Improper colourings of outerplanar graphs

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Abstract

A (k, d)-colouring of a graph G is a vertex colouring with at most k colours such that every monochromatic connected component has maximum degree at most d. Thus, a $(k, 0)$-colouring corresponds to a proper k-colouring. It is well known that every planar graph admits $(3, 2)$-colourings, but there is no d for which every planar graph admits $(1, d)$-colourings or $(2, d)$-colourings. Moreover, deciding whether a planar graph admits a $(2, 1)$-colouring is NP-complete. It is also known that every outerplanar graph admits $(3, 0)$ and $(2, 2)$-colourings, although some outerplanar graphs do not admit $(2, 1)$-colourings, with triangles playing an important role. In this work, we prove that deciding the existence of $(2, 1)$-colourings of apart-plane graphs – in which every pair of triangles is disjoint – is also NP-complete. Moreover, we present a polynomial-time algorithm for deciding whether an outerpath graph – an outerplane graph for which the weak dual is isomorphic to a path – admits $(2, 1)$-colourings. This fully characterises the $(2, 1)$-colourability of outerpath graphs and provides new insights into the structural role of triangles in defective colouring problems.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We denote $e = uv$ an edge whose *endpoints* are u and v and denote $d(v)$ the *degree* of v. The *neighbourhood* of v is $N(v) = \{u \in V(G) : uv \in E(G)\}$ and its elements are the *neighbours* of v.

A *vertex colouring* of a graph G is an assignment of *colours* from a colour set $\mathcal{C}$ to the vertices of G. Given a vertex colouring π, a *monochromatic subgraph* is a maximal subgraph of G induced by the vertices of a single colour. A connected component of a monochromatic subgraph is a *monochromatic component*. A (k, d)*-colouring* is a vertex colouring using at most k colours so that the maximum degree of any monochromatic component is at most d. In particular, a $(k, 0)$-colouring corresponds to a *proper k-colouring* – that is, it is a colouring in which every pair of adjacent vertices are assigned a different colour.

Defective colourings were introduced independently by Andrews and Jacobson [1], Harary and Jones [2], and Cowen et al. [3] around 1985. Cowen et al. [3] proved that every planar graph has $(3, 2)$-colourings – note that the Four Colour Theorem implies that every planar graph has a $(4, 0)$-colouring. Nevertheless, the authors proved that there is no d for which every planar graph admits $(1, d)$- or $(2, d)$-colourings and that some planar graphs do not admit $(3, 1)$-colourings. They also showed that every outerplanar graph admits $(2, 2)$-colourings, and that there exist outerplanar graphs that do not admit $(2, 1)$-colourings.

Cowen et. al. [3] provide a linear time algorithm that finds a $(3, 2)$-colouring for planar graphs. However, deciding whether a planar graph has $(3, 1)$-colourings, or even $(2, k)$-colourings, for any $k > 0$, is NP-complete [4]. Jonck Jr. [5] showed that every outerplanar graph with at most three

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(a) Faces f_1 and f_4 are end faces. (b) Weak dual of the graph in Fig. 1a.

Figure 1: An outerpath graph (left) and its weak dual (right).

triangular faces admits $(2,1)$-colourings, also showing that there exist outerplanar graphs with more than three triangles that do not admit $(2,1)$-colourings. Furthermore, Jonck Jr. and Campos [5, 6] characterised maximal outerplanar graphs that admit $(2,1)$-colourings: these graphs comprise a subclass of maximal outerplanar graphs whose weak dual is a path. These results propelled us to investigate whether the adjacency of triangles is related to the $(2,1)$-colourability of planar and outerplanar graphs.

In this work, we prove that deciding whether a planar graph has $(2,1)$-colourings is still NP-complete even for a subclass of outerplanar graphs in which all triangles are (vertex) disjoint. We also provide a polynomial time algorithm to decide whether any outerplanar graph whose weak dual is a path admits $(2,1)$-colourings, expanding the results of Jonck Jr. and Campos [5, 6].

2 Preliminaries

Let G be a plane graph and G' be its dual. Denote f_y the face of G that corresponds to vertex y of G^*, and denote $\partial(f_y)$ the boundary of f_y. If $\partial(f_y) \cong K_3$ for a face f_y, then f_y is called a *triangle* or *triangular face*. We also denote f_y by abc, meaning that $V(\partial(f_y)) = \{a,b,c\}$ and $E(\partial(f_y)) = \{ab, bc, ca\}$. Similarly, if $\partial(f_y) \cong C_4$, then $f_y = abcd$ is called a *square* or *square face*, with $V(\partial(f_y)) = \{a,b,c,d\}$ and $E(\partial(f_y)) = \{ab, bc, cd, da\}$. Two faces are *adjacent* if their boundaries have an edge in common, and are *apart* if their boundaries are vertex disjoint.

An *outerplane graph* G is a simple plane graph in which all vertices lie on the boundary of its outer face. If every pair of two distinct triangles of G is vertex-disjoint, G is called an *apart-outerplane graph*. Let G' be the dual of G. Suppose that u is the universal vertex corresponding to the outer face of G. Graph $G^* = G' - u$ is called *weak dual*. For outerplane graphs, its weak dual is isomorphic to a forest [7]. In particular, when G^* is isomorphic to a path, then G is called an *outerpath* graph and the faces of G corresponding to the leaves of this path are called *end faces*. Fig. 1 illustrates these concepts.

Let G be a graph endowed with a (k,d)-colouring π. The *defect* $\xi_\pi(v)$ of a vertex $v \in V(G)$ is the number of its neighbours that share the same colour of v, that is, $|\{x : x \in N(v) \text{ and } \pi(x) = \pi(v)\}|$. If G is a subgraph of G', then, π is called a *partial colouring* of G' and the vertices in $V(G') \setminus V(G)$ are *uncoloured vertices* of G'. Moreover, the *partial defect* of a vertex $v \in V(G')$ in π is the number of its neighbours in G that are coloured and have the same colour of v.

3 Main results

First, we prove that the $(2,1)$-colouring problem remains NP-complete even when restricted to apart-planar graphs – this result is established in Theorem 4. Next, we present and prove the correctness of the algorithm that decides whether an outerpath graph admits a $(2,1)$-colouring.

Definition 1. *Let ϕ be a boolean formula in 3-CNF. The* associated graph $G(\phi)$ *has one vertex v_x for each variable x of ϕ and one vertex v_c for each clause c of ϕ. There is an edge between v_x*

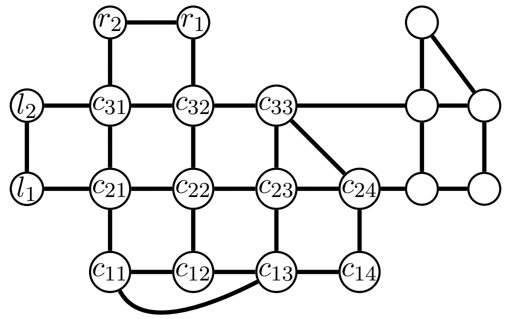


Figure 2: The literal gadget: labelled vertices play special role in the gadget.

and v_c if and only if x or $\neg x$ appears in c. The boolean formula ϕ is called planar *if its associated graph $G(\phi)$ is planar.*

Definition 2. *[8] In the* Positive Planar 1-in-3-SAT Problem (PP-1-in-3-SAT)*, we are given a collection ϕ of clauses containing exactly three variables together with a planar embedding of its associated graph $G(\phi)$. The problem is to decide whether there exists an assignment of truth values to the variables of ϕ such that exactly one variable in each clause is true.*

Definition 3. *In the* Apart-Planar $(2,1)$-colouring Problem ($(2,1)$-COL-AP)*, we are given a apart-planar graph G and have to decide whether G has a $(2,1)$-colouring.*

Theorem 4. *The apart-planar $(2,1)$-colouring problem is NP-complete.*

Proof. First, note that the problem belongs to NP. Given a colouring π of a given graph G, we can verify whether π is a $(2,1)$-colouring by checking, for each vertex, how many of its neighbours share the same colour. This verification can be done in $O(|V(G)| \cdot |E(G)|)$ time.

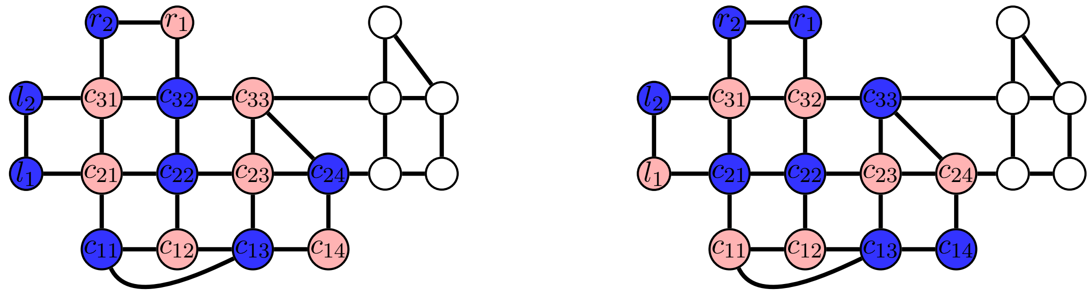
In order to prove our result, we reduce an instance of PP-1-in-3-SAT to an instance of $(2,1)$-COL-AP. Given a formula ϕ and its associated graph $G(\phi)$, we build an apart-planar graph G, by replacing each element of $G(\phi)$ with a gadget. We start by describing the gadgets.

The *literal gadget* replaces each variable vertex of $G(\phi)$ and it is illustrated in Fig. 2. We distinguish some vertices of the literal gadget, partitioning them into three groups: *core vertices* $(c_{11}, \ldots, c_{33})$ – the subgraph induced by these vertices is called the *core of the gadget*; *link vertices* (l_1, l_2); and *repeater vertices* (r_1, r_2).

We first show that there exist only two distinct $(2,1)$-colourings of the core of the gadget. First observe that, in any $(2,1)$-colouring π of the gadget, the endpoints of the edge $c_{24}c_{33}$ receive different colours – it happens because of the unlabelled vertices of the gadget. Since $\pi(c_{24}) \neq \pi(c_{33})$, we conclude that either $\pi(c_{23}) = \pi(c_{33})$ or $\pi(c_{23}) = \pi(c_{24})$. Each of these two options corresponds to a distinct colouring of the core and, in each case, the colours of the remaining core vertices are uniquely determined, up to flipping all colours, as we show in the following.

Let π_{T} be a $(2,1)$-colouring of the gadget such that $\pi_{\mathrm{T}}(c_{23}) = \pi_{\mathrm{T}}(c_{33}) = 1$. This implies that $\pi_{\mathrm{T}}(c_{22}) = \pi_{\mathrm{T}}(c_{32}) = 2$ and, therefore, $\pi_{\mathrm{T}}(c_{21}) = \pi_{\mathrm{T}}(c_{31}) = 1$. All remaining vertices of the core, except for c_{14}, are adjacent to at least one of the vertices whose colours have already been determined, all with defect 1. Consequently, each of them must receive the opposite colour of their already coloured neighbour. Finally, vertex c_{14} is adjacent to c_{24} and c_{13}, both of which with colour 2; so it must be assigned colour 1, completing the colouring of the core. Additionally, the colours of the link and repeater vertices are uniquely determined by this partial colouring.

Now, let π_{F} be the second $(2,1)$-colouring of the literal gadget. In this case, $\pi_{\mathrm{F}}(c_{23}) = \pi_{\mathrm{F}}(c_{24}) = 1$. Thus, $\pi_{\mathrm{F}}(c_{14}) = \pi_{\mathrm{F}}(c_{13}) = 2$ and $\pi_{\mathrm{F}}(c_{11}) = \pi_{\mathrm{F}}(c_{12}) = 1$. Finally, all the remaining vertices



(a) Colouring π_{T}.

(b) Colouring π_{F}.

Figure 3: The two possible $(2,1)$-colourings.

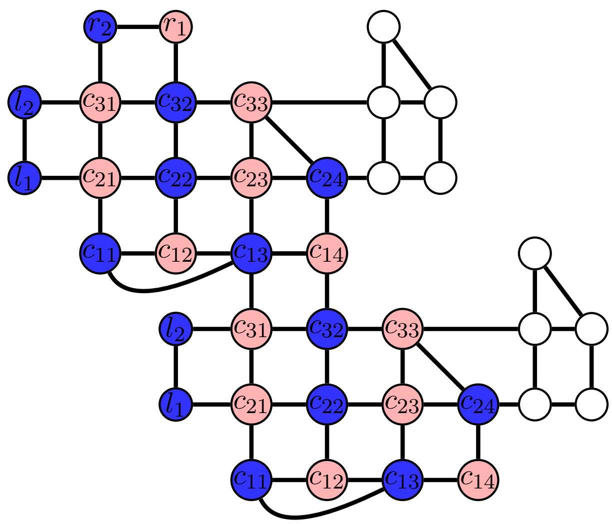


Figure 4: Example of chaining two literal gadgets.

of the core are incident with a vertex with defect 1 and must have the other only possible colour. Again, the colours of the link and repeater vertices are uniquely determined. Fig. 3 shows π_{T} and π_{F}.

Now, we analyse the repeater and link vertices. Recall that the colours of these vertices are already determined in π_{T} and π_{F}. Call the *logical value* of an edge *true* if its endpoints have the same colour and *false* otherwise. Observe that the logical value of $c_{23}c_{33}$ is true in π_{T} and false in π_{F}, which explains the subscript of the colouring names. Also, if a core of a literal gadget is endowed with a $(2,1)$-colouring π_{T} or π_{F}, we say that their logical value is true or false accordingly. Finally, note that link edge l_1l_2 always has the same logical value $c_{23}c_{33}$ whilst repeater edge r_1r_2 has the opposite.

This structure allows us to chain gadgets together. The colours of vertices c_{13} and c_{14} in the core are forced to match the colours of the repeater vertices. Therefore, by identifying the repeater vertices of one gadget with c_{13} and c_{14} vertices of the second gadget, we ensure that the cores of both gadgets receive the same colouring and, thus, the same truth value. This process, illustrated in Fig. 4, can be repeated to construct a gadget with different link edges all with the same logical value. We call a gadget constructed from x such copies a *degree x literal gadget.* We replace each variable vertex v_x in the graph $G(\phi)$ with a degree $d(v_x)$ literal gadget.

We now describe the two remaining gadgets: one for clause vertices and one for the edges that link clause vertices to literal vertices. The *clause gadget* is a triangle, and there is one such gadget for each clause vertex of the input graph $G(\phi)$. The *edge gadget* is a ladder $P_3\Box P_2$[1]. The new graph G is constructed by replacing: the clause vertices by clause gadgets; the literal vertices v_x by degree $d(v_x)$ literal gadgets; and each edge $v_{c_i}x_{x_j}$ by an edge gadget, identifying two degree 2 adjacent vertices to two vertices of the clause gadget and the remaining two degree 2 adjacent vertices to two adjacent link vertices of the degree $d(v_{x_j})$ literal gadget. Note that, in order to the resulting graph to be planar, the identification between link vertices and gadget vertices must be performed following the cyclic order induced by the planar embedding of graph G.

Since each element of $G(\phi)$ is replaced by a gadget with a constant number of vertices and edges, the total size of G is a linear function of the size of $G(\phi)$. Thus, the entire construction is computable in polynomial time.

In order to conclude the proof, we now show that ϕ is satisfiable if and only if G has a $(2,1)$-colouring.

First, assume ϕ is satisfiable. Let $x_1,\ldots,x_s$ be the set of variables assigned the value true. Each of these variables corresponds to a literal gadget in G. We colour the graph based on this assignment. For each literal gadget corresponding to variables $x_1,\ldots,x_s$ with value true, we choose π_{T}. For all other literal gadgets (those corresponding to false variables), we choose π_{F}. The colouring of the literal gadgets determines the truth values of the link edges, which, forces the colouring of the ladder and triangle gadgets. By definition, each clause has precisely one true literal. Thus, each clause gadget (a triangle) is connected to exactly one literal gadget with a true link edge. This ensures that each clause gadget has exactly one edge whose vertices share the same colour; and the defect of each vertex in G is at most one. This satisfies the $(2,1)$-colouring constraints for the entire graph G. Therefore, a satisfying assignment for ϕ directly yields a $(2,1)$-colouring for G.

Conversely, let π be a $(2,1)$-colouring of G. We define a truth assignment for the variables of ϕ. A variable x_i is assigned true if the truth value of its corresponding literal gadget in G is true; otherwise, it is false. Now, we show this is a satisfying assignment for ϕ. By construction, each clause gadget (a triangle) has exactly one edge whose vertices share the same colour, giving that edge a truth value of true, which is propagated to the edge gadget and, then, to exactly one of the link edges of the clause gadgets (recall that this implies all link vertices have the same truth value in the same gadget). Therefore, every clause is satisfied, concluding the proof.

□

4 Algorithm for outerpath graphs

Next, we establish our results about deciding whether an outerpath graph admits a $(2,1)$-colouring: Lemma 5, Theorem 10 and Corollary 11. First, we present Lemma 5, that shows that large faces do not play a role in deciding the existence of $(2,1)$-colourings of outerpath graphs. Then, we present Algorithm 1, that decides, in polynomial time, whether an outerpath graph G, whose maximum induced cycle has size at most six, has $(2,1)$-colourings. Theorem 10 proves the linear time on $|V(G^*)|$ complexity and correctness of Algorithm 1. Finally, Corollary 11 sums up all these ideas. We conclude this article with Corollary 12, which establishes that Lemmas 5 through 9 can be used to prove that all apart outerpath graphs admit $(2,1)$-colourings.

Let G be an outerpath graph. For the remaining of this text all faces considered are internal faces, i.e., faces different from the unlimited face of G. Let f be a (internal) face of G. We say that f is a *large face* if $|E(\partial(f))| \geq 7$. Let $V_L = \{v \in V(G) : v \text{ incides only in large faces}\}$. We call

[1]Graph $P_3\Box P_2$ denotes the Cartesian product of paths P_3 and P_2.

$H \subseteq G$ an *L-reduced* graph if $H = G - V_L$; that is, H is obtained by removing the vertices of V_L. Note that H may not be connected.

Lemma 5. *Let G be an outerpath graph and H be its L-reduced subgraph. Then, G has a $(2,1)$-colouring if and only if H has a $(2,1)$-colouring.*

Proof. Let G and H be as stated in the hypothesis. If $G = H$ the result follows. Suppose $G \neq H$. First, note that if G admits a $(2,1)$-colouring π, then the restriction of π to H is also a $(2,1)$-colouring of H. Now, suppose π is a $(2,1)$-colouring of H. Then, π is a partial $(2,1)$-colouring of G. Let $G_0 = H$ and $\pi_0 = \pi$. We construct pairs (G_0, π_0), (G_1, π_1), ..., (G_k, π_k) such that: at each step, G_{i+1} is obtained from G_i by colouring the uncoloured vertices of a large face of G_i; π_{i+1} is a $(2,1)$- colouring of G_{i+1}; $G_k \cong G$.

Suppose $i > 0$ and $G_i \cong G$. Let f be any large face with some uncoloured vertices. Note that, since G is an outerpath graph, at most two edges of $\partial(f)$ are incident with other faces. Therefore, there exist at least three uncoloured vertices of π_i and at most two pairs of adjacent vertices coloured: u and v; and x and y. If no such coloured pairs exists, choose any pair of adjacent uncoloured vertices of $V(\partial(f))$ for that purpose and colour its vertices arbitrarily, keeping π_i as a partial $(2,1)$-colouring.

Adjust notation so that P_{ux} and P_{yv} are paths of $\partial(f)$ with internal vertices uncoloured. Suppose, without loss of generality, that $|E(P_{ux})| \leq |E(P_{yv})|$. If $|E(P_{ux})| \geq 2$, then uv and xy lie in different connected components of G_i. If $\pi_i(u) \neq \pi_i(x)$, swap all colours of the component that includes xy. Therefore, if $|E(P_{ux})| \geq 2$, we can assume that $\pi_i(u) = \pi_i(x)$. Let $x' \in V(P_{ux})$ and $y' \in V(P_{yv})$ such that $xx', yy' \in E(G)$. Assign to x' (resp. y') a colour distinct from that of x (resp. y). For the remaining vertices, starting from the other ends of P_{ux} and P_{yv}, proceed along each path in linear order, colouring the vertices, alternating the colours. Thus, vertices u, v, x, and y and their adjacent vertices in P_{ux} and P_{yv} have distinct colours. Moreover, by construction, every vertex in P_{ux} and P_{yv} has defect at most 1. Therefore, (G_{i+1}, π_{i+1}) satisfies all requirements. At the end of the process, π_k is a $(2,1)$-colouring of $G_k \cong G$ and the result follows. □

Let H be the L-reduced graph of G and let H' be one of its connected components. Note that H' is not necessarily 2-connected. Moreover, since H' is obtained from an outerpath graph G, its faces and blocks have an order inherited from the canonical order of the faces of G. A *canonical block ordering* for the blocks of H' is $B_1, B_2, \ldots, B_k$, following the canonical order of the faces of G, such that $V(B_i) \cap V(B_{i+1}) \neq \emptyset$, $0 \leq i < k$. By construction, each of its blocks is an outerpath graph. Observe that the canonical ordering of the faces ensures that the last face of B_i is adjacent to the first face of B_{i+1}.

In order to present Algorithm 1, we first introduce some additional notation. Let G be an outerpath graph and π be a $(2,1)$-colouring of G. For an edge $uv \in E(G)$, we define $P_\pi(uv) = (\alpha, \beta)$, with $\alpha \in \{S, D\}$, $\beta \in \{00, 01, 10, 11\}$ such that $\alpha = S$ ("same") if $\pi(u) = \pi(v)$, and $\alpha = D$ ("distinct") otherwise; and $\beta = ij$ if $\xi(u) = i$ and $\xi(v) = j$. Define $\mathcal{P}(uv) = \{P_\pi(uv) :$ π is a $(2,1)$-colouring of $G\}$ as the *possibilities* of uv.

Since G^* is isomorphic to a path, let this path be $v_1 v_2 \ldots v_k$. Define the *canonical ordering* of the faces of G as $f_1, f_2, \ldots, f_k$, in which f_i of G corresponds to v_i of G^*. For $1 < i < k$, let $u_i v_i = E(f_{i-1}) \cap E(f_i)$ and $x_i y_i = E(f_i) \cap E(f_{i+1})$ be the *entry edge* and *exit edge* of f_i, respectively. For $i = 1$, the exit edge is $E(f_1) \cap E(f_2)$ and the entry edge can be any edge incident with the outer face; for $i = k$, define the entry edge and exit edge symmetrically. The i^{th} edge of the face f is the edge at distance i from its entry edge when traversing $\partial(f)$ in clockwise order, and β is obtained also considering the clockwise order.

Let f be a face of an outerpath graph G whose longest induced cycle is at most 6 and let e be its entry edge. We show that $\mathcal{P}(e)$ uniquely determines $\mathcal{P}(e')$, for all $e' \in E(\partial(f))$. This correspondence is established in Lemmas 6 through 9.

Lemma 6. *Let G be an outerpath graph endowed with a partial $(2,1)$-colouring π and e be an entry edge of a triangular face f. Suppose the endpoints of e are the only coloured vertices of f. Then, Table 1 establishes the correspondence between $P_\pi(e)$ and $\mathcal{P}(e')$, $e' \in E(\partial(f))$, $e' \neq e$.*

Table 1: Possibilities for the i^{th} edge of a triangular face.

$P_\pi(e)$ \ $e' \in E(\partial(f))$	1^{st} edge	2^{nd} edge
S-11	D-10	D-01
D-00	S-11 D-01	D-10 S-11
D-01	D-11	S-11
D-10	S-11	D-11
D-11	$\emptyset$	$\emptyset$

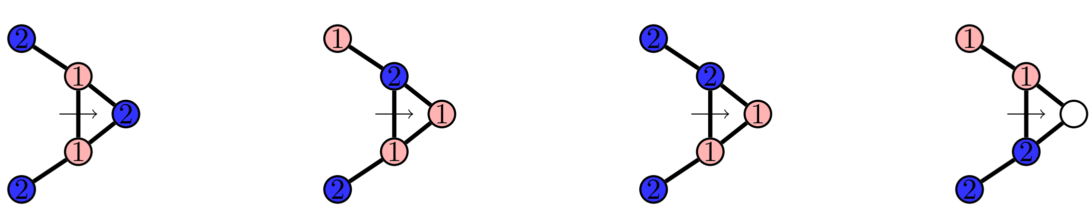
Proof. Let G, e and f be as defined. Suppose π is a partial $(2,1)$-colouring of G. First, note that, in any coloured triangular face of G, exactly one edge must have both endpoints assigned the same colour.

Suppose $P_\pi(e) =$ S-11. Since the endpoints of e have the same colour and defect 1, we conclude that the third vertex has the other colour and defect 0. In clockwise order, we have that $P_\pi(1^{\text{st}}$ edge$) =$ D-10 and $P_\pi(2^{\text{nd}}$ edge$) =$ D-01.

Now, suppose $P_\pi(e) =$ D-00. Then, either the 1^{st} edge or the 2^{nd} edge must have its endpoints with the same colour. Thus, we have that either $P_\pi(1^{\text{st}}$ edge$) =$ S-11 and $P_\pi(2^{\text{nd}}$ edge$) =$ D-10 or $P_\pi(1^{\text{st}}$ edge$) =$ D-01 and $P_\pi(2^{\text{nd}}$ edge$) =$ S-11. Therefore, $\mathcal{P}(1^{\text{st}}$ edge$) = \{\text{S-11}, \text{D-01}\}$ and $\mathcal{P}(2^{\text{nd}}$ edge$) = \{\text{D-10}, \text{S-11}\}$.

Next, suppose $P_\pi(e) = D - 01$. Again, one of the 1^{st} edge or 2^{nd} edge has its endpoints with the same colour. However, in this case, this edge must be the one that shares the endpoint with defect 0. We conclude that $P_\pi(2^{\text{nd}}$ edge$) =$ S-11, which completely determines the possibility for the 1^{st} edge, establishing $P_\pi(1^{\text{st}}$ edge$) =$ D-11. The case in which $P_\pi(e) =$ D-10 is symmetric.

Finally, suppose $P_\pi(e) =$ D-11. In this case, it is not possible to assign a colour to the remaining vertex of $\partial(f)$ so as to extend π; every possible assignment would result in a vertex with defect 2. These cases are illustrated in Fig. 5, in which the entry edge is indicated by an arrow. □



(a) (b) (c) (d)

Figure 5: Possibilities for the edges of a triangular face.

Lemma 7. *Let G be an outerpath graph endowed with a partial $(2,1)$-colouring π and e be an entry edge of a square face f. Suppose the endpoints of e are the only coloured vertices of f. Then, Table 2 establishes the correspondence between $P_\pi(e)$ and $\mathcal{P}(e')$, $e' \in E(\partial(f))$, $e' \neq e$.*

Table 2: Possibilities for the i^{th} edge of a square face.

$P_\pi(e)$ \ $e' \in E(\partial(f))$	1^{st} edge	2^{nd} edge	3^{rd} edge
S-11	D-11	S-11	D-11
D-00	D-00 S-11	D-00 D-11	D-00 S-11
D-01	D-10	D-00	D-00
D-10	D-00	D-00	D-01
D-11	D-10	D-00	D-01

Proof. The proof is analogous to the proof of Lemma 6. □

Lemma 8. *Let G be an outerpath graph endowed with a partial $(2,1)$-colouring π and e be an entry edge of a pentagonal face f. Suppose the endpoints of e are the only coloured vertices of f. Then, Table 3 establishes the correspondence between $P_\pi(e)$ and $\mathcal{P}(e')$, $e' \in E(\partial(f))$, $e' \neq e$.*

Table 3: Possibilities for the i^{th} edge of a pentagonal face. Line D-10 is symmetric to D-01, like is shown in table 2.

$P_\pi(e)$ \ $e' \in E(\partial(f))$	1^{st} edge	2^{nd} edge	3^{rd} edge	4^{th} edge
S-11	D-10	D-00	D-00	D-01
D-00	S-11 D-01 D-00	S-11 D-00 D-01 D-10	D-00 D-10 D-01 S-11	D-00 D-10 S-11
D-01	D-10 D-11	S-11 D-00 D-01	D-10 D-01 S-11	D-00 S-11 D-10
D-11	D-11 D-10	S-11 D-01	D-10 S-11	D-01 D-11

Proof. The proof is analogous to the proof of Lemma 6. □

Lemma 9. *Let G be an outerpath graph endowed with a partial $(2,1)$-colouring π and e be an entry edge of a hexagonal face f. Suppose the endpoints of e are the only coloured vertices of f. Then, Table 4 establishes the correspondence between $P_\pi(e)$ and $\mathcal{P}(e')$, $e' \in E(\partial(f))$, $e' \neq e$.*

Table 4: Possibilities for the i^{th} edge of a hexagonal face. Line D-10 is symmetric to D-01, like is shown in table 2.

$e' \in E(\partial(f))$ / $P_\pi(e)$	1^{st} edge	2^{nd} edge	3^{rd} edge	4^{th} edge	5^{th} edge
S-11	D-10 D-11	D-01 S-11 D-00	S-11 D-10 D-01	D-10 D-00 S-11	D-01 D-11
D-00	D-00 S-11 D-01	D-00 D-10 D-01 S-11 D-11	D-00 D-01 S-11 D-10 D-11	D-00 S-11 D-11 D-10 D-01	D-00 D-10 S-11
D-01	D-10 D-11	D-00 S-11 D-01	D-00 D-11 S-11 D-10	D-00 S-11 D-01 D-11	D-00 D-10 S-11
D-11	D-10 D-11	D-00 S-11	D-00 D-11	D-00 S-11	D-01 D-11

Proof. The proof is analogous to the proof of Lemma 6. □

Algorithm 1 takes as input an outerpath graph G and determines whether G admits a $(2, 1)$-colouring. It consists of three nested loops. The third loop runs inside of function isColourable, called in the first line of the second loop. The algorithm starts by getting the L-reduced subgraph H of G. Each loop targets different structures of the graph: the outer loop iterates over each component; the second, over blocks within each component; and finally, the innermost loop iterates over each face. Faces are visited according to the canonical ordering of G. For each face and entry edge e, given $\mathcal{P}(e)$, the algorithm computes the possibilities of the exit edge, considering all possible colourings of the subgraph of H induced by the faces visited until that moment. For each connected component, since no face has been visited at the beginning of the algorithm, the initial possibilities set is $\{\text{S-11}, \text{D-00}, \text{D-01}, \text{D-10}, \text{D-11}\}$, that is, its vertices may assume any colour and any defect. At each iteration, to obtain the possibilities of the exit edge, function Restriction takes the possibilities of the entry edge and computes the union of the table rows associated with its entries.

isColourable $(G, e_{in}, e_{last}, p_{in})$

```
f_1, f_2, ..., f_k ← canonical order of the faces of G
for i = 1 to k − 1 do
    e_out ← E(f_i) ∩ E(f_{i+1})
    p_out ← Restriction(f_i, e_in, e_out, p_in)
    if (p_out = {D-11}) ∧ (f_{i+1} is a triangular face) then
        return {}
    end if
    e_in ← e_out
    p_in ← p_out
end for
return Restriction(f_k, e_in, e_last, p_in)
```

Algorithm 1

Input: an outerpath graph G
Output: TRUE if there is a $(2,1)$-colouring of G; and FALSE otherwise.

1: $H \leftarrow$ L-reduced graph from G
2: **for** each connected component H' of H **do**
3: $\quad \mathcal{B} \leftarrow$ canonical ordering of H' into blocks $B_1, B_2, \ldots, B_k$
4: $\quad p_{ini} \leftarrow \{\text{S-11}, \text{D-00}, \text{D-01}, \text{D-10}, \text{D-11}\}$
5: $\quad v_{cut} \leftarrow V(B_1) \cap V(B_2)$
6: $\quad e_{ini} \leftarrow e \in E(G)$ incident with the outer face and first face of B_1
7: $\quad e_{last} \leftarrow e \in E(G)$ in the boundary of last face of B_1 and incident with v_{cut}
8: $\quad$ **for** $i = 1$ **to** k **do**
9: $\quad\quad p_{out} \leftarrow$ ISCOLOURABLE$(B_i, e_{ini}, e_{last}, p_{ini})$
10: $\quad\quad$ **if** $(p_{out} = \{\})$ **then**
11: $\quad\quad\quad$ **return** FALSE
12: $\quad\quad$ **end if**
13: $\quad\quad$ **if** v_{cut} has defect 0 in any $P_\pi(e_{last}) \in p_{out}$ **then**
14: $\quad\quad\quad p_{ini} \leftarrow \{\text{S-11}, \text{D-00}, \text{D-01}, \text{D-10}, \text{D-11}\}$
15: $\quad\quad$ **else**
16: $\quad\quad\quad p_{ini} \leftarrow \{\text{D-10}, \text{D-11}\}$ ▷ Assuming v_{cut} is always the first vertex in e_{ini}
17: $\quad\quad$ **end if**
18: $\quad\quad$ **if** $i < k$ **then**
19: $\quad\quad\quad e_{ini} \leftarrow e \in E(G)$ incident with v_{cut} and $e \in \partial(\text{outer face}) \cap \partial(\text{first face of } B_{i+1})$
20: $\quad\quad\quad$ **if** $i < k - 1$ **then**
21: $\quad\quad\quad\quad v_{cut} \leftarrow V(B_{i+1}) \cap V(B_{i+2})$ and also in $V(B_{i+2})$
22: $\quad\quad\quad$ **else**
23: $\quad\quad\quad\quad v_{cut} \leftarrow v \in V(G)$ in the last face of $V(B_{i+1})$
24: $\quad\quad\quad$ **end if**
25: $\quad\quad\quad e_{last} \leftarrow e \in E(G)$ incident with the last face of B_{i+1}, the outer face, and v_{cut}
26: $\quad\quad$ **end if**
27: $\quad$ **end for**
28: **end for**
29: **return** TRUE

Theorem 10. *Let function* ISCOLOURABLE *from Algorithm 1 be executed on an outerpath graph G with circumference at most 6 and let e_{last} be an edge of the last face of the canonical order of G. Then, the algorithm returns the set of possibilities for e_{last}. Moreover, it runs in linear time.*

Proof. Let G be an outerpath graph with circumference at most 6 and $f_1, \ldots, f_l$ be the canonical order of the faces of G. At iteration i of the main loop, let H_i denote the subgraph of G induced by $\partial(f_1), \ldots, \partial(f_i)$, following the canonical order. We first prove that, at iteration i of the algorithm, set p_{out}, computed by function RESTRICTION in line 4 is the set possibilities of the exit edge e_{out} of f_i. The proof proceeds by induction on the number k of iterations.

For the base case ($k = 1$), subgraph H_1 contains only face f_1. Thus, the result follows from Lemmas 6, 7, 8, and 9 since function RESTRICTION consults their tables directly. Now, assume the claim holds for iteration k and consider iteration $k + 1$. In line 9, set p_{in} receives set p_{out} that has been just calculated at iteration k. By the induction hypothesis, in any $(2,1)$-colouring of H_k, the exit edge of face f_k has possibilities $\mathcal{P}(e_{in})$ as set p_{in}. Thus, for each element of p_{in}, Lemmas 6 to 9 provide the corresponding possibilities set, and their union is p_{out}, for the exit edge of f_{k+1}. Therefore, for any $(2,1)$-colouring of H_{k+1}, set p_{out} of edge e_{out} respects the union of all possibilities derived from the tables in these lemmas, that is, it is equal to the set $\mathcal{P}(e_{out})$.

Hence, if the algorithm reaches the final face without being interrupted by the conditional in line 5, it returns $\mathcal{P}(e_{out})$. In this case, there exists at least one valid $(2,1)$-colouring of G, by the definition, since at each iteration the set of possibilities is nonempty. On the other hand, suppose the conditional in line 5 is true at iteration i. Then, for every $(2,1)$-colouring of H_i, the exit edge e_{out} has endpoints with distinct colours, each with defect 1, i.e. $\mathcal{P}(e_{out}) = \{(D,11)\}$. Moreover, this edge is also in $\partial(f_{i+1})$ that includes an uncoloured vertex v. By Lemma 6, for any exit edge of f_{i+1}, its possibilities set is empty. Therefore, no $(2,1)$-colouring is possible for G and $\mathcal{P}(e_{out}) = \{\}$.

We now analyse the time complexity of function isColourable when executed on an outerpath graph G with n vertices. The dual of G can be constructed in linear time. Thus, the canonical order for the faces of G can be obtained in linear time (line 1). In the main loop (lines 2 to 10), each vertex is visited at most four times since the boundary of each face is visited once when it is the next face (f_{i+1}) and again when it is the current one (f_i).

Function Restriction first uses the degree of f_i to select the appropriate table and determines the column to consult based on the position of the exit edge relative to the entry edge. Then, for each element of p_{in}, it outputs the union of the possibilities sets contained in the corresponding cells. Since there are at most five elements, the union can be computed in constant time using an indexed list.

The remaining operations in the loop, including table lookups and possibilities set updates, are performed in constant time per iteration. Since the loop iterates once per face, and the number of faces in a planar embedding of an outerpath graph is linear in the number of vertices, the entire algorithm runs in linear time. □

Corollary 11. *Let Algorithm 1 be executed on an outerpath graph G. Then, the algorithm returns* true *if and only if G admits a $(2,1)$-colouring. Moreover, it runs in polynomial time.* □

Corollary 12. *Every apart outerpath graph admits $(2,1)$-colourings.* □

References

[1] J. A. Andrews and M. S. Jacobson. On a generalization of chromatic number. *Congressus Numerantium*, 47:33–48, 1985.

[2] F. Harary and K. Jones. Conditional colorability ii: Bipartite variations. *Congressus Numerantium*, 50:205–218, 1985.

[3] L. J. Cowen, R. H. Cowen, and D. R. Woodall. Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency. *Journal of Graph Theory*, 10(2):187–195, 1986.

[4] Lenore Cowen, Wayne Goddard, and C Esther Jesurum. Defective coloring revisited. *Journal of Graph Theory*, 24(3):205–219, 1997.

[5] A. L. J. Junior. Sobre $(2,1)$-colorações em grafos exoplanares. Master dissertation, Unicamp, 2023.

[6] A. L. J. Junior and C. N. Campos. Sobre (2, 1)-colorações em grafos exoplanares maximais. In *Anais do VI Encontro de Teoria da Computação*, pages 58–61. SBC, 2021.

[7] Rowland Leonard Brooks. On colouring the nodes of a network. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 37, pages 194–197. Cambridge University Press, 1941.

[8] Wolfgang Mulzer and Günter Rote. Minimum-weight triangulation is NP-hard. *Journal of the ACM (JACM)*, 55(2):1–29, 2008.