested in the space $\mathcal{P}^{d,n}/\mathbf{S}^{n-1}$, consisting of the polynomial functions on $\mathbf{R}^n$ of degree d, restricted to the sphere $\mathbf{S}^{n-1} = \{\, x \in \mathbf{R}^n : |x| = 1 \,\}$.

Observe that polynomials which are distinct in $\mathbf{R}^n$ can be identical when restricted to the sphere $\mathbf{S}^{n-1}$. Therefore, the dimension of $\mathcal{P}^{d,n}/\mathbf{S}^{n-1}$ is smaller than that of $\mathcal{P}^{d,n}$. The following Theorem is fundamental for the caracterization of $\mathcal{P}^{d,n}/\mathbf{S}^{n-1}$:

Theorem 1. *Every polynomial in $\mathcal{P}^{d,n}$, $n \geq 1$, is equivalent (modulo $\mathbf{S}^{n-1}$). to a unique polynomial in $\mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n}$.*

Proof.

Existence: Let p be a polynomial in $\mathcal{P}^{d,n}$. We will show that all terms of p with degree $\leq d-2$ can be replaced by terms of degree d and $d-1$. Let $c_{i_1 i_2 \ldots i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$ be a term of p such that $i_1 + i_2 + \ldots + i_n = k \leq d-2$. Since x is a point of $\mathbf{S}^{n-1}$ we have $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Then,

$$c_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n} \;=\; c_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n} (x_1^2 + \cdots + x_n^2)$$

Thus, a term of degree $k \leq d-2$ can be replaced by n terms of degree $k+2 \leq d$, while maintaining equivalence modulo $\mathbf{S}^{n-1}$. If we keep applying this substitution until there are no more terms of degree $\leq d-2$, we obtain a polynomial q with terms of degree d and $d-1$. Then q is a polynomial in $\mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n}$ such that $p \equiv q \pmod{\mathbf{S}^{n-1}}$.

Unicity: Let p be a polynomial in $\mathcal{P}^{d,n}$. Suppose that there exist $q_1, q_2 \in \mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n}$ with $q_1 \equiv p$ and $q_2 \equiv p$. Since $\equiv$ is transitive, we must have $q_1 \equiv q_2$, i. e.

$$(q_1 - q_2)/\mathbf{S}^{n-1} = 0$$

Since $\mathbf{S}^{n-1}$ is an algebric variety [5, 4], the minimal equation of $\mathbf{S}^{n-1}$ must be a factor of $q_1 - q_2$, that is,

$$q_1 - q_2 = R(x_1^2 + x_2^2 + \ldots + x_n^2 - 1)$$

where R is some polynomial on $\mathbf{R}^n$ of degree $n-2$. Since the polynomial $x_1^2 + x_2^2 + \ldots + x_n^2 - 1$ has terms whose degrees differ by 2, and, on the other hand,

$$q_1 - q_2 \in \mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n}$$

we can conclude that $R = 0$, i.e. $q_1 = q_2$. Thus, there is at most one polynomial in $\mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n}$ that is equivalent to p modulo $\mathbf{S}^{n-1}$. $\square$

Corollary 2. $\mathcal{H}^{d,n}/\mathbf{S}^{n-1} \subseteq \mathcal{H}^{d+2,n}/\mathbf{S}^{n-1}$.

Corollary 3. *If* $p, q \in \mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n}$ *and* $p \equiv q \pmod{\mathbf{S}^{n-1}}$, *then* $p = q$.

Corollary 4. $\mathcal{H}^{d-1,n}/\mathbf{S}^{n-1} \cap \mathcal{H}^{d,n}/\mathbf{S}^{n-1} = \{0\}$

As a consequence of Theorem 1,

$$\mathcal{P}^{d,n}/\mathbf{S}^{n-1} = (\mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n})/\mathbf{S}^{n-1} = \mathcal{H}^{d-1,n}/\mathbf{S}^{n-1} \oplus \mathcal{H}^{d,n}/\mathbf{S}^{n-1}$$

It follows that

$$\dim((\mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n})/\mathbf{S}^{n-1}) = \dim(\mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n}) = \binom{d+n-1}{n}$$

4 Derivatives of Spherical Polynomials

If f is a function from $\mathbf{S}^{n-1}$ to $\mathbf{R}$, we denote by $\bar{\nabla} f$ its gradient with respect to directions tangent to $\mathbf{S}^{n-1}$. For this paper, we can define $\bar{\nabla} f$ as a vector of $\mathbf{R}^n$, tangent to $\mathbf{S}^{n-1}$, such that the derivative of f at a point $u \in \mathbf{S}^{n-1}$, in the direction of a unit vector v tangent at u, is $v \cdot (\bar{\nabla} f(u))$.

If $f = F/\mathbf{S}^{n-1}$ for some differentiable function F from $\mathbf{R}^n$ to $\mathbf{R}$, then $\bar{\nabla} f$ is merely the projection of ∇F (the ordinary gradient of F) onto the sphere. That is, for any point $u \in \mathbf{S}^{n-1}$,

$$(\bar{\nabla} f)(u) = ((\nabla F)(u) - ((\nabla F)(u) \cdot u)\, u)\,/\mathbf{S}^{n-1} \tag{1}$$

Let's denote by $[v]_\alpha$ the αth component of a vector $v \in \mathbf{R}^n$.

Theorem 5. *If* f *belongs to* $\mathcal{H}^{d,n}/\mathbf{S}^{n-1}$, *then* $[\bar{\nabla} f]_\alpha \in \mathcal{H}^{d+1,n}/\mathbf{S}^{n-1}$ *for each* $\alpha \in \{1, \ldots, n\}$.

Proof. If f belongs to $\mathcal{H}^{d,n}/\mathbf{S}^{n-1}$, for $d \geq 1$, then $f = F/\mathbf{S}^{n-1}$ for some $F \in \mathcal{H}^{d,n}$. It is easy to check that $[\nabla F]_\alpha = \partial F/\partial x_\alpha$ is in $\mathcal{H}^{d-1,n}$. Therefore, the right-hand side of formula (1) lies in $\mathcal{H}^{d-1,n}/\mathbf{S}^{n-1} + \mathcal{H}^{d+1,n}/\mathbf{S}^{n-1}$. By Corollary 2, $\mathcal{H}^{d-1,n}/\mathbf{S}^{n-1}$ is actually a subspace of $\mathcal{H}^{d+1,n}/\mathbf{S}^{n-1}$. □

Corollary 6. *If* f *belongs to* $\mathcal{P}^{d,n}$, *then* $[\bar{\nabla} f]_\alpha \in \mathcal{P}^{d+1,n}/\mathbf{S}^{n-1}$, *for each* α.

5 Piecewise Polynomial Functions on $\mathbf{S}^2$

We now proceed to extend Theorem 1 to the piecewise polynomial functions defined on the sphere $\mathbf{S}^2$. Let T a decomposition of $\mathbf{R}^3$ into trihedra $T_1, T_2, \ldots, T_n$, with non empty interiors and a common vertex at the origin of $\mathbf{R}^n$. Observe that the set $T_i \cap \mathbf{S}^2$, for $1 \leq i \leq n$, is a spherical triangle, whose sides are arcs of great circles. Therefore, T determines a triangulation of $\mathbf{S}^2$, which we denote by $T/\mathbf{S}^2$.

For such a trihedral decomposition T, we define the following function spaces from $\mathbf{R}^3$ to $\mathbf{R}$:

$$\mathcal{P}^d[T] = \{\ p : (\forall i)\ p/T_i \in \mathcal{P}^{d,3}/T_i\ \}$$
$$\mathcal{H}^d[T] = \{\ h : (\forall i)\ h/T_i \in \mathcal{H}^{d,3}/T_i\ \}$$

By Theorem 1, we can immediately conclude that

$$\mathcal{P}^d[T]/\mathbf{S}^2 = \mathcal{H}^{d-1}[T]/\mathbf{S}^2 + \mathcal{H}^d[T]/\mathbf{S}^2$$

Furthemore, we can show the following:

Theorem 7. *If p, q are functions in $\mathcal{H}^{d-1}[T] + \mathcal{H}^d[T]$, then $p \equiv q \pmod{\mathbf{S}^2}$ if and only if $p = q$.*

Proof. Let p, $q \in \mathcal{H}^{d-1}[T] + \mathcal{H}^d[T]$. For $1 \leq i \leq n$, let p_i, q_i be functions of $\mathcal{H}^{d-1,3} + \mathcal{H}^{d,3}$ such as $p/T_i = p_i/T_i$ e $q/T_i = q_i/T_i$. Then $p \equiv q$ means that $p_i/(\mathbf{S}_2 \cap T_i) = q_i/(\mathbf{S}^2 \cap T_i)$, and, therefore, $(p_i - q_i)/(\mathbf{S}^2 \cap T_i) = 0$. Since $T_i \cap \mathbf{S}^2$ is a 2-dimensional subset of $\mathbf{S}^2$, and $\mathbf{S}^2$ is an irreducible argebraic variety in $\mathbf{R}^3$, we can conclude that $(p_i - q_i)/\mathbf{S}^2 = 0$. According to the Corollary 3, $p_i - q_i = 0$. Since this equality holds for all trihedra T_i, we conclude that $p = q$. ☐

This Theorem has the following consequences:

Corollary 8.

$$\mathcal{H}^{d-1}[T]/\mathbf{S}^2 \cap \mathcal{H}^d[T]/\mathbf{S}^2 = \{0\}/\mathbf{S}^2$$

Corollary 9.

$$\mathcal{P}^d[T]/\mathbf{S}^2 = \mathcal{H}^{d-1}[T]/\mathbf{S}^2 \oplus \mathcal{H}^d[T]/\mathbf{S}^2$$

6 Continuity Constraints

Finally, we extend Corollary 9 to piecewise polynomial functions, subject to continuity restrictions across the edges of the triangulation.

We say that a function from $\mathbf{S}^{n-1}$ to $\mathbf{R}$ is *continuous to order zero* if it is continuous in the ordinary sense; and is *continuous to order k*, for $k > 0$, if it is continuous, and each component of its spherical gradient is continuous to order $k-1$. We denote by $\mathbf{C}_k(\mathbf{S}^2)$ the set of all functions from $\mathbf{S}^2$ to $\mathbf{R}$ that are continuous to order k.

For a decomposition T of $\mathbf{R}^3$ into non-degenerate trihedra, as in Sect. 5, we define the function spaces

$$\mathcal{P}^d_k[T] = \{\ p : p \in \mathcal{P}^d[T]\ \wedge\ p/\mathbf{S}^2 \in \mathbf{C}_k(\mathbf{S}^2)\ \}$$
$$\mathcal{H}^d_k[T] = \{\ h : h \in \mathcal{H}^d[T]\ \wedge\ h/\mathbf{S}^2 \in \mathbf{C}_k(\mathbf{S}^2)\ \}$$

Our goal is to show that $\mathcal{P}^d_k[T]/\mathbf{S}^2$ is the direct sum of $\mathcal{H}^{d-1}_k[T]/\mathbf{S}^2$ and $\mathcal{H}^d_k[T]/\mathbf{S}^2$. In other words, imposing kth-order continuity on $\mathcal{P}^d[T]/\mathbf{S}^2$ is equivalent to independently imposing kth-order continuity on each of the two subspaces $\mathcal{H}^{d-1}[T]/\mathbf{S}^2$ and $\mathcal{H}^d[T]/\mathbf{S}^2$.

Let's consider first the case $k = 0$:

Theorem 10.

$$\mathcal{P}_0^d[T]/\mathbf{S}^2 = \mathcal{H}_0^{d-1}[T]/\mathbf{S}^2 \oplus \mathcal{H}_0^d[T]/\mathbf{S}^2$$

Proof. ($\supseteq$): Trivial.
($\subseteq$): Let p be a function in $\mathcal{P}_0^d[T]/\mathbf{S}^2$. Let T_i and T_j be adjacent trihedra from T, and let p_i and p_j be functions of $\mathcal{P}^{d,3}$ such that $p/T_i = p_i/T_i$ and $p/T_j = p_j/T_j$. By Corollary 9,

$$p = h^d + h^{d-1}$$

with $h^d \in \mathcal{H}^d[T]/\mathbf{S}^2$ and $h^{d-1} \in \mathcal{H}^{d-1}[T]/\mathbf{S}^2$. (Here '$d$' is an index and not an exponent.) Since $p/T_i = p_i/T_i$, p_i can be written as

$$p_i = h_i^d + h_i^{d-1}$$

where $h_i^d/T_i \in \mathcal{H}^{d,3}/T_i$ and $h_i^{d-1}/T_i \in \mathcal{H}^{d-1,3}/T_i$. Analogously,

$$p_j = h_j^d + h_j^{d-1}$$

where $h_j^{d-1}/T_j \in \mathcal{H}^{d-1,3}/T_j$ and $h_j^d/T_j \in \mathcal{H}^{d,3}/T_j$.

Let w be the arc shared by the spherical triangles $T_i \cap \mathbf{S}^2$ and $T_j \cap \mathbf{S}^2$, and let c be the circle which contains the arc w. Since $p \in \mathbf{C}_0(\mathbf{S}^2)$, we have $p/w = p_i/w = p_j/w$, and therefore $(p_i - p_j)/w = 0$. Given that w is a one-dimensional subset of c, and c is an irreducible variety, we conclude that $(p_i - p_j)/c = 0$.

We can assume, without loss of generality, that c is the circle $x^2 + y^2 = 1$, contained in the plane π with equation $z = 0$. Since $(p_i - p_j)/\pi \in \mathcal{P}^{d,2}$ and $(p_i - p_j)/c = 0$, we can, by Corollary 8, conclude that

$$(h_i^d - h_j^d)/c = 0$$
$$(h_i^{d-1} - h_j^{d-1})/c = 0$$

Therefore,

$$h_i^d/c = h_j^d/c$$
$$h_i^{d-1}/c = h_j^{d-1}/c$$

We conclude that $h^d \in \mathcal{H}_0^d[T]$ and $h^{d-1} \in \mathcal{H}_0^{d-1}[T]$. □

Let's now prove the general case:

Theorem 11. *For any $k \geq 0$,*

$$\mathcal{P}_k^d[T]/\mathbf{S}^2 = \mathcal{H}_k^{d-1}[T]/\mathbf{S}^2 \oplus \mathcal{H}_k^d[T]/\mathbf{S}^2$$

Proof. ($\supseteq$): Trivial.
($\subseteq$): We prove this part by induction on k. The case $k = 0$ is Theorem 10, so let's assume $k > 0$.

Let p be a function in $\mathcal{P}^d_k[T]/\mathbf{S}^2$. By definition, p is continuous, and ∇p is continuous of order $k-1$. By Corollary 6, $[\nabla p]_\alpha$ belongs to $\mathcal{P}^{d+1}[T]/\mathbf{S}^2$, and therefore to $\mathcal{P}^{d+1}_{k-1}[T]/\mathbf{S}^2$. By induction,

$$[\nabla p]_\alpha \in \mathcal{H}^d_{k-1}[T]/\mathbf{S}^2 \oplus \mathcal{H}^{d+1}_{k-1}[T]/\mathbf{S}^2 \tag{2}$$

On the other hand, by Theorem 1, $p = h^d + h^{d-1}$, where $h^d \in \mathcal{H}^d[T]/\mathbf{S}^2$ and $h^{d-1} \in \mathcal{H}^{d-1}[T]/\mathbf{S}^2$. In the interior of each triangle of T, the spherical gradient of p is then

$$\nabla p = \nabla h^d + \nabla h^{d-1}$$

By Theorem 2,

$$[\nabla h^d]_\alpha \in \mathcal{H}^{d+2}[T]/\mathbf{S}^2 \tag{3}$$

$$[\nabla h^{d-1}]_\alpha \in \mathcal{H}^{d+1}[T]/\mathbf{S}^2 \tag{4}$$

Comparing equation (2) with equations (3) and (4), we conclude that

$$[\nabla h^d]_\alpha \in \mathcal{H}^{d+2}_{k-1}[T]/\mathbf{S}^2 \tag{5}$$

$$[\nabla h^{d-1}]_\alpha \in \mathcal{H}^{d+1}_{k-1}[T]/\mathbf{S}^2 \tag{6}$$

Since p is continuous, Theorem 10 implies that h^d and h^{d-1} are continuous, too. With equations (5-6), we conclude that $h^d \in \mathcal{H}^d_k[T]$ and $h^{d-1} \in \mathcal{H}^{d-1}_k[T]$. □

7 Bases for Spherical Splines

The results above show that $\mathcal{P}^d_k[T]/\mathbf{S}^2$, the space of piecewise polynomial functions restricted to the sphere with order-k continuity, is the direct sum of the spaces $\mathcal{H}^d_k[T]$ and $\mathcal{H}^{d-1}_k[T]$, restricted to $\mathbf{S}^2$.

Alfeld, Neamtu and Schumaker [1–3] have recently obtained an explicit basis for the space $\mathcal{H}^d_k[T]$, in terms of Bernstein-Bézier polinomials. In view of Theorem 11, their construction also gives a basis for $\mathcal{P}^d_k[T]/\mathbf{S}^2$.

8 Conclusion

We believe that the space $\mathcal{P}^d_k[T]/\mathbf{S}^2$ is a better choice than $\mathcal{H}^d_k[T]/\mathbf{S}^2$ for function approximation on the sphere. For one thing, $\mathcal{P}^r_k[T] \subseteq \mathcal{P}^d_k[T]$ when $r \leq d$, while $\mathcal{H}^r_k[T] \subseteq \mathcal{H}^d_k[T]$ only when $d-r$ is even. In particular, $\mathcal{P}^d_k[T]$ includes the functions which are constant on $\mathbf{S}^2$, for all d; whereas $\mathcal{H}^d_k[T]$ only contains such functions when d is even.

References

1. Peter Alfeld, Marian Neamtu, and Larry L. Schumaker. Bernstein-Bézier polynomials on circle, spheres, and sphere-like surfaces. *Computer Aided Geometric Design Journal*, 13:333–349, 1996.
2. Peter Alfeld, Marian Neamtu, and Larry L. Schumaker. Dimension and local bases of homogeneous spline spaces. *SIAM Journal of Mathematical Analysis*, 27(5):1482–1501, September 1996.
3. Peter Alfeld, Marian Neamtu, and Larry L. Schumaker. Fitting scattered data on sphere-like surfaces using spherical splines. *Journal of Computational and Applied Mathematics*, 73:5–43, 1996.
4. W. Fulton. *Algebraic Curves: An Introduction to Algebraic Geometry.* W. A. Benjamin, 1969.
5. E. Kunz. *Introduction to Commutative Algebra and Algebraic Geometry.* Birkhauser, 1993.