Neighbour-distinguishing edge-labellings of Powers of Paths

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Abstract

Given a graph G, the pair (π, c_π) is a neighbour-distinguishing k-edge-labelling if $\pi : E(G) \to \{1, \ldots, k\}$ such that, for every $v \in V(G)$, $c_\pi(v) = \sum_{u \in N(v)} \pi(uv)$ and $c_\pi(x) \neq c_\pi(y)$ for every edge $xy \in E(G)$. The least k for which it has been shown that every graph admits a neighbour-distinguishing k-edge-labelling is three. The $1, 2, 3$-Conjecture, proposed in 2004 by Karoński et al., states that every graph has a neighbour-distinguishing 3-edge-labelling. This conjecture has been recently proved by Keusch and published May 2024. In 2017, Luiz and Campos verified the $1, 2, 3$-Conjecture for powers of paths and conjectured that a neighbour-distinguishing 2-edge-labelling could be built for powers of paths not isomorphic to complete graphs. In this work, we prove this conjecture.

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. As usual, we denote an edge $e \in E(G)$ by uv, with u and v its *endpoints*. The *degree* of a vertex $v \in V(G)$ is denoted by $d(v)$. If $d(u) \neq d(v)$ for every pair $u, v \in V(G)$, then G is said to be *irregular*. The *neighbourhood* of v is $N(v) = \{u : uv \in E(G)\}$. Given a simple graph G and two vertices u and v of $V(G)$, the *distance* $dist_G(u, v)$ between these vertices is the size of a shortest path between them. If no such a path exists, $dist_G(u, v) = \infty$.

A *vertex colouring* of G is a mapping $c : V(G) \to \mathcal{C}$, with $\mathcal{C}$ a set of *colours*. If $c(u) \neq c(v)$ for every pair u, v of adjacent vertices, then c is said to be *proper*. If G has a proper vertex-colouring with $|\mathcal{C}| = k$, then G is k*-colourable*.

For a set $\mathcal{L}$ of labels and $\mathcal{C}$ a set of colours, a pair (π, c_π) for which $\pi : E(G) \to \mathcal{L}$ and $c_\pi : V(G) \to \mathcal{C}$, with $c_\pi(v) = \sum_{u \in N(v)} \pi(uv)$, is an *edge-labelling*. If $\mathcal{L} = \{1, \ldots, k\}$, we say that (π, c_π) is a k*-edge-labelling*. Moreover, if c_π is proper, then (π, c_π) is called a *neighbour-distinguishing edge-labelling* (NDEL) or *neighbour-distinguishing* k*-edge-labelling* (k-NDEL). The least k for which G has a k-edge-labelling is denoted $\chi^e_\Sigma(G)$. Note that no simple graph with isolated edges has an NDEL since vertices incident with an isolated edge always have the same colour for any (π, c_π). A simple graph with no isolated edges is called a *nice graph*.

Neighbour-distinguishing edge-labellings were introduced in 2004 by M. Karoński et al. [1] as a variation of the problem of finding the least number of non-loop edges that could be added to a graph so as to make it irregular. In the same article it is shown that there exist neighbour-distinguishing 3-edge-labellings for complete graphs and for 3-colourable graphs. Based on these results, the authors posed the $1, 2, 3$-*Conjecture* that states that every nice graph G has a 3-NDEL.

Since it was posed, the $1, 2, 3$-Conjecture has drawn significant attention from researchers, resulting in numerous published works, many of them focusing on boundary cases for the conjecture [2, 3, 4, 5, 6]. In 2024, R. Keusch proved the 1,2,3-Conjecture [7]. The proof, however, does

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not provide a polynomial-time algorithm for constructing a 3-NDEL. Additionally, several compelling questions about NDELs remain open for further investigation. One of them, investigated in this paper, is finding graphs that have a 2-NDEL.

It is straightforward to determine whether a graph has a 1-NDEL, as these are precisely graphs in which $d(u) \neq d(v)$, for $uv \in E(G)$. However, determining which graphs have a 2-NDEL is more challenging. Dudek and Wajc [8] proved that deciding whether a given graph has a 2-NDEL is NP-complete. This result opens a new research approach: investigating this problem for specific classes of graphs. For instance, Thomassen et al.[9] characterized all bipartite graphs that have neighbour-distinguishing 2-edge-labellings. Escuadro et al. [10] proved that for $2 \leq k < 7$, C_n^k has 2-NDELs if and only if it is not isomorphic to K_n. Moreover, in the same paper, the authors showed that powers of paths also have 2-NDELs when $k \geq 2$ and $n \geq k(k+1)$.

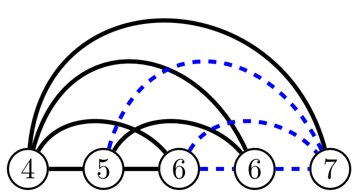
The k^{th} *power of a graph* G is a graph G^k with vertex set $V(G^k) = V(G)$ and two vertices being adjacent if and only if $dist_G(u,v) \leq k$. We denote by P_n the path graph of order n. A power G^k of a path graph $G \cong P_n$ is a *power of paths* and is denoted P_n^k. The *linear ordering* of $V(P_n^k)$ is $(v_0, v_1, \ldots, v_{n-1})$, such that $v_i v_{i+1} \in E(P_n)$ for $0 \leq i \leq n-2$. The *reach* between vertices u and v in $V(P_n^k)$ is $dist_{P_n}(u,v)$. Observe that P_n^1 is isomorphic to graph P_n. On the other hand, if we have that $k \geq n-1$, then P_n^k is isomorphic to a complete graph. Similarly to powers of paths, a power G^k of a cycle $G \cong C_n$ is a *power of cycle* and is denoted C_n^k. Observe that C_n^1 is isomorphic to graph C_n and, when $k \geq \left\lfloor \frac{n}{2} \right\rfloor$, C_n^k is isomorphic to K_n.

Since powers of cycles are regular graphs, they do not admit 1-NDELs. It was proven by Luiz et al. [11] that every power of cycles has a 3-NDEL. The only power of a path that admits a 1-NDEL is P_3, as every other power of a path contains at least two adjacent vertices with the same degree. Furthermore, no complete graph admits a 2-NDEL, as this would require a partition into two simple irregular subgraphs, which is impossible. Thus, the only power of paths that can admit 2-NDELs has $k < n-1$. Luiz et al. [11] also proved that every power of paths has a 3-NDEL. The authors further conjectured that every power of a path not isomorphic to a complete graph admits a 2-NDEL. In this work, we prove this conjecture, formally stated as Theorem 1.

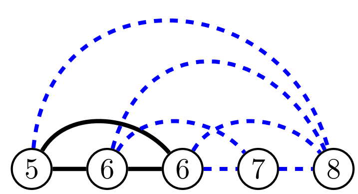
Theorem 1. *[11] Let G be a power of paths such that $G \ncong K_n$. Then, $\chi_\Sigma^e(G) \leq 2$.*

1.1 Preliminaries

In order to prove our main result, we define a family of edge-labellings for complete graphs, with $\mathcal{L} = \{1, 2\}$, which we call type-l edge-labelling, $l \in \mathbb{N}^*$. Let $V(K_n) = \{v_0, \ldots, v_{n-1}\}$. A *type-$l$ edge-labelling* (π, c_π) is defined as follows: $\pi(v_i v_j) = 2$ if $j > n - (l+i)$; $\pi(v_i v_j) = 1$ otherwise; and $c_\pi(v_j) = \sum_{u \in N(v)} \pi(uv_j)$. Fig. 1 shows examples of type-l edge-labellings for K_5. Blue dashed edges are labelled 2 and the remaining 1. Type-l labellings are used to build a 2-NDEL for power of paths in the proof of Theorem 4.



(a) Type-1.



(b) Type-2.

Figure 1: Edge-labellings for K_5. Vertex labels are their colours.

It is not difficult to see that complete graphs do not have 2-NDELs. Therefore, type-l edge-labellings are not neighbour-distinguishing. However, they present an interesting, and useful, property: following the linear ordering of the vertex indices, the colours appear in non-decreasing order with exactly one block of consecutive vertices with the same colour. Proposition 1.3 and Corollary 1.4 establish these results.

Proposition 2. *Let $G \cong K_n$ endowed with a type-l edge-labelling. Then:*

$$c_\pi(v_i) = \begin{cases} d(v_i) + i + (l-1) & \text{if } i \leq \lfloor \frac{n-l}{2} \rfloor; \\ d(v_i) + i + (l-2) & \text{otherwise.} \end{cases}$$

□

Corollary 3. *Let $G \cong K_n$ endowed with a type-l edge-labelling, $l \leq 2$. Then, every vertex has a distinct colour except for $v_{\lfloor \frac{n-l}{2} \rfloor}$ and $v_{\lfloor \frac{n-l}{2} \rfloor + 1}$.* □

2 Main results

In this section, we provide a proof for Theorem 1. In order to show that $\chi^e_\Sigma(P^k_n) \leq 2$, we partition $V(P^k_n)$ into blocks and use type-l labellings for these blocks. Then, we adjust the labelling so as to obtain a 2-NDEL for P^k_n.

Theorem 4. *Let G be a power of paths such that $G \ncong K_n$. Then, $\chi^e_\Sigma(G) \leq 2$.*

Proof. Let $G \cong P^k_n$ for $n \geq 3$. Since $G \ncong K_n$, we conclude that $k \leq n-2$. Consider the canonical ordering of its vertices $(v_0, v_1, v_2, \ldots, v_{n-1})$. We construct a 2-NDEL for G dividing the construction into two cases: $n \geq 3k+1$; and $n < 3k+1$.

Case 1. $n \geq 3k+1$

Let $\mathcal{B} = \{B^0, B^1, \ldots, B^{q+2}\}$ be a partition of $V(P^k_n)$ into *blocks* such that: two blocks have size k; q of them have size $k+1$; and one has size $k+1+r$, in which $q = \left\lfloor \frac{n-(3k+1)}{k+1} \right\rfloor$ and $r = (n-(3k+1)) \bmod (k+1)$. Each block B^i comprises a set of consecutive vertices in the following way:

(i) block B^0 is comprised of vertices $v_0, v_1, \ldots, v_{k-1}$, representing the first k vertices of the ordering;

(ii) block B^i, for $1 \leq i \leq q$, consists of vertices $v_{ki}, v_{ki+1}, \ldots, v_{ki+k-1}$;

(iii) block B^{q+1} is formed by $k+1+r$ vertices with indices from $(k+1)(q+1)-1$ to $[(k+1)(q+1)-1]+(k+r)$;

(iv) finally, block B^{q+2} has vertices $v_{n-k}, v_{n-k+1}, \ldots, v_{n-1}$.

Denote by v^j_i the i^{th} vertex of block B^j. Fig. 2 exemplifies this block partition on P^3_{15}. Note that every vertex in B^i, for $1 \leq i \leq q+1$, has degree $2k$, while vertices in B^0 and B^{q+2} have degree at most $2k-1$.

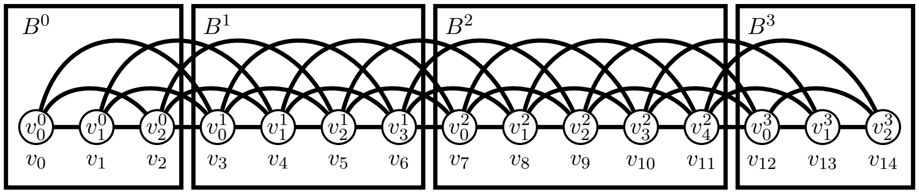


Figure 2: Graph P^3_{15} with a block partition.

The set of edges of G is partitioned into two parts: edges whose endpoints are in the same block and edges whose endpoints are in consecutive blocks. We call the latter *linking edges*. Note that, by construction, there are no edges that link nonconsecutive blocks. Block B^{q+1} has at least $k+1$ vertices. Due to its size, block B^{q+1} is decomposed into two sub-blocks, B_{pre} and B_{suf}, ensuring that B_{pre} contains the first $k+1$ vertices of B^{q+1}, while B_{suf} contains the remaining r vertices. The vertices in B_{pre} are assigned indices from $(k+1)(q+1)-1$ to $[(k+1)(q+1)-1]+k$, and the vertices in B_{suf} have indices ranging from $[(k+1)(q+1)-1]+(k+1)$ to $[(k+1)(q+1)-1]+(k+r)$. For example, in Fig. 2, $B_{pre} = \{v^2_0, v^2_1, v^2_2, v^2_3\}$ and $B_{suf} = \{v^2_4\}$. To facilitate subsequent references within each sub-block, we relabel the vertices of B_{pre} using the superscript p and those of B_{suf} using the superscript s. In the case of the previous example, $B_{pre} = v^2_0, v^2_1, v^2_2, v^2_3$ is relabeled as $B_{pre} = v^p_0, v^p_1, v^p_2, v^p_3$, and $B_{suf} = v^2_4$ becomes $B_{suf} = v^s_0$.

Now, we are ready to build a 2-NDEL (π, c_π) for G. First, suppose there is a (π, c_π) satisfying the following properties.

(i) For v^j_i in $B^1, \ldots, B^{q+1}$:

$$c_\pi(v^j_i) = \begin{cases} 2k+1+i & \text{if } 1 \le j \le q \text{ and } 0 \le i \le k; \\ 2k+1+i & \text{if } j = q+1 \text{ and } v^j_i \in B_{pre}; \\ k+i & \text{if } j = q+1 \text{ and } v^j_i \in B_{suf}. \end{cases}$$

(ii) for $v^j_i \in \{B^0, B^{q+2}\}$, $c_\pi(v^j_i) \in \{d(v^j_i), d(v^j_i)+1\}$; also, $c_\pi(v^0_{i-1}) < c_\pi(v^0_i) < c_\pi(v^0_{i+1})$ and $c_\pi(v^{q+2}_{i-1}) > c_\pi(v^{q+2}_i) > c_\pi(v^{q+2}_{i+1})$, whenever v^0_{i-1}, v^0_{i+1}, v^{q+2}_{i-1}, and v^{q+2}_{i+1} exist.

Now, we show that such c_π is a proper colouring. Let v^j_i and v^t_l be two adjacent vertices of G. First note that if $j = t = 0$ or $j = t = q+2$, $c_\pi(v^j_i) \neq c_\pi(v^t_l)$ by definition. Now, suppose $j = 0$ and $t = 1$ or $j = q+2$ and $t = q+1$. Then, $d(v^j_i) < 2k$ and $c_\pi(v^t_l) \ge 2k+1$. Thus, $c_\pi(v^j_i) \le d(v^j_i) + 1 < 2k+1 \le c_\pi(v^t_l)$. We conclude that $c_\pi(v^j_i) \neq c_\pi(v^t_l)$ in this case.

It remains to analyse the cases in which v^j_i and v^t_l belong to blocks $B^1, \ldots,$ B^{q+1}. We first approach the case in which $1 \le j, t \le q$. If $j = t$, then $c_\pi(v^j_i) \neq c_\pi(v^t_l)$, as the colours depend on the distinct values of i and l within the same block. Suppose $j \neq t$. Without loss of generality, we can assume $t = j+1$. In this case, $c_\pi(v^j_i) = c_\pi(v^t_l)$ only when $i = l$. This implies that the reach between v^j_i and v^t_l is $k+1$, meaning that there are k vertices between them in the same block. Consequently, these vertices are non-adjacent.

At last, consider the case in which either v^j_i or v^t_l belongs to B^{q+1}. If $v^j_i, v^t_l \in B_{pre}$ or $v^j_i, v^t_l \in B_{suf}$ or, even, $v^j_i \in B^q$ and $v^t_l \in B_{pre}$, the result follows by the same argument of the previous case. Suppose, then, that $v^j_i \in B_{pre}$ and $v^t_l \in B_{suf}$, and that v^j_i and v^t_l are adjacent. Note that $c_\pi(v^j_i) = 2k+1+i$ and $c_\pi(v^t_l) = k+l$. Suppose that $2k+1+i = k+l$. Then, $l - i = k+1$. However, this assumption leads to a contradiction, as v^j_i and v^t_l are adjacent, implying that $l - i \le k$.

To complete the proof of this case, we show that there exists (π, c_π) satisfying (i) and (ii). Initially, we label the subgraph induced by each block of $\mathcal{B} \setminus \{B^{q+1}\}$ with a type-2 labelling. For block B^{q+1}, its sub-blocks B_{pre} and B_{suf} also each receive a type-2 labelling individually. Finally, we need to assign labels to the linking edges. We choose a matching M contained in this set of edges to receive label 2, while the remaining linking edges receive label 1.

Suppose that the subgraphs induced by the blocks of $\mathcal{B}$ are endowed with a type-2 labelling as indicated above. Let $\alpha = \left\lfloor \frac{(k+1)}{2} \right\rfloor$ and $\beta = \left\lfloor \frac{(|B_{suf}|)}{2} \right\rfloor$. Consider, for one moment, that all linking edges have label 1. It remains to define M, the subset of linking edges that receive label 2. Let $M = M' \cup M''$, such that M' contains edges with at least one endpoint in B^i, $0 \le i \le q$, and M'' contains edges with both endpoints in $B^{q+1} \cup B^{q+2}$.

In each block B^j, $1 \le j \le q$, there are edges in M'. One endpoint of these edges is incident with vertices from v^j_α to v^j_{k-1}, the other may be in B^{j-1} or B^{j+1}, depending on the parity of j. The edges of M' may be defined as: $M' = \{v^0_{k-1}v^1_\alpha\} \cup \{v^{2j-1}_{i+1}v^{2j}_i : 1 \le j \le \lfloor \frac{q+1}{2} \rfloor \text{ and } \alpha \le i \le k-1\} \cup \{v^{2j}_k v^{2j+1}_\alpha : 1 \le j \le \lceil \frac{q-1}{2} \rceil\}$. Fig. 3 illustrates set M'.

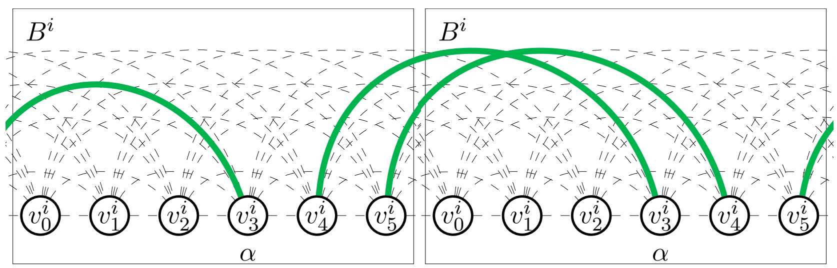


Figure 3: Illustration of the edges M', green and solid lines, in the graph P^k_n: block B^j, with odd j (left); block B^j with even j (right).

In order to define M'' – edges with endpoints in B_{pre}, B_{suf}, and possibly B^{q+2} – we consider the parity of $q+1$ and the value of β. In the case that $q+1$ is even, there are $r - \beta$ edges in M''. These edges are $v^p_k v^s_\beta$ and edges $v^s_{\beta+1+i}v^{q+2}_i$, with $0 \le i \le r - \beta - 2$. Edge $v^p_k v^s_\beta$ has reach $\beta + 1$. The remaining edges have reach $r - \beta - 1$. Fig. 4 illustrates this case.

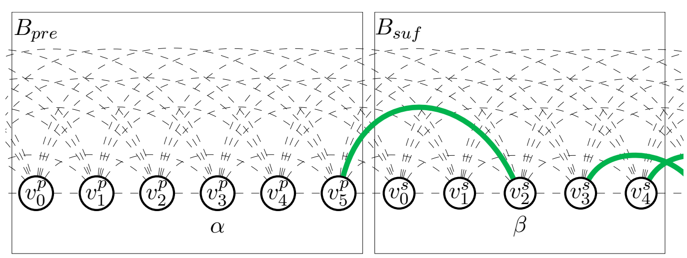


Figure 4: Illustration of block B^{q+1} in P^5_n, $q+1$ even, with solid green edges belonging to M''.

If $q+1$ is odd, there are $k - \alpha$ edges in M'', each with a reach of $k - \alpha + \beta$. Suppose v^s_β denote the vertex with index x in the canonical ordering of G, that is $v^s_\beta = v_x$. The edges of M'' are $v^p_i, v_{x+i-(\alpha+1)}$, with $\alpha + 1 \le i \le k$. The reason for employing the general canonical ordering with vertex v_x instead of v^s_β is that, depending on the number of vertices in B_{suf}, the endpoint of the edges may lie in either B_{suf} or B^{q+2} if, at any point, $\beta + i - (\alpha+1) > r - 1$. Fig. 5 illustrates this case. Note that the first edge has endpoint in a vertex of B_{suf} and the second in B^{q+2}.

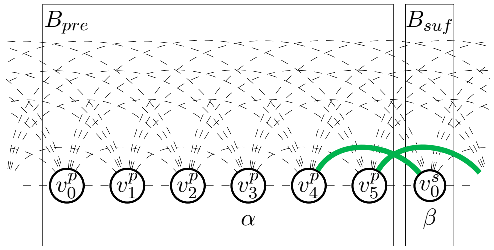


Figure 5: Block B^{q+1} with odd $q+1$ in P_n^5, green solid edges represent linking edges belonging to set M''.

Additionally, when k is odd and $r = k$, $v_{r-1}^s v_0^{q+2} \in M''$. Fig. 6 shows an example of M'' with the additional edge.

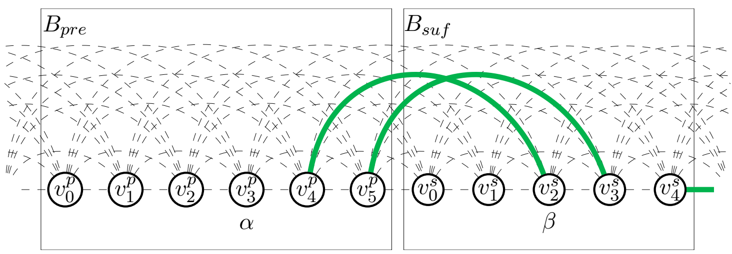


Figure 6: Block B^{q+1} in P_n^5 for the case in which $r = k$; green edges belong to M''.

In summary, the edges of M'' are:

$$M'' = \begin{cases} \{v_k^p v_\beta^{suf}\} \cup \{v_{(\beta+1)+i}^s v_i^{q+2} : 0 \le i \le r-(\beta+1)\} \text{ if } q+1 \text{ is even;} \\ \{v_{(\alpha+1)+i}^p v_{\beta+i}^s : 0 \le i \le r-(\beta+1)\} \cup \\ \ \{v_{(\alpha+r-\beta)+i}^p v_i^{q+2} : 0 \le i \le k-(\alpha+r-\beta)\} \text{ if } q+1 \text{ is odd, and either } k \text{ is even or } r<k; \\ \{v_{(\alpha+1)+i}^p v_{\beta+i}^s : 0 \le i \le r-(\beta+2)\} \cup \{v_{r-1}^{suf} v_0^{q+2}\} \text{ if } q+1 \text{ is odd, } k \text{ is odd and } r=k. \end{cases}$$

Finally, it remains to show that c_π from the labelling (π, c_π) satisfies the properties (i) and (ii) defined previously. Recall that the colour of a vertex in a 2-NDEL is its degree added to the number of edges incident with it labelled 2. Thus, for a given $v \in V(G)$, we count the number of edges labelled 2 incident with v.

First, consider the vertices in blocks B^j, $1 \le j \le q$. Each of these blocks receives a type-2 labelling. So, by Proposition 2, the vertices v_i^j with $i \le \left\lfloor \frac{|B^j|-2}{2} \right\rfloor$ are incident with $i+1$ edges labelled 2; and when $i > \left\lfloor \frac{|B^j|-2}{2} \right\rfloor$ they are incident with i edges labelled 2. Since $|B^j| = k+1$, we have that $\left\lfloor \frac{|B^j|-2}{2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor = \alpha - 1$. Thus, vertices v_i^j, with $i < \alpha$, are incident with $i+1$ edges labelled 2, and vertices v_i^j, with $i \ge \alpha$, are incident with i edges labelled 2. For these blocks, it remains to consider vertices that are incident with edges of matching M. In this case, those are only edges of M'. By the definition of M', each vertex v_i^j with $i < \alpha$ is not incident with any edge of M, and every vertex v_i^j with $i \ge \alpha$ is incident with one edge of M'. Therefore, the colour of v_i^j is $d(v_i^j) + i + 1 = 2k + i + 1$ when $i < \alpha$. If $i \ge \alpha$, the colour of v_i^j is $d(v_i^j) + i + 1 = 2k + i + 1$. Therefore, $c_\pi(v_i^j) = 2k + i + 1$, $0 \le i \le k$, satisfying the properties (i) and (ii).

Now, consider the block B_{pre}. This block always receives a type-2 labelling. Since $|B_{pre}| = |B^j|$, $1 \leq j \leq q$, then, by Proposition 2, vertices v_i^p, with $i < \alpha$, are incident with $i+1$ edges labelled 2; and vertices v_i^p, with $i \geq \alpha$, are incident with i edges labelled 2. Additionally, vertices in this block are incident with edges of matching M. When $q+1$ is even, vertices v_i^p, $\alpha \leq i \leq k-1$, are incident with edges of M and v_k^p is incident with one edge of M''. When $q+1$ is odd, vertex v_α^p is incident with one edge of M', and v_i^p, $\alpha+1 \leq i \leq k$, is incident with one edge of M''. Thus, the colour of v_i^p is $d(v_i^p)+i+1 = 2k+i+1$ when $i < \alpha$. If $i \geq \alpha$, the colour of v_i^p is $d(v_i^p)+i+1 = 2k+i+1$. Therefore, $c_\pi(v_i^p) = 2k+i+1$, and since $v_i^p = v_i^{q+1}$, we have that $c_\pi(v_i^{q+1}) = 2k+i+1$, for $v_i^{q+1} \in B_{pre}$. This way, properties (i) and (ii) are satisfied.

The block B_{suf} also receives a type-2 labelling. Thus, by Proposition 2, vertices v_i^s, with $i \leq \left\lfloor \frac{|B_{suf}|-2}{2} \right\rfloor$, are incident with $i+1$ edges labelled 2; and vertices v_i^s, with $i > \left\lfloor \frac{|B_{suf}|-2}{2} \right\rfloor$, are incident with i edges labelled 2. Since $|B_{suf}| = r$, we have that $\left\lfloor \frac{|B_{suf}|-2}{2} \right\rfloor = \left\lfloor \frac{r-2}{2} \right\rfloor = \beta - 1$. Thus, vertices v_i^s, with $i < \beta$, are incident with $i+1$ edges labelled 2; and vertices v_i^s, with $i \geq \beta$, are incident with i edges labelled 2. Additionally, the vertices of this block are also incident with edges of matching M, more precisely, with edges of M''. By the definition of M'', each vertex v_i^s with $i < \beta$ is not incident with edges of M, and every vertex v_i^s with $i \geq \beta$ is incident with one edge of M''. Thus, the colour of v_i^s is $d(v_i^s)+i+1 = 2k+i+1$ when $i < \beta$. If $i \geq \beta$, the colour of v_i^s is $d(v_i^s)+i+1 = 2k+i+1$. Thus, $c_\pi(v_i^s) = 2k+i+1$. Since there was a relabelling of vertices in B_{suf}, then $v_i^s = v_{i+k+1}^{q+1}$. Therefore, we have that $c_\pi(v_i^{q+1}) = k+i$ when $v_i^{q+1} \in B_{suf}$, satisfying properties (i) and (ii).

Blocks B^0 and B^{q+2} have not been labelled with a type-2 labelling. So, if vertices of these blocks are incident with an edge labelled 2, those edges are from M. This implies that each of these vertices is incident with at most one edge labelled 2.

In the case of block B^0, the only vertex incident with an edge of M is v_{k-1}^0, which is incident with an edge of M'; every other vertex in it is not incident with any edge labelled 2. So, $c_\pi(v_{k-1}^0) = d(v_{k-1}^0)+1$, and every other vertex v_i^0, with $i < k-1$, has colour $d(v_i^0)$. Moreover, we know that the degrees of those vertices are in increasing order. Thus, their colours are also increasing. With that, the colour of all vertices in this block satisfies (i) and (ii).

Finally, for the block B^{q+2}, we have that its degrees are in decreasing order. However, the number of vertices incident with edges of M can change from 0 to $\alpha - 1$. Yet, we know that all those edges belong to M''. Let t be the number of edges of M'' incident with vertices of B^{q+2}. By the definition of M'', these edges are incident with vertices of block B^{q+1}, following the canonical ordering and starting at vertex v_0^{q+2}. We conclude that only vertices from v_0^{q+2} to v_{t-1}^{q+2} have (exactly) one edge incident with it labelled 2. The remaining vertices have no incident edge labelled 2. Therefore, $c_\pi(v_i^{q+2}) = d(v_i^{q+2})+1$ when $0 \leq i \leq t-1$, and $c_\pi(v_i^{q+2}) = d(v_i^{q+2})$ when $i > t-1$. Since the degrees of vertices of this block are in decreasing order, their colours are also in decreasing order, satisfying properties (i) and (ii). Thus, the result follows.

Case 2. $n < 3k+1$

In this case, we partition $V(P_n^k)$ into three blocks: B^1, B^2 and B^3. Blocks B^1 and B^3 both have the same size, $|B^1| = |B^3| = \frac{n-|B^2|}{2}$. Block B^2 has size k or $k+1$, and its vertices induce a complete subgraph in P_n^k. If $n \equiv k \pmod 2$, then $|B^2| = k$; otherwise, $|B^2| = k+1$. Note that $n - |B^2| \equiv 0 \pmod 2$ to allow for B^1 and B^3 to have the same size. Each block B^i is a set of consecutive vertices, following the canonical ordering. Block B^1 is composed by vertices $v_0, \ldots, v_{|B_1|-1}$. Block B^2 is composed by vertices $v_{|B^1|}, \ldots, v_{n-|B^3|}$. The remaining vertices belong to B^3. Denote by v_i^q the i^{th} vertex of block B^q, with $q \in \{1,3\}$ following the non-decreasing order of its degrees. For block B^2, denote its i^{th} vertex by v_j^2, with j following the ordering $(0, 2, 4, \ldots, |B^2|, \ldots, 5, 3, 1)$. That is, the first half of the vertices v_j^2 are indexed by even numbers in increasing order, while the second half are indexed by odd numbers in decreasing order. This ensures that the vertex degrees, when arranged in the order of their indices, form a non-decreasing sequence. Moreover, by construction, $d(v_{2x}^2) = d(v_{2x+1}^2)$, with $0 \leq x < \left\lfloor \frac{|B^2|}{2} \right\rfloor$. Note that, in B^1, vertices are ordered in the canonical order; in B^3, vertices are ordered in the inverse of the canonical order; and, in B^2, vertices do not have a specific ordering related to the canonical ordering, alternating inside the block. Fig. 7 exemplifies this notation in P_8^3.

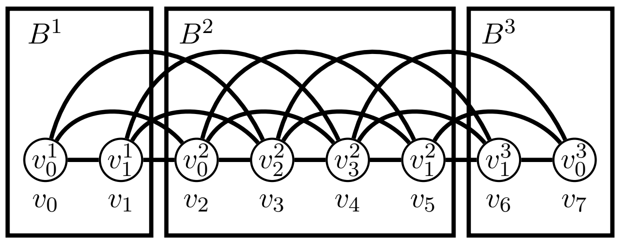
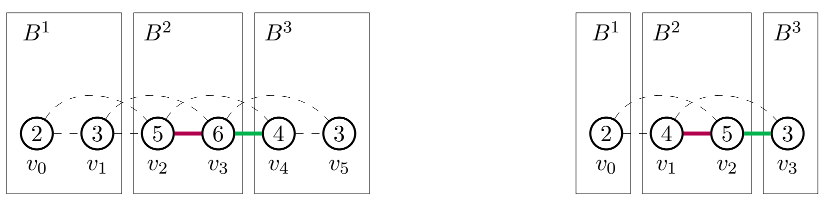


Figure 7: Graph P_8^3 with a block partition.

There are only two graphs for which $|B^2| = 2$. Since $k > 1$, this only occurs when $k = 2$, as $|B^2| = k$ or $|B^2| = k + 1$. Besides that, since $|B^2|$ is even, n must also be even due to the construction of B^2. Finally, $n < 3k + 1 = 7$ by the restriction of the case, and $n > 2$ by the definition of power of paths. So, the only valid values of n for this case are $n = 4$ and $n = 6$. Therefore, $|B^2| = 2$ only for P_4^2 or P_6^2. Fig. 8 shows a 2-NDEL for both cases. Consider, from this moment forwards, that $|B^2| \geq 3$.



(a) Labelling for P_6^2.

(b) Labelling for P_4^2.

Figure 8: Graphs with $|B^2| = 2$ endowed with a 2-NDEL; only coloured edges have label 2.

Let $\alpha = \left\lfloor \frac{|B^2|-1}{2} \right\rfloor$. Suppose, initially, that there is a 2-NDEL (π, c_π) for G satisfying the following properties.

(i) For v_i^j belonging to B^2:

$$c_\pi(v_i^2) = \begin{cases} d(v_i^j) + i + 1 & \text{if } i \in \{0,1\} \text{ and } |B^2| = k+1; \\ d(v_i^j) + i & \text{if either } 2 \leq i < \alpha - 1, \text{ or } i \in \{0,1\} \text{ and } |B^2| = k; \\ d(v_i^j) + (i+1) & \text{if } i \geq \alpha - 1. \end{cases}$$

(ii) For v_i^j belonging to B^1 or B^3:

$$c_\pi(v_i^j) = \begin{cases} d(v_{\alpha-1}^2) - 1 + \alpha & \text{if } i = |B^1| - 1 = |B^3| - 1; \\ d(v_i^j) + 1 & \text{if } |B^1| - (d(v_{\alpha-1}^2) - d(v_{|B^1|-1}^1) - (|B^2| - k)) \leq i < |B^1| - 1; \\ d(v_i^j) & \text{otherwise.} \end{cases}$$

We prove, as follows, that c_π is a proper vertex colouring. Initially, note that vertices in B^1 and B^3 have increasing degrees in relation to their indices inside the block. By construction, the colour of vertices $v^1_{|B^1|-1}$ and $v^3_{|B^3|-1}$ is greater then the colour of the remaining vertices in $B^1 \cup B^3$, since $d(v^2_{\alpha-1}) > d(v^j_i) + 1$, for $i < |B^j| - 1$, with $j \in \{1, 3\}$, and the value of α is at least 1. The colours of the remaining vertices of B^1 and B^3 are increasing in relation to their indices by definition. So, adjacent vertices in B^1 (in B^3) have distinct colours. Moreover, by construction, two vertices in distinct blocks, one in B^1 and the other in B^3, are not adjacent. We conclude that $B^1 \cup B^3$ does not have adjacent vertices with the same colour.

The colour of vertices $v^1_{|B^1|-1}$ and $v^3_{|B^3|-1}$ is $d(v^2_{\alpha-1}) - 1 + \alpha$. Since the degrees of vertices in B^2 are in non-decreasing order of their indices, we have $d(v^2_i) \leq d(v^2_j)$, with $i < j$, which implies, $d(v^2_i) + i < d(v^2_j) + j$. Thus, we have that $c_\pi(v^2_i) = d(v^2_i) + i < d(v^2_{\alpha-1}) - 1 + \alpha$ when $i < \alpha - 1$. Moreover, $d(v^2_{\alpha-1}) + \alpha - 1 < d(v^2_i) + (i+1) = c_\pi(v^2_i)$, when $i \geq \alpha - 1$. So, the colour of vertices $v^1_{|B^1|-1}$ and $v^3_{|B^3|-1}$ is greater than the colour of vertex v^2_i when $i < \alpha - 1$ and lower when $i \geq \alpha - 1$. For $v \in B^2$ and $u \in (B^1 \cup B^3) \setminus \{v^1_{|B^1|-1}, v^3_{|B^3|-1}\}$, since the degree of vertices in B^1 (B^3) is increasing, we have that $d(v) > d(v^1_{|B^1|-1}) = d(v^3_{|B^3|-1}) \geq d(u) + 1$. So, $c_\pi(v) \geq d(v) > d(u) + 1 \geq c_\pi(u)$. Thus, every pair of adjacent vertices, one in $B^1 \cup B^3$ and the other in B^2, have distinct colours. Finally, when $|B^2| = k$, since the colour of vertex v^2_i in B^2 is $d(v^2_i) + i$, with $i < \alpha - 1$, or $d(v^2_i) + (i+1)$, with $i \geq \alpha - 1$, and the degree of the vertices is non-decreasing, we have that the colour of vertices in B^2 increases with each vertex. When $|B^2| = k+1$, the only difference is on vertices v^2_0 and v^2_1, that have colour $d(v^2_i) + i + 1$. Since $|B^2| = k+1$, then $|B^1| = |B^3| < k$, as $n \leq 3k$. Vertex v^2_0 is adjacent to every vertex in $|B^1| \cup |B^2|$ but to no vertex in $|B^3|$, since there are k vertices between v^2_0 and the first vertex of B^3. Recall that, by construction, $d(v^2_0) = d(v^2_1)$. Vertex v^2_2 is adjacent to every vertex in B^1 and, to at least one vertex in B^3. Thus, $d(v^2_2) > d(v^2_0)$. Therefore, $d(v^2_0) = d(v^2_1) < d(v^2_i)$ and $c_\pi(v^2_0) < c_\pi(v^2_1) = d(v^2_1) + 2 < d(v^2_i) + i = c_\pi(v^2_i)$, for every $2 \leq i \leq |B^2| - 1$, and we have that the colour of vertices in B^2 still increases. We deduce that no pair of adjacent vertices in G has the same colour in c_π.

To conclude the proof, we construct (π, c_π) to satisfy (i) and (ii). Initially, we label the subgraph induced by block B^2 with a modified type-1 labelling. This modified labelling has an additional edge labelled 2 when $|B^2|$ is odd and two additional edges labelled 2 when $|B^2|$ is even. Following this, we select a set of 2α linking edges to receive label 2, in which α of those edges have exactly one endpoint in $v^1_{|B^1|-1}$ and α have exactly one endpoint in $v^3_{|B^3|-1}$. The remaining linking edges receive label 1. If $d(v^2_{\alpha-1}) - d(v^j_{|B^j|-1}) - (|B^2| - k) \geq 2$, $j \in \{1, 3\}$, we assign label 2 to edges with one endpoint in $v^1_{|B^1|-1}$ (in $v^3_{|B^3|-1}$) and the other in the remaining vertices of B^1 (of B^3) in decreasing order of its indices, until the colour of $v^1_{|B^1|-1}$ ($v^3_{|B^3|-1}$) is equal to $d(v^2_{\alpha-1}) - 1 + \alpha$. In other words, we assign label 2 to $d(v^2_{\alpha-1}) - d(v^1_{|B^1|}) - 1$ edges with both ends in B^1 and incident with $v^1_{|B^1|-1}$ and the same number of edges with both ends in B^3 and incident with $v^3_{|B^3|-1}$. Finally, we show that the 2-edge labelling built satisfies properties (i) and (ii) and, therefore, is a 2-NDEL.

Consider a type-1 edge labelling of B^2. Recall that $\alpha = \left\lfloor \frac{|B^2|-1}{2} \right\rfloor$. Adjust the labelling as follows: $\pi(v^2_{\alpha-1}v^2_{\alpha+1}) = 2$ if $|B^2| \equiv 1 \pmod 2$; $\pi(v^2_\alpha v^2_{\alpha+1}) = \pi(v^2_{\alpha-1}v^2_{\alpha+2}) = 2$ if $|B^2| \equiv 0 \pmod 2$; $\pi(v^1_{|B^1|-1}v^2_0) = \pi(v^3_{|B^3|-1}v^2_1) = 2$ if $|B^2| = k+1$. These edges are called *adjusted* and are exemplified in Fig. 9.

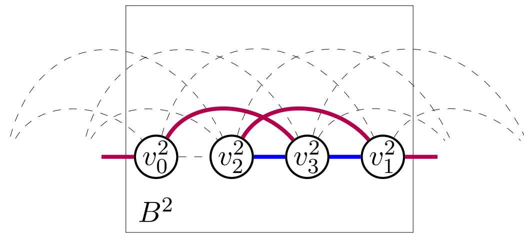


Figure 9: Block B^2 of a P_8^3 with an adjusted type-1 labelling. Continuous edges have label 2; edges with reach 2 are adjusted.

To conclude the labelling, it is necessary to assign labels to linking edges, as well as to edges from $E(G[B^1])$ and $E(G[B^3])$. Except by edges listed below, all remaining edges receive label 1. For $j \in \{1,3\}$,

(a) $v^j_{|B^j|-1}v^2_{|B^2|-\alpha+i}$, $1 \leq i \leq \alpha - 1$;

(b) $v^j_{|B^j|-1}v^j_{|B^j|-i}$, $2 \leq i \leq t$ and $t = d(v^2_{\alpha-1}) - d(v^j_{|B^j|-1}) - (|B^2| - k)$;

(c) $v^1_{|B^1|-1}v^2_{|B^2|-\alpha}$;

(d) $v^3_{|B^3|-1}v^2_{|B^2|-\alpha-1}$.

We finish the proof by showing that c_π of the constructed labelling satisfies (i) and (ii) as defined previously. Recall that the colour of a vertex in a 2-NDEL is its degree added to the number of its incident edges labelled 2. Thus, given $v \in V(G)$, we count the number of edges labelled 2 incident with v.

First, consider the block B^2. This block receives a type-1 labelling with some adjusted edges labelled 2. By Proposition 2, every vertex v^2_i, with $i \leq \left\lfloor \frac{|B^2|-1}{2} \right\rfloor$, is incident with i edges labelled 2; and with $i > \left\lfloor \frac{|B^2|-1}{2} \right\rfloor$, is incident with $i-1$ edges labelled 2. By definition, $\alpha = \left\lfloor \frac{|B^2|-1}{2} \right\rfloor$. Thus, every vertex v^2_i, with $i \leq \alpha$, is incident with i edges labelled 2; and with $i > \alpha$ is incident with $i-1$ edges labelled 2, just by the type-1 labelling. For these vertices, it remains to consider the adjusted labels and linking edges labelled 2.

Consider the vertex v^2_i, with $i < \alpha - 1$. Note that $|B^2| - \alpha - 1 = \alpha$ when $|B^2|$ is even, and $|B^2| - \alpha - 1 = \alpha + 1$ when $|B^2|$ is odd. So, $|B^2| - \alpha - 1 + l > \alpha - 1$, for $0 \leq l \leq \alpha$. Thus, this vertex is not incident with linking edges labelled 2. Moreover, this vertex is also not an endpoint of any adjusted edge, except for v^2_0 and v^2_1 when $|B^2| = k+1$. Therefore, this vertex has colour $d(v^2_i) + i$ or $d(v^2_i) + i + 1$ when $i \in \{0,1\}$ and $|B^2| = k+1$. When $i = \alpha - 1$, this vertex is incident with i edges labelled 2 by the type-1 edge labelling and to an adjusted edge: $v^2_{\alpha-1}v^2_{\alpha+2}$ when $|B^2|$ is even; or $v^2_{\alpha-1}v^2_{\alpha+1}$ when $|B^2|$ is odd. Therefore, its colour is $d(v^2_i) + i + 1$. When $i = \alpha$, v^2_i is incident with i edges labelled 2 by the type-1 labelling and one adjusted edge: $v^2_\alpha v^2_{\alpha+1}$ if $|B^2|$ is even; or with a linking edge with endpoint in $v^3_{|B^3|-1}$ if $|B^2|$ is odd. Therefore, its colour is $d(v^2_i) + i + 1$. When $i = \alpha + 1$, v^2_i is incident with $i - 1$ edges labelled 2 by the type-1 labelling and one adjusted edge: $v^2_\alpha v^2_{\alpha+1}$ if $|B^2|$ is even; or $v^2_{\alpha-1}v^2_{\alpha+1}$ if $|B^2|$ is odd. Moreover, v^2_i is also incident with one linking edge with endpoint in: $v^3_{|B^3|-1}$ if $|B^2|$ is even; or $v^1_{|B^1|-1}$ if $|B^2|$ is odd. Therefore, its colour is $d(v^2_i) + i + 1$. When $i = \alpha + 2$, v^2_i is incident with $i - 1$ edges labelled 2 by the type-1 labelling. Additionally, v^2_i is incident with: an adjusted edge $v^2_{\alpha-1}v^2_{\alpha+2}$ and a linking edge with endpoint in $v^1_{|B^1|-1}$ if $|B^2|$ is even; or two linking edges if $|B^2|$ is odd. Therefore, its colour is $d(v^2_i) + i + 1$.

When $i > \alpha + 2$, this vertex is incident with $i-1$ edges labelled 2 by the type-1 labelling, and with two linking edges $v_i^2 v_{|B^1|-1}^1$ and $v_i^2 v_{|B^3|-1}^3$. Therefore, its colour is $d(v_i^2) + i + 1$. We conclude that $c_\pi(v_i^2) = d(v_i^2) + i$ when $i < \alpha - 1$, and $c_\pi(v_i^2) = d(v_i^2) + i + 1$ when $i \geq \alpha - 1$, satisfying properties (i) and (ii).

Except for the edges indicated in (a), (b), (c) and (d), edges incident with vertices in B^1 and B^3 receive label 1. The vertices $v_{|B^1|-1}^1$ and $v_{|B^3|-1}^3$ are incident with α linking edges labelled 2 each, as mentioned in item (a). Additionally, these vertices are incident with $t-1$ edges labelled 2, for $t = d(v_{\alpha-1}^2) - d(v_{|B^1|-1}^1) - (|B^2| - k)$, as stated in item (b) and to one adjusted edge when $|B^2| = k+1$. Therefore, the colour of $v_{|B^1|-1}^1$ is $d(v_{|B^1|-1}^1) + \alpha + d(v_{\alpha-1}^2) - d(v_{|B^3|-1}^1) - 1 = d(v_{\alpha-1}^2) - 1 + \alpha$. Similarly, the colour of $v_{|B^3|-1}^3$ is $d(v_{|B^3|-1}^3) + \alpha + d(v_{\alpha-1}^2) - d(v_{|B^3|-1}^3) - 1 = d(v_{\alpha-1}^2) - 1 + \alpha$. The vertex v_i^j, for $j \in \{1, 3\}$ and $|B^j| - t \leq i < |B^j| - 1$, is incident with exactly one edge labelled 2 and endpoint in $v_{|B^j|-1}^j$. Thus, for this vertex, $c_\pi(v_i^j) = d(v_i^j) + 1$. Finally, the remaining vertices of $B^1 \cup B^3$ do not have any edge labelled 2 incident with them and, therefore, have colour $d(v_i^j)$. We conclude that the vertices of B^1 and B^3 satisfy properties (i) and (ii), and the result follows. □

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