(2,1)-total number of complete equipartite graphs

M. M. Omai *C. N. Campos* *Atílio G. Luiz*

Technical Report - IC-25-05 - Relatório Técnico
September - 2025 - Setembro

UNIVERSIDADE ESTADUAL DE CAMPINAS
INSTITUTO DE COMPUTAÇÃO

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M. M. Omai *　　C. N. Campos †　　Atílio G. Luiz‡

Abstract

We investigate the $(2, 1)$-total labelling for complete equipartite graphs $K_{r\times n}$. Motivated by the conjecture of Havet and Yu, which states that every graph G satisfies $\lambda_2^t(G) \leq \Delta(G) + 3$, we provide constructive labellings that support this conjecture for almost all cases of $K_{r\times n}$ with $r \geq 3$ and $n \geq 2$. The only exception is when r is even and n is odd, for which we establish a new upper bound of $\Delta(G) + r + 2$.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The vertices and the edges of G are called *elements* of G. The *degree* of a vertex $v \in V(G)$ is denoted $d(v)$. We call v a *k-vertex* when $d(v) = k$. The *maximum degree* of G is denoted by $\Delta(G)$.

In the Frequency Channel Assignment Problem [8], one must assign integers, representing frequency channels, to transmitters geographically spread. In order to avoid interference, transmitters that are very close must be assigned channels at least two units apart, whilst transmitters that are close but not adjacent must be assigned distinct channels. The main goal is to minimize the number of distinct channels required. Although this classical problem was introduced in 1980, it remains relevant today, particularly in the context of modern mobile networks [2].

Motivated by this problem, Griggs and Yeh [9] introduced $L(2, 1)$-labellings in which adjacent vertices must receive integers that differ by at least two and vertices at distance two receive distinct integers. In 2008, Havet and Yu [1] proposed a variant of this problem, known as $(2, 1)$-total labelling in which labels are assigned to both vertices and edges.

A k-$(2, 1)$-*total labelling* of a simple graph G is a function $\pi\colon V(G) \cup E(G) \to \{0, 1, \ldots, k\}$ for which the following properties hold: $|\pi(uv) - \pi(u)| \geq 2$ and $|\pi(uv) - \pi(v)| \geq 2$ for $uv \in E(G)$; $\pi(uv) \neq \pi(uw)$ for $uv, uw \in E(G)$; and $\pi(u) \neq \pi(v)$ for $uv \in E(G)$. If a labelling π does not satisfy any of these properties, then we say that there is a *conflict* in π. The least integer k for which G admits a k-$(2, 1)$-total labelling is denoted by $\lambda_2^t(G)$ and called $(2, 1)$-*total number*.

The problem of determining the $(2, 1)$-total number of a graph is, in general, computationally hard. In fact, the problem is NP-hard even when $\Delta(G) = 3$. For non-bipartite k-regular graphs the complexity remains an open question, with polynomial-time results known only for specific classes, such as 2-regular graphs. For bipartite graphs G, Havet and Thomassé [3] showed that deciding whether $\lambda_2^t(G) = \Delta(G) + 1$ is NP-complete for almost all values of $\Delta(G)$, except for some restricted cases, as in $\Delta(G) = 3$, for which a polynomial-time algorithm is given. Despite this difficulty, the $(2, 1)$-total number has been determined for some classes of graphs such as complete graphs [1, 5], near-ladder graphs [15], generalized Petersen graphs [14], among others [11, 12, 13, 15].

*Inst. de Computação, Unicamp, 13083-852 Campinas, SP. m228269@dac.unicamp.br.
†Inst. de Computação, Unicamp, 13083-852 Campinas, SP. cnc@unicamp.br.
‡Campus de Quixadá, UFC, 63902-580 Quixadá, CE. gomes.atilio@ufc.br.

In their seminal work, Havet and Yu [1] proved that $\lambda_2^t(G) \leq 6$ holds for graphs G with $\Delta(G) \leq 3$, also showing that $\lambda_2^t(K_4) = 6$. Additionally, the authors proved that, for graphs with $\Delta(G) \geq 2$, $\lambda_2^t(G) \leq 2\Delta$ and proposed the following conjecture for graphs with $\Delta(G) > 3$.

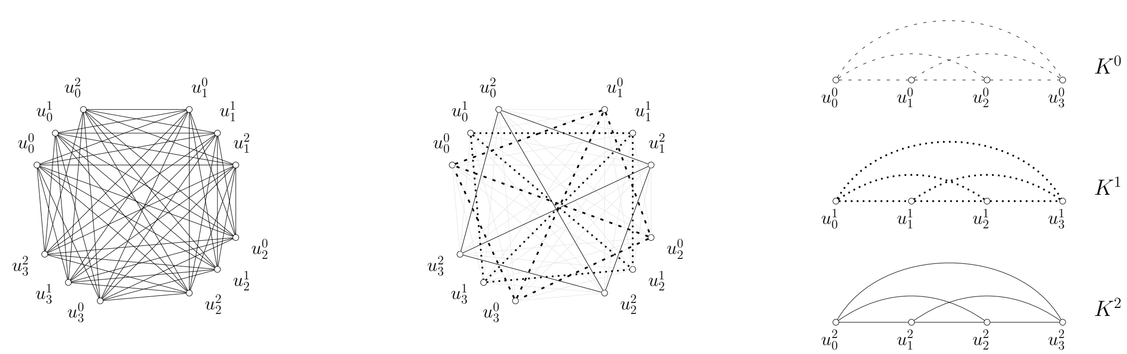
Conjecture 1. *(Havet and Yu [1]) Let G be a graph with $\Delta(G) > 3$. Then, $\lambda_2^t(G) \leq \Delta(G) + 3$.*

Furthermore, Havet and Yu [1] proved that there exists a constant Δ_0 such that for every graph G with $\Delta(G) \geq \Delta_0$, $\lambda_2^T(G) \leq \Delta(G) + 2\log_{10}\Delta(G) + 4$, which provides support for Conjecture 1. Motivated by Conjecture 1, we investigate $(2,1)$-total labellings on complete equipartite graphs $K_{r\times n}$, a class that generalizes complete bipartite graphs by allowing r parts of n vertices each instead of just two. It is noteworthy that when $r \geq 3$, these graphs are regular non-bipartite graphs for which the complexity remains open, as discussed earlier.

For a complete equipartite graph $K_{r\times n}$, if $r = 1$, then $\lambda_2^t(K_{1\times n}) = 0$ since $E(K_{1\times n}) = \emptyset$. Moreover, if $r = 2$, $K_{2\times n}$ is a complete bipartite graph, for which $\lambda_2^t(K_{2\times n}) = \Delta(K_{2\times n}) + 2$ [1]. On the other hand, if $n = 1$, $K_{r\times 1}$ is isomorphic to the complete graph K_r for which the $(2,1)$-total number is also known [1, 5]. Therefore, we focus on complete equipartite graphs $K_{r\times n}$ with $r \geq 3$ and $n \geq 2$, showing that the Conjecture 1 holds for most cases of $K_{r\times n}$ with $r \geq 3$ and $n \geq 2$. We also establish a new upper bound of $\Delta(K_{r\times n}) + r + 2$ for the specific case where r is even and n is odd.

2 Preliminaries

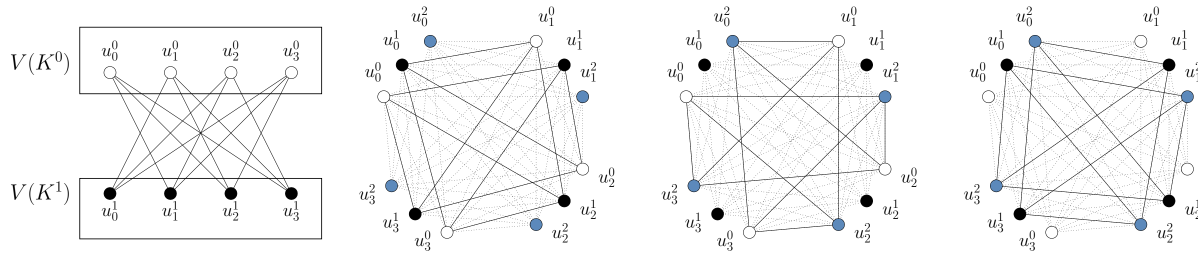
Let $G = K_{r\times n}$ with $r \geq 3$ parts $V_0, \ldots, V_{r-1}$, each of size $n \geq 2$, with $V_\ell = \{u_\ell^i : 0 \leq i < n\}, 0 \leq \ell < r$. Define subgraphs $K^i = G[\{u_0^i, \ldots, u_{r-1}^i\}]$, $0 \leq i < n$, which are isomorphic to the complete graph K_r. Thus, $K_{r\times n}$ contains n disjoint copies of K_r as induced subgraphs. In this work, we frequently refer to elements that play the same role in each copy of K_r. Therefore, we define two vertices u_ℓ^i and u_ℓ^j, $i \neq j$ as *corresponding vertices* – note that corresponding vertices belong to different copies of K_r whilst belonging to the same part of $K_{r\times n}$. Analogously, two edges $u_\ell^i u_p^i$ and $u_s^j u_t^j \in K_{r\times n}$ are called *corresponding edges* if $\{\ell, p\} = \{s, t\}$ and $i \neq j$. Figure 1 illustrates these concepts; observe, in Figure 1(a), that u_0^0 and u_0^1 are corresponding vertices and $u_0^0 u_1^0$ and $u_0^1 u_1^1$ are corresponding edges of $K_{4\times 3}$.



(a) (b) (c)

Figure 1: Graph $K_{4\times 3}$ with (a) vertex notation, (b) subgraphs K^0, K^1 and K^2 highlighted, and (c) the subgraphs displayed separately.

Let $B^{ij} \subseteq K_{r\times n}$ be the maximal bipartite graph with bipartition $\{V(K^i), V(K^j)\}$, $0 \leq i < j < n$. By construction, each B^{ij} is a $(r-1)$-regular graph since there is no edge between corresponding vertices since these vertices belong to the same part of $K_{r\times n}$. Figure 2 shows B^{01}, B^{02} and B^{12} for $K_{4\times 3}$. Note that $E(K_{r\times n}) = \bigcup_{i=0}^{n-1}[(\bigcup_{j=i+1}^{n-1} E(B^{ij})) \cup E(K^i)]$, that is, every edge of $K_{r\times n}$ either belongs to some $E(K^i)$ or to some $E(B^{ij})$.



(a) Graph B^{01}. (b) Graphs B^{01} (left), B^{02} (center) and B^{12} (right) marked.

Figure 2: Bipartite subgraphs B^{01}, B^{02} and B^{12} highlighted.

We define a *canonical decomposition* $[\mathcal{K}, \mathcal{B}]$ of $K_{r\times n}$ as the union of two families $\mathcal{K}$ and $\mathcal{B}$ of subgraphs of $K_{r\times n}$. The family $\mathcal{K}$ contains subgraphs K^i, $0 \le i < n$, and $\mathcal{B}$ consists of the $(r-1)$-regular bipartite subgraphs B^{ij}, $0 \le i < j < n$. Considering the previous examples, for $K_{4\times 3}$, $\mathcal{K}$ is shown in Figure 1(b), also in 1(c), and $\mathcal{B}$ is shown in Figure 2(b). Let G_R be the underlying simple graph obtained from $\mathcal{K} \cup \mathcal{B}$ by contracting all the vertices of K^i into a single vertex v_i. Graph G_R is called the *representative graph* of $K_{r\times n}$. Observe that each vertex $v_i \in V(G_R)$ represents a component $K^i \subseteq \mathcal{K}$, and each edge $v_iv_j \in E(G_R)$ represents a bipartite graph $B^{ij} \in \mathcal{B}$. The representative graph G_R of $K_{4,3}$ is depicted in Figure 3.

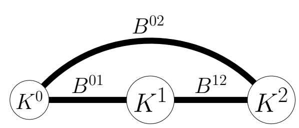
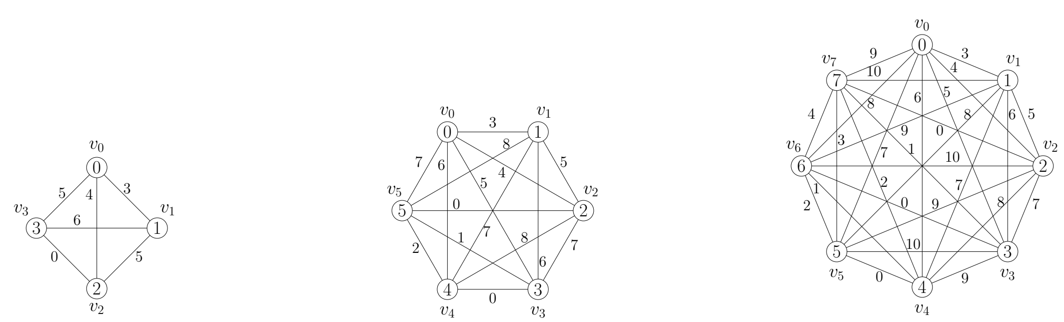


Figure 3: Representative graph of $K_{4\times 3}$.

Since a complete equipartite graph can be decomposed into complete graphs and regular bipartite graphs, it is important to describe $(2,1)$-total labellings of complete graphs, which we refer to as *standard labellings*. For $n \in 4, 6, 8$, these standard labellings are $(n+2)$-$(2,1)$-total labellings, which are intentionally non-optimal, as illustrated in Figure 4.



(a) 6-$(2,1)$-total labelling of K_4 (b) 8-$(2,1)$-total labelling of K_6 (c) 10-$(2,1)$-total labelling of K_8.

Figure 4: Standard labellings for K_n, $n \in \{4, 6, 8\}$.

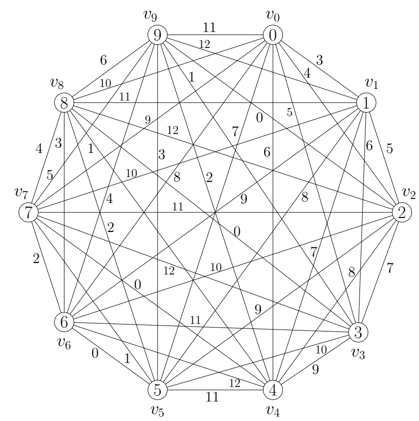
For n even and $n \geq 10$, the standard labelling defined in Figure 5(a) is exactly the $(n+2)$-$(2,1)$-total labelling given by Havet and Yu [1] and it is illustrated in Figure 5(b).

- For each vertex v_i, $i \in [0, n-1]$:

  $\tau(v_i) = i$

- For each edge $v_i v_j$, $i, j \in [0, n-1]$:

  $\tau(v_i v_j) = (\tau(v_i) + \tau(v_j) + 2) \bmod (n+3)$



(a) (b)

Figure 5: (a) Definition of the standard labelling for even $n \geq 10$; (b) standard labelling of K_{10}.

Now, we consider n odd. For $n = 3$: $\tau(v_0) = 4$, $\tau(v_1) = 2$, $\tau(v_2) = 0$, $\tau(v_0 v_1) = 0$, $\tau(v_1 v_2) = 4$, and $\tau(v_2 v_0) = 2$. For $n \geq 5$, the standard labelling τ is exactly the $(n+1)$-$(2,1)$-total labelling defined by Chia et al. [5]. The vertex labels are: $\tau(v_0) = 0$ and $\tau(v_i) = i + 2$ for $i > 0$. In order to define the edge labels of K_n, let $E'_i = \{v_{(i+j+1) \bmod n} v_{(i-j-1) \bmod n} \colon j \in [1, \frac{n-3}{2}]\}$ and

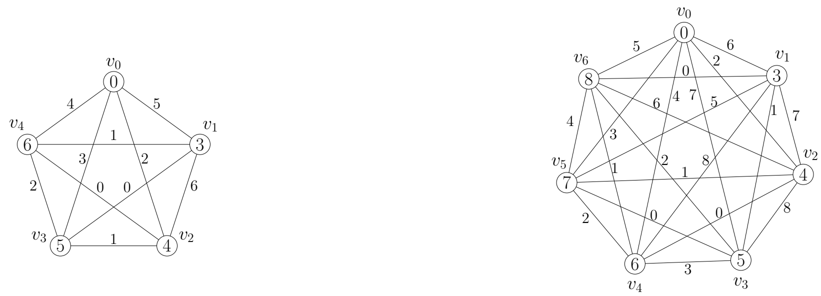
$$E_i = \begin{cases} (E'_0 \cup \{v_0 v_{n-2}\}) \setminus \{v_2 v_{n-2}\}, & \text{if } i = 0; \\ E'_1 \cup \{v_0 v_2\}, & \text{if } i = 1; \\ E'_i, & \text{if } i \in [2, n-1]; \\ \{v_{(4j-2) \bmod n} \, v_{(4j) \bmod n} : j \in [1, \frac{n-1}{2}]\}, & \text{if } i = n; \\ (\{v_{(4j) \bmod n} \, v_{(4j+2) \bmod n} : j \in [1, \frac{n-1}{2}]\} \cup \{v_2 v_{n-2}\}) \setminus \{v_0 v_{n-2}\}, & \text{if } i = n+1. \end{cases}$$

Finally, for each edge $e \in E(K_n)$, let $\tau(e)$ be defined as follows:

$$\tau(e) = \begin{cases} 3, & \text{if } e \in E_0; \\ 2, & \text{if } e \in E_1; \\ i+2, & \text{if } e \in E_i, i \in [2, n-1]; \\ 0, & \text{if } e \in E_n; \\ 1, & \text{if } e \in E_{n+1}. \end{cases}$$

Figure 6 illustrates the standard labellings of K_5 and K_7.

Since complete equipartite graphs admit a canonical decomposition, our approach is to label each component of the decomposition separately. The complete subgraphs of $\mathcal{K}$ receive standard labellings, whereas the edges of the regular bipartite subgraphs are coloured in accordance with classical colouring results, applied both to the bipartite graphs individually and to the associated representative graph.



(a) 6-(2, 1)-total labelling of K_5. (b) 8-(2, 1)-total labelling of K_7.

Figure 6: Standard labellings of K_n, $n \in \{5, 7\}$.

A *k-total colouring* c of a graph G is an assignment of k distinct colours to each vertex and each edge of G such that no pair of adjacent or incident elements receive the same colour. The *total chromatic number* of G, denoted by $\chi''(G)$, is the smallest integer k for which G admits a k-total colouring. If the colours are assigned just to the edges of G, then c is a *k-edge colouring*, and the smallest integer k for which G admits a k-edge colouring is called *chromatic index* of G, denoted by $\chi'(G)$. Given a colouring, a *colour class* is the set of elements assigned the same colour. When restricted to the edges of G, each colour class forms an independent set of edges, also known as a *matching*.

Several classical results concerning edge and total colourings are repeatedly used throughout this work. For bipartite graphs G, König shows that $\chi'(G) = \Delta(G)$ [4]. For complete graphs, Vizing [6] establishes that the chromatic index equals $n - 1$ when n is even and n when n is odd. Regarding total colourings, Behzad et al.[7] determine that the total chromatic number of a complete graph is n when n is even, and $n + 1$ when n is odd.

3 Main Result

In Theorem 3, we show that $\lambda_2^t(K_{r\times n}) \leq \Delta(K_{r\times n}) + 3$ for the complete equipartite graphs $K_{r\times n}$ with $r \geq 3$ and $n \geq 2$, except when r is even and n is odd. In this case, we establish a different upper bound of $\Delta(K_{r\times n}) + r + 2$. In order to prove this result, we first present Proposition 2.

Proposition 2. *Let K_n be a complete graph. If n is odd, then K_n has a $(n+2)$-$(2,1)$-total labelling, such that the labels 3 and 4 do not occur in any vertex.*

Proof. Let K_n be the complete graph with odd n. In order to prove the statement, we construct a labelling π that is a $(n+2)$-$(2,1)$-total labelling, such that the set of vertex labels is $[0, n+1]\setminus\{3, 4\}$. For $n = 3$ and $n = 5$, labelling π is given in Figure 7.

For $n \geq 7$, labelling π is constructed based on the standard labelling τ of K_n, which is a $(n+1)$-$(2,1)$-total labelling. Initially, assign $\pi(x) = \tau(x)$ for all $x \in V(K_n) \cup E(K_n)$. Observe that, by the definition of τ, $\pi(v) \in \{0, 3, 4, \ldots, n + 1\}$ for all $v \in V(K_n)$. In particular, it is known that $\pi(v_1) = 3$ and $\pi(v_2) = 4$. Therefore, to conclude the proof, we redefine the labels of v_1 and v_2 and, next, adjust the labels of certain edges to avoid conflicts.

First, consider v_1 and its incident edges. By the definition of τ, $\tau(v_1) = 3$, $\tau(v_1v_3) \in \{0, 1\}$, and $\tau(v_1v_{n-1}) \in \{0, 1\}$, with $\tau(v_1v_3) \neq \tau(v_1v_{n-1})$. Also, there does not exist v_1v_j, $0 \leq j < n$, assigned label 2. Thus, assign $\pi(v_1) = 1$, observing that no other vertex has been assigned this label. Now, we modify the labels of v_1v_3 and v_1v_{n-1} so as to guarantee the properties of a $(2, 1)$-total labelling.

(a) 5-$(2,1)$-total labelling π for K_3. (b) 7-$(2,1)$-total labelling π for K_5.

Figure 7: Labelling π for K_3 and K_5; labels 3 and 4 do not appear as vertex labels.

In order to solve the conflicts between edges v_1v_3, v_1v_{n-1} and vertex v_1, we redefine the labels of v_1v_3 and v_1v_{n-1}. First, assign v_1v_3 a new label $n+2$. Note that this label differs by at least two from $\pi(v_1) = 1$ and $\pi(v_3) = 5$. Now, assign v_1v_{n-1} label 3. By construction, $\pi(v_{n-1}) = n+1$. Therefore, since $n \geq 7$, we have $|(n+1) - 3| \geq 2$.

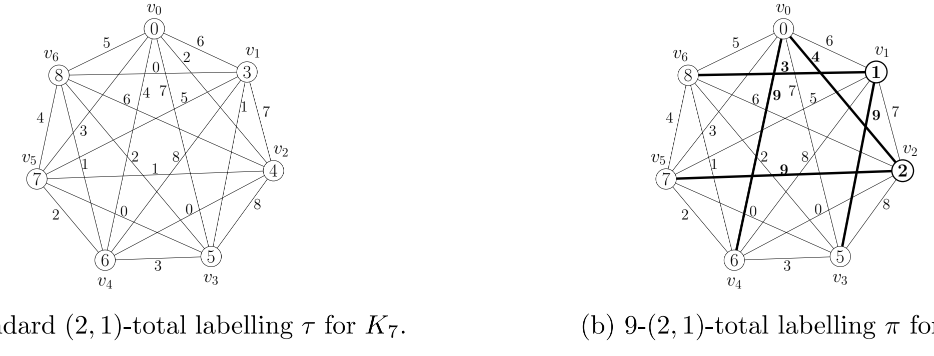
By construction, the only edges that received label 3 in τ belong to E_0. Therefore, it suffices to show that v_{n-1} is not an endpoint of any edge in E_0. By definition, $E_0 = \{(v_{(j+1) \bmod n} v_{(-j-1) \bmod n} : j \in [1, \frac{n-3}{2}]\} \cup \{v_0v_{n-2}\}) \setminus \{v_2v_{n-2}\}$. Note that $(j+1) \bmod n$ is $n-1$ only if $j = (n-2)+tn$, for an integer t. Also, $(n-3)/2 < n-2$ and $-2 < 1$. On the other hand, note that $(n-1) = (-j-1) \bmod n$ only when $j = tn$ for an integer t. Since $j \in [1, \frac{n-3}{2}]$, $n > \frac{n-3}{2}$ and $0 < 1$, it follows that no such j exists. Therefore, we conclude that v_jv_{n-1}, $0 \leq j < n$ does not belong to E_0, and the result follows.

Now, we adjust the label of v_2. Assign label 2 to v_2 since no vertex has been assigned this label yet. By the definition of τ, $\tau(v_0v_2) = 2$ and $\tau(v_2v_{n-2}) = 1$. Moreover, $\tau(v_2v_j) \neq 3, 0 \leq j < n$, since $\tau(v_2) = 4$. Therefore, we adjust the labels of three edges, v_0v_2, v_0v_4 and v_2v_{n-2}, so as to guarantee that π is a $(2,1)$-total labelling.

First, assign label $n+2$ to v_0v_4 and v_2v_{n-2}. Label $n+2$ has only been assigned to edge v_1v_3, after adjusting the label of vertex v_1. Therefore, the set of edges with label $n+2$ is v_0v_4, v_2v_{n-2} and v_1v_3. Observe that these edges do not have endpoints in common. Moreover, by the definition of π, we have $\pi(v_0) = 0$, $\pi(v_4) = 6$, $\pi(v_2) = 2$ and $\pi(v_{n-2}) = n$. Therefore, the difference between the label assigned to edges v_2v_{n-2} and v_0v_4 and the labels assigned to their endpoints is at least 2.

Finally, assign v_0v_2 label 4. Note that, by construction, the only edges that received label 4 in τ belong to $E_2 = \{v_{(j+3) \bmod n} v_{(1-j) \bmod n} : j \in [1, \frac{n-3}{2}]\}$. Therefore, the only edge whose ends are v_0 or v_2 is v_0v_4, which is obtained when $j = 1$ in $v_{(j+3) \bmod n} v_{(1-j) \bmod n}$. The label of this edge has just been redefined to $n+2$. Thus, there is no conflict and the result follows. $\square$

Figure 8 illustrates K_7 endowed with the labelling established in the preceding proposition.



(a) Standard $(2,1)$-total labelling τ for K_7. (b) 9-$(2,1)$-total labelling π for K_7.

Figure 8: Labelling π for K_7. The labels of the elements in bold were modified.

Theorem 3. *Let $K_{r\times n}$ be a complete equipartite graph with $r \geq 3$ and $n \geq 2$. Then, $\lambda_2^t(K_{r\times n}) \leq \Delta(K_{r\times n}) + 3$ except when n is odd and r is even, in which case $\lambda_2^t(K_{r\times n}) \leq \Delta(K_{r\times n}) + r + 2$.*

Proof. Let $G = K_{r\times n}$ be a complete equipartite graph with $r \geq 3$ and $n \geq 2$ and consider its canonical decomposition $[\mathcal{K}, \mathcal{B}]$, recalling that $\Delta(K_{r\times n}) = n(r-1)$. In order to prove our result, we construct a $(2,1)$-total labelling π for $K_{r\times n}$ by labelling $\mathcal{K}$ and $\mathcal{B}$ separately. We construct four distinct labellings, each corresponding to one of the cases defined by the parity of r and n.

Case 1. $r \equiv 0 \pmod 2$ *and* $n \equiv 0 \pmod 2$

We first label $\mathcal{K}$. Since each $K^i \subseteq \mathcal{K}$ is an even complete graph of order r, let τ^i be the standard $(r+2)$-$(2,1)$-total labelling of K^i, $0 \leq i < n$. Define π as the union of all τ^i, $0 \leq i < n$. As the subgraphs K^i are pairwise disjoint, it follows that π restricted to $\mathcal{K}$ preserves the properties of a $(2,1)$-total labelling. Since $\mathcal{K}$ is a spanning subgraph of G, every vertex of G is assigned a label. Moreover, each vertex $v \in V(G)$ has $\pi(v) \in \{0, \ldots, r-1\}$.

It remains to label the edge set of $\mathcal{B}$. Each $B^{ij} \subseteq \mathcal{B}$ is an $(r-1)$-regular bipartite graph and, consequently, admits an $(r-1)$-edge colouring. Observe that whenever $\{i,j\} \cap \{k,\ell\} = \emptyset$, B^{ij} and $B^{k\ell}$ are disjoint, allowing the use of the same label set on both. By identifying each B^{ij} with an edge of the representative graph G_R, we conclude that $\chi'(G_R) = n-1$ corresponds to the number of distinct label sets required. Thus, consider an $(n-1)$-edge colouring of G_R and associate each colour class with a distinct label set of size $r-1$, chosen as an interval of consecutive integers.

In order to avoid conflicts with labels already used in $\mathcal{K}$, we start the label sets of $\mathcal{B}$ at $r+3$. Recall that the maximum label of a vertex in $\mathcal{K}$ is $r-1$ and of an edge is $r+2$. Therefore, this choice guarantees that the difference between the label of an edge in $\mathcal{B}$ and the label of its endpoints is at least two and that adjacent edges from $\mathcal{K}$ and $\mathcal{B}$ receive distinct labels. Also note that whenever B^{ij} and B^{ik} share a vertex, their representative edges in G_R are adjacent and, thus, lie in different colour classes, with disjoint label sets. Therefore, adjacent edges in $\mathcal{B}$ always receive distinct labels. We conclude that $(n-1)(r-1)$ new labels are used for $\mathcal{B}$, ranging from $r+3$ to $(r+3)+(n-1)(r-1)-1$. Therefore, π is a $(\Delta(G)+3)$-$(2,1)$-total labelling of G.

Figure 9 illustrates the construction of π for graph $K_{4\times 4}$.

Case 2. $r \equiv 1 \pmod 2$ *and* $n \equiv 0 \pmod 2$

In this case, the construction of π follows the same steps as in Case 1. Since r is odd, the standard labelling τ^i used on each K^i, $0 \leq i < n$, is an $(r+1)$-$(2,1)$-total labelling. By construction, label $r+1$ is always assigned to a vertex of K_i. The labelling of $\mathcal{B}$ proceeds as in Case 1 starting from $r+3$, which ensures the properties of a $(2,1)$-total labelling. Therefore, the edges of $\mathcal{B}$ receive new labels from $r+3$ to $n(r-1)+3$, and we conclude that π is a $(\Delta(K_{r\times n})+3)$-$(2,1)$-total labelling.

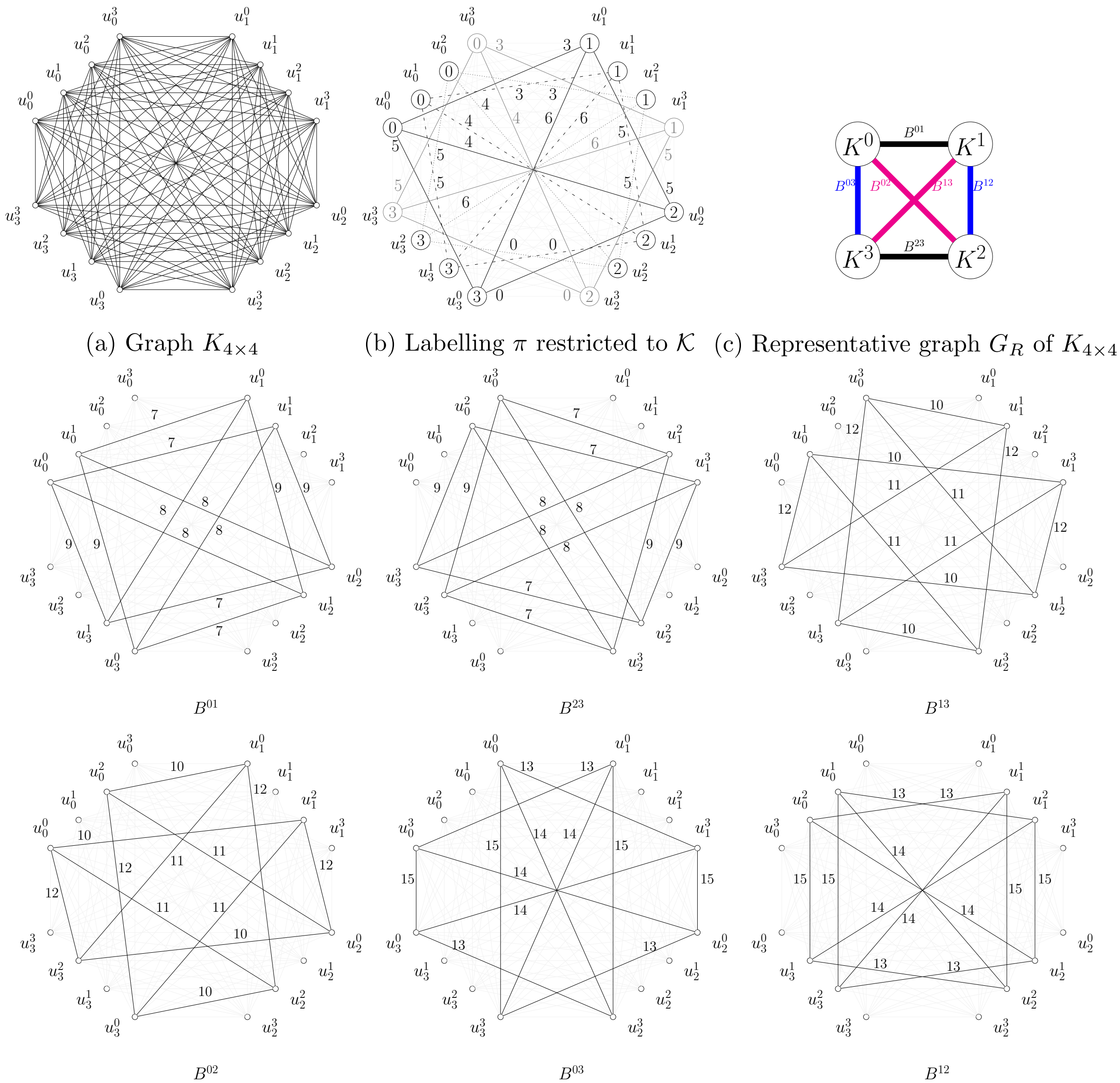
Figure 10 illustrates the construction of π for graph $K_{3\times 4}$.

Case 3. $r \equiv 1 \pmod 2$ *and* $n \equiv 1 \pmod 2$

First, we label $\mathcal{K}$. Let τ^i be the $(r+2)$-$(2,1)$-total labelling of K^i given by Proposition 2. For each element $x \in V(K^i) \cup E(K^i)$, if $\tau^i(x) \in \{4, 5, \ldots, r+2\}$ define $\pi(x) = \tau^i(x) + i(r-1)$, refer to these as *variable labels*; otherwise, if $\tau^i(x) \in \{0,1,2,3\}$, let $\pi(x) = \tau^i(x)$, these are called *fixed labels.*

Observe that π is a total labelling of $\mathcal{K}$ since the shift of the original labels of some elements yields new values still not used. Also, by construction, there exists an edge x in $\mathcal{K}$ such that $\tau^{n-1}(x) = r+2$. Therefore, π uses labels ranging from 0 to $r+2+(n-1)(r-1) = n(r-1)+3$.

Now, we label the edges of $\mathcal{B}$. Observe that all fixed labels appear in every K^i. Denote L^i the set of variable labels that occur in K^i. Note that $L^i \cap L^j = \emptyset$ for all $i \neq j$ since the sets L^i are



(a) Graph $K_{4\times 4}$ (b) Labelling π restricted to $\mathcal{K}$ (c) Representative graph G_R of $K_{4\times 4}$

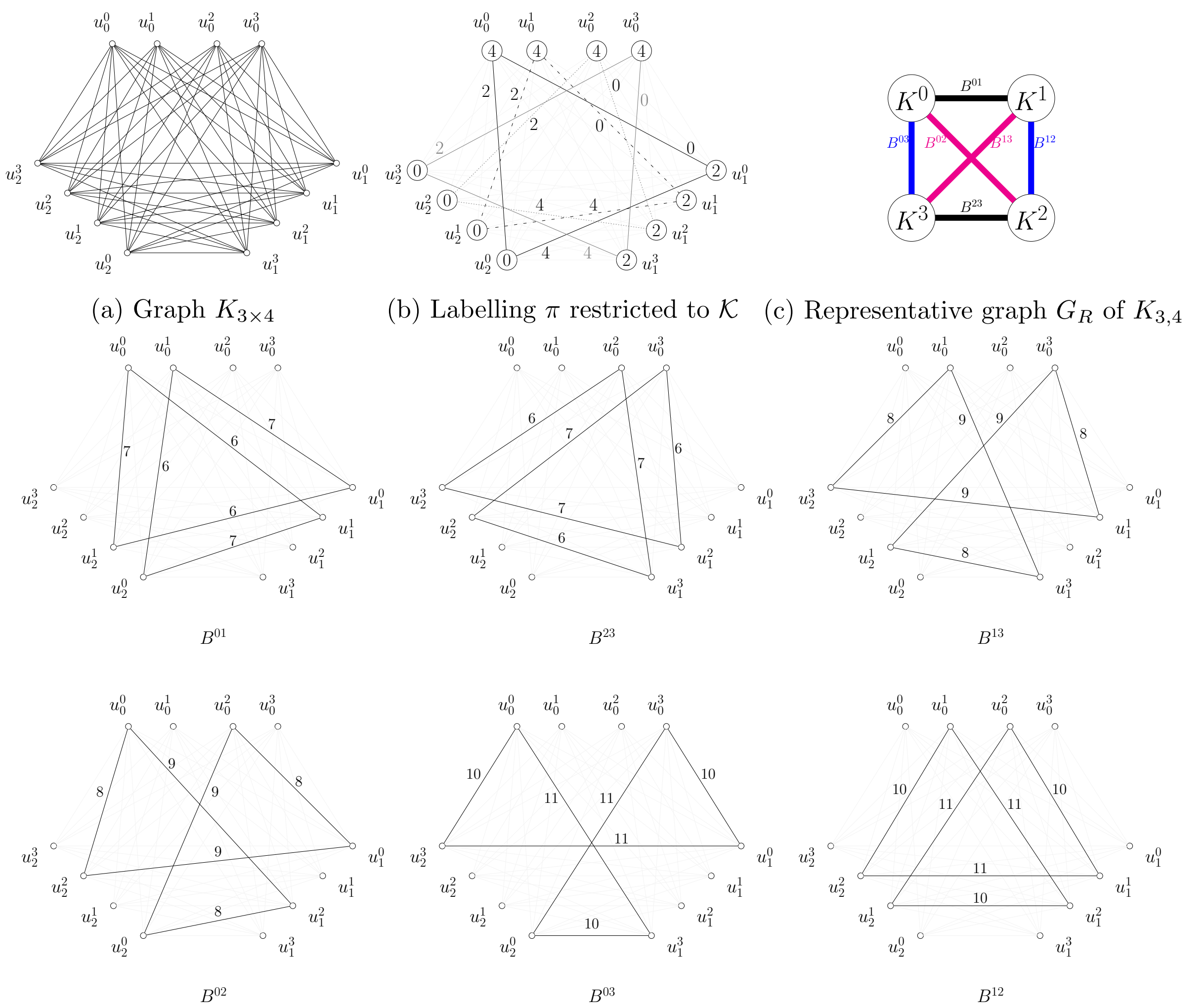
(d) Labelling π restricted to $E(\mathcal{B})$. Edges of B^{01} and B^{23} receive the labels $7, 8, 9$; Edges of B^{02} and B^{13} receive the labels $10, 11, 12$; and of B^{03} and B^{12} receive the labels $13, 14, 15$.

Figure 9: Illustration of labelling π for graph $K_{4\times 4}$.

obtained by distinct shifts of size $r-1$. Then, the values in set L^i are assigned to selected edges of $\mathcal{B}$, $0 \leq i \leq n-1$, following a total colouring of G_R as a guide. In Case 1, an edge colouring of G_R was sufficient as a guide. However, in this case, since certain labels from π are reused in the edges of some B^{ij}, we needed to analyze both the adjacencies and the incidences between elements. Therefore, the total colouring turned out to be a more suitable framework.

Let φ be an n-total colouring of G_R. Let S be a colour class of φ and $v_k \in S$. Since G_R is isomorphic to a complete graph, the other elements of S are edges. Then, associate L^k to each B^{ij} such that $v_i v_j \in S$. Considering $|L^i| = r-1$, this assignment is performed as in the previous cases, using an $(r-1)$-edge colouring of the bipartite graphs as guide.

Recall that each B^{ij} is incident with both K^i and K^j. By construction, the label set used in



(a) Graph $K_{3\times 4}$ (b) Labelling π restricted to $\mathcal{K}$ (c) Representative graph G_R of $K_{3,4}$

(d) Labelling π restricted to $E(\mathcal{B})$. Edges of B^{01} and B^{23} receive the labels $6, 7$; Edges of B^{02} and B^{13} receive the labels $8, 9$; and of B^{03} and B^{12} receive the labels $10, 11$.

Figure 10: Illustration of labelling π for graph $K_{3\times 4}$.

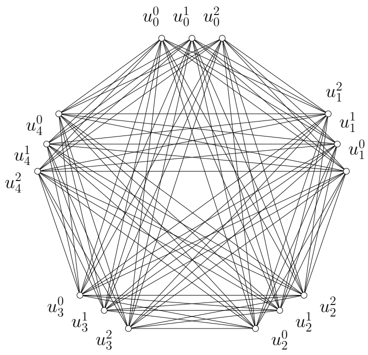
B^{ij} is disjoint from those assigned to K^i and K^j. Therefore, there does not exist conflict between the label of $e \in E(B^{ij})$ and the label of the elements of K^i or K^j.

Considering the fixed labels that occurs only in $\mathcal{K}$, there is no conflict with the labels assigned to the edges of $\mathcal{B}$ since the smallest label assigned to an edge of $\mathcal{B}$ is 4 and, by construction, label 3 is not used for any vertex. Thus, we conclude that the difference between the smallest label assigned to an edge in $\mathcal{B}$ and any fixed label assigned to a vertex in $\mathcal{K}$ is at least 2. Now, for the variable label sets, recall that by construction each pair K^i and K^j with $i \neq j$ has variable label sets L^i and L^j that differ by at least $r-1$. Since $r \geq 3$, it follows that $r - 1 \geq 2$ and, therefore, the label of every edge of $\mathcal{B}$ differs by at least 2 from the other labels that occur in $\mathcal{K}$. Therefore, we conclude that π is a $(2, 1)$-total labelling, and the result follows.

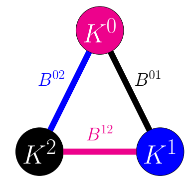
Figure 11 illustrates the construction of π for graph $K_{5\times 3}$.

Case 4. $r \equiv 0 \pmod 2$ *and* $n \equiv 1 \pmod 2$*.*

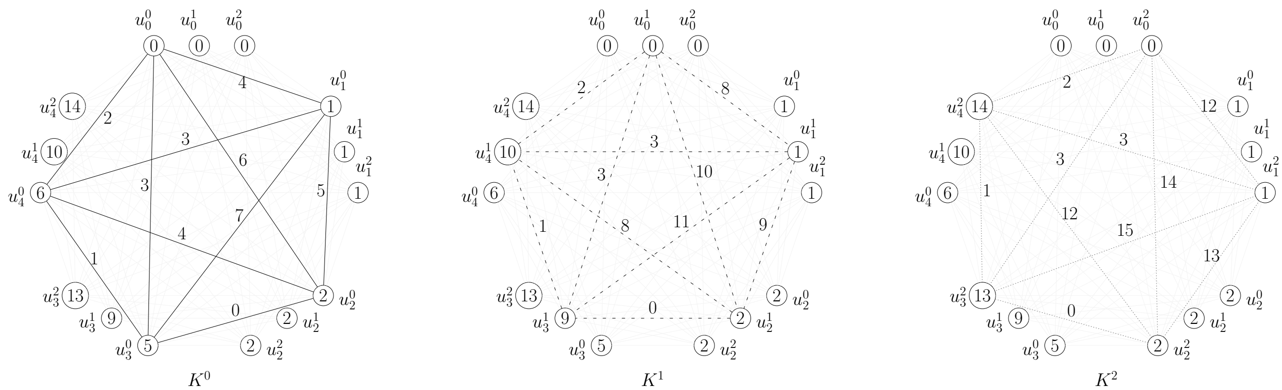
In this case, the construction of π follows the same steps as in Case 1. Since r is even, the



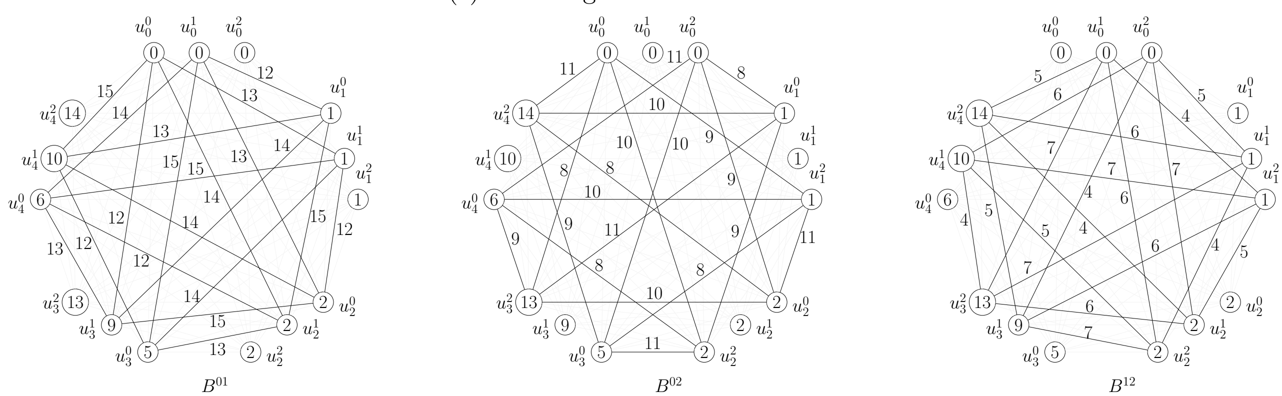
(a) Graph $K_{5,3}$



(b) Representative graph G_R of $K_{5,3}$



(c) Labelling π restricted to $\mathcal{K}$



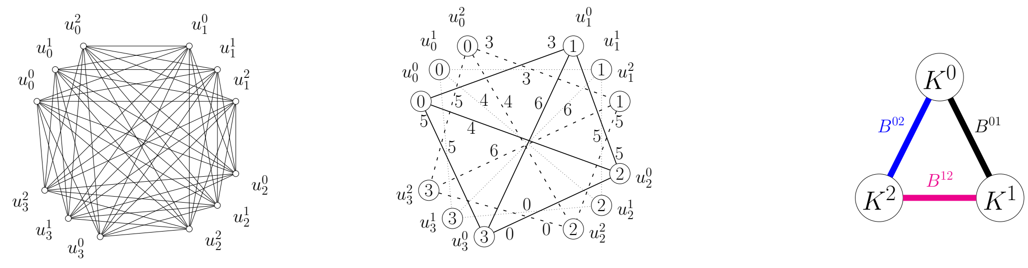
(d) Labelling π restricted to $E(\mathcal{B})$. Edges of B^{01} receive the labels $12, 13, 14, 15$; Edges of B^{02} receive the labels $8, 9, 10, 11$; and of B^{12} receive the labels $4, 5, 6, 7$.

Figure 11: Illustration of labelling π for graph $K_{5,3}$.

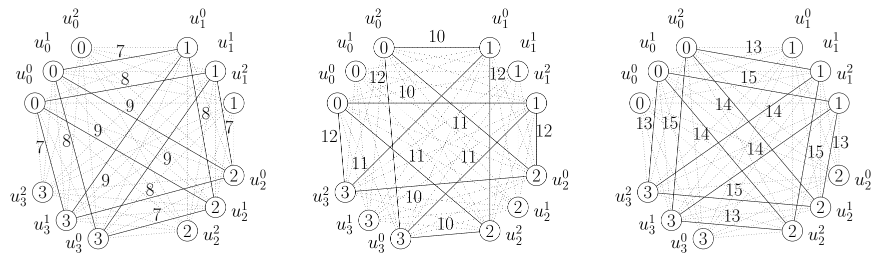
standard labelling τ^i used on each K^i, $0 \le i < n$, is an $(r+2)$-$(2,1)$-total labelling. By construction, $r+1$ and $r+2$ are not assigned to any vertex. The labelling of $\mathcal{B}$ proceeds as in Case 1 starting from $r+3$, which ensures the properties of $(2,1)$-total labelling. Moreover, as n is odd, the representative graph G_R has odd order and $\chi'(G_R) = n$. Consequently, the edges of $\mathcal{B}$ receive new labels from $r+3$ to $n(r-1)+r+2$. Therefore, π is a $(\Delta(K_{r\times n})+r+2)$-$(2,1)$-total labelling.

Figure 12 illustrates the construction of π for graph $K_{4\times 3}$.

□



(a) Graph $K_{4\times 3}$ (b) Labelling π restricted to $\mathcal{K}$ (c) Representative graph G_R of $K_{4\times 3}$



(d) Labelling π restricted to $E(\mathcal{B})$. Edges of B^{01} receive the labels $7, 8, 9$; Edges of B^{02} receive the labels $10, 11, 12$; and of B^{12} receive the labels $13, 14, 15$.

Figure 12: Illustration of labelling π for graph $K_{4\times 3}$.

4 Concluding Remarks

In this work, we studied $(2,1)$-total labellings of complete equipartite graphs $K_{r\times n}$. Based on their canonical decomposition into complete and bipartite subgraphs, we presented $(2,1)$-total labellings for these graphs that provide upper bounds for the $(2,1)$-total number of this class. Our main contribution was to show that Havet and Yu's conjecture holds for almost all complete equipartite graphs with $r \geq 3$ and $n \geq 2$, with the exception of the case where r is even and n is odd, for which we established a different upper bound of $\Delta(G) + r + 2$. Although upper bounds have been theoretically established, our experimental results – obtained via an ILP implementation – did not reveal any instance requiring a span grater than $\Delta(K_{r\times n}) + 2$. Since the equipartite complete graphs are regular graphs, it is known that $\lambda_2^t(K_{r\times n}) \geq \Delta(K_{r\times n}) + 2$ [1]. These observations, both empirical and theoretical, provide strong motivation for the following conjecture.

Conjecture 4. *For every complete equipartite graph $K_{r\times n}$ with $r \geq 3$ and $n \geq 2$, $\lambda_2^t(K_{r\times n}) = \Delta(K_{r\times n}) + 2$.*

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