Some families of 0-rotatable graceful caterpillars

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Abstract

A graceful labelling of a tree T is an injective function $f\colon V(T) \to \{0, 1, \ldots, |E(T)|\}$ such that $\{|f(u) - f(v)|\colon uv \in E(T)\} = \{1, 2, \ldots, |E(T)|\}$. A tree T is said to be 0-rotatable if, for any $v \in V(T)$, there exists a graceful labelling f of T such that $f(v) = 0$. In this work, it is proved that the following families of caterpillars are 0-rotatable: caterpillars with a perfect matching; caterpillars obtained by identifying a central vertex of a path P_n with a vertex of K_2; caterpillars obtained by linking one leaf of the star $K_{1,s-1}$ to a leaf of a path P_n with $n \geq 3$ and $s \geq \lceil \frac{n}{2} \rceil$; and caterpillars with diameter five or six. These results reinforce the conjecture that all caterpillars with diameter at least five are 0-rotatable.

1 Introdução

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A *labelling* of G is an injective function $f\colon V(G) \to \mathbb{Z}_{\geq 0}$. Under labelling f, the *label* of a vertex $v \in V(G)$ is $f(v)$, and the *(induced) label* of an edge $uv \in E(G)$ is the absolute difference of the labels of its ends, $|f(u) - f(v)|$. Given a labelling f of G, denote by L_V^f the set of vertex labels under f and denote by L_E^f the set of induced edge labels under f. Labelling f is a *graceful labelling* if $L_V^f \subseteq \{0, 1, \ldots, |E(G)|\}$ and $L_E^f = \{1, \ldots, |E(G)|\}$. We say that G is *graceful* if it has a graceful labelling. A labelling f of G is an *α-labelling* if f is graceful and there exists an integer $k \in \{0, 1, \ldots, |E(G)|\}$ such that, for each edge $uv \in E(G)$, either $f(u) \leq k < f(v)$, or $f(v) \leq k < f(u)$.

In 1967, Rosa [10] introduced four types of labellings of graphs, among them graceful labellings and α-labellings, and posed the *Graceful Tree Conjecture*, which states that all trees are graceful. Rosa proved that the Graceful Tree Conjecture is a strengthened version of the well-known Ringel-Kotzig Conjecture which states that K_{2m+1} has a cyclic decomposition into subgraphs isomorphic to a given tree T with m edges. The Graceful Tree Conjecture is a very important open problem in Graph Theory, with hundreds of papers about it [6].

As soon as one starts investigating graceful labellings of trees, it becomes clear the importance of knowing how to construct graceful labellings with the label 0 appearing in a given vertex. There are at least two results in the literature that stress the importance of label 0 in a graceful labelling of a tree T: first, it is easy to grow a gracefully labelled tree T by adding k new leaves to the 0-labelled vertex and expand the graceful labelling by assigning labels $|E(T)|+1, \ldots, |E(T)|+k$ to these new leaves. Second, Huang et al. [8] showed that it is possible to combine any tree with an

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α-labelling and any tree with a graceful labelling, by identifying the vertices labelled 0, such that the resultant tree is graceful. A tree T is 0-*rotatable* if, for any $v \in V(T)$, there exists a graceful labelling f of T such that $f(v) = 0$.

The importance of the 0-rotatability of trees was first noted by Rosa in his seminal paper [10], in which the author stated, without proof, that all paths are 0-rotatable. Ten years later, the author published a proof of this result [11]. Meanwhile, in 1969, some examples of non-0-rotatable trees were discovered [5]. As an example, the smallest non-0-rotatable tree is shown in Figure 1. Posteriorly, Chung and Hwang [4] investigated the 0-rotatability of a product of trees called Δ-*construction* and proved that if two trees T and T' are 0-rotatable, then their product $T\Delta T'$ is also 0-rotatable. Using this result, the authors showed that every caterpillar whose non-leaf vertices have the same degree is 0-rotatable.

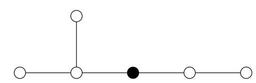


Figure 1: This tree does not have a graceful labelling that assigns label 0 to the black vertex.

In 2004, Bussel [2] showed that all trees with diameter at most three are 0-rotatable. Additionally, the author showed that there exist non-0-rotatable trees with diameter four. In fact, he completely determined the non-0-rotatable trees of diameter four. In order to do this, the author used the following result:

Theorem 1 (Bussel [2])**.** *Let T be a tree of diameter four such that its center v has degree two. Let v_1, v_2 be the vertices adjacent to v and m_1, m_2 be the number of leaves adjacent to v_1, v_2, respectively. Assume $m_1 \geq m_2$. The tree T has a graceful labelling f with $f(v) = 0$ if and only if there exist integers x and r such that $m_1 = (m_2 + 2 - x)(r - 1) - x$, with: (i) x, r not both odd; (ii) $2 \leq r \leq |E(T)|/2$; and (iii) $0 \leq x \leq \min\{r - 1, m_2\}$.* □

Let $\mathcal{D}$ denote the class of diameter-four trees whose center has degree two and that do not satisfy the conditions of Theorem 1. Let $\mathcal{D}'$ be the class of trees built by identifying a leaf of an arbitrary path P_n, $n \geq 1$, with the center of a tree in $\mathcal{D}$. Bussel [2] proved that, given a tree T with diameter four, T is 0-rotatable if and only if $T \notin \mathcal{D}'$. Additionally, he showed that all trees with at most 14 vertices and that are not 0-rotatable belong to the class $\mathcal{D}'$. Thus, based on these results, the author posed the following conjecture:

Conjecture 2 (Bussel [2])**.** *The class $\mathcal{D}'$ contains all non-0-rotatable trees.*

From the time it was first studied, 0-rotatability of trees has been considered a possible way to approach the Graceful Tree Conjecture, and also a challenging problem by itself. Even for arbitrary caterpillars the result is not known. In fact, note that, if Conjecture 2 is true, then it implies that every caterpillar with diameter at least five is 0-rotatable.

In this work we investigate the Conjecture 2 restricted to caterpillars and prove that the following families of caterpillars are 0-rotatable: (i) caterpillars with a perfect matching; (ii) caterpillars obtained by identifying a central vertex of a path P_n with a vertex of K_2; (iii) caterpillars obtained by linking one leaf of the star $K_{1,s-1}$ to a leaf of a path P_n, $n \geq 3$ and $s \geq \lceil \frac{n}{2} \rceil$; and (iv) caterpillars with diameter five or six. These results reinforce the conjecture that every caterpillar with diameter at least five is 0-rotatable. In particular, the last two families show that, for each integer $d \geq 5$, there exist 0-rotatable caterpillars with diameter d and arbitrary number of vertices.

In the next section, we present additional definitions as well as classic results and techniques that are used in our proofs. The main results are presented in Section 3.

2 Preliminaries

A *matching* M of a graph G is a set of pairwise nonadjacent edges of G. A vertex $v \in V(G)$ is *saturated* by M if v is incident with an edge of M. If M saturates all the vertices of G, then M is a *perfect matching*. Given a tree T with perfect matching M, the *contree* of T is the tree T' obtained from T by contracting all the edges of M.

Broersma and Hoede [1] introduced the concept of strongly graceful labellings of trees, defined as follows. Let T be a tree with a perfect matching M. A labelling f of T is *strongly graceful* if f is a graceful labelling and if $f(u) + f(v) = |E(T)|$ for every edge $uv \in M$. The authors proved that the Graceful Tree Conjecture is true if and only if every tree with a perfect matching has a strongly graceful labelling. They also proved the following result.

Lemma 3 (Broersma and Hoede [1])**.** *Let T be a tree with a perfect matching M and $uv \in M$, $u, v \in V(T)$. Let T' be the contree of T and let $x \in V(T')$ be the vertex corresponding to edge uv. If T' has a graceful labelling f', with $f'(x) = 0$, then T has two strongly graceful labellings f_1 and f_2, such that: (i) $f_1(u) = 0$ and $f_1(v) = |E(T)|$; (ii) $f_2(u) = |E(T)|$ and $f_2(v) = 0$.* □

Given a graceful labelling f of a tree T, the *complementary labelling* of f is the labelling $\overline{f}$ defined by $\overline{f}(v) = |E(T)| - f(v)$ for each $v \in V(T)$. Note that the complementary labelling is also a graceful labelling since: (i) $f(v)$ is an injection from $V(T)$ to $\{0, \ldots, |E(T)|\}$; and (ii) for each $uv \in E(T)$, $|\overline{f}(u) - \overline{f}(v)| = |(|E(T)| - f(u)) - (|E(T)| - f(v))| = |f(v) - f(u)|$.

A tree T is a *path* P_n if its vertices can be arranged in a linear sequence such that two vertices are adjacent if and only if they are consecutive in the sequence. The next lemmas are related to α-labellings and graceful labellings of paths and are used in Section 3.

Lemma 4 (Rosa [11])**.** *Let P_n be a path, $n \geq 1$, and let $v \in V(P_n)$. Then,*

(i) there exists an α-labelling f of P_n such that $f(v) = 0$ if and only if v is not the central vertex of P_5.

(ii) if v is the central vertex of P_5, then P_5 has a graceful labelling f such that $f(v) = 0$. □

Lemma 5 (Cattell [3])**.** *Let P_n be a path and $v \in P_n$. For any $i \in \{0, \ldots, n-1\}$, there exists a graceful labelling f of P_n with $f(v) = i$ whenever at least one of the following conditions is true:*

(i) n is even;

(ii) $n \equiv 5$ or $9 \pmod{12}$;

(iii) given a bipartition $\{X, Y\}$ of P_n with $|X| \geq |Y|$, $v \in X$;

(iv) $i \neq \frac{n-1}{2}$. □

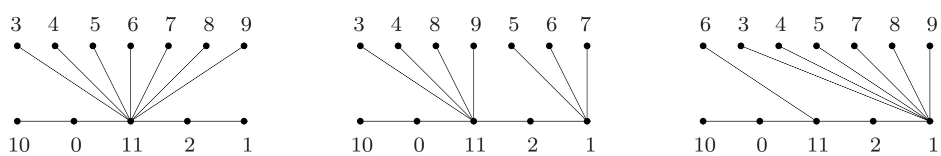
Let T be a tree and $v \in V(T)$. Denote by $N_k(v)$ the set of neighbours of v with degree k. The *distance* $d(u, v)$ between two vertices $u, v \in V(T)$ is the number of edges in the unique path connecting u and v in T. The *eccentricity* of a vertex $u \in V(T)$ is defined as $\epsilon(u) = \max\{d(u, v) \colon v \in V(T)\}$, the *diameter* as $diam(T) = \max_{v \in V(T)} \epsilon(v)$, and the *radius* as $radius(T) = \min_{v \in V(T)} \epsilon(v)$. A vertex $v \in V(T)$ is a *central vertex* of T if $\epsilon(v) = radius(T)$. A *spine* of T is a path $P \subset T$ such that its ends have maximum eccentricity in T. Given a tree T with spine P, we say that T is a *caterpillar* if all vertices of T are either contained in P, or are at distance exactly one from P. The next result states that every caterpillar has an α-labelling.

Lemma 6 (Rosa [10]). *Let T be a caterpillar and $v \in V(T)$ be a vertex which either has maximum eccentricity or is adjacent to a vertex of maximum eccentricity. Then, T has an α-labelling f such that $f(v) = 0$.* □

Let u, v, w be distinct vertices of a tree T, such that w is adjacent to u. We call *transfer* the operation of deleting edge wu from T and adding edge wv. After the transfer operation, we say that vertex w has been *transferred* or *moved* from u to v. For any two distinct vertices u and v of a gracefully labelled tree T, the notation $u \to v$ means that we moved some vertices incident with vertex u to vertex v. We say that a transfer $u \to v$ applied to a graceful tree is *safe* if the resulting tree is also graceful. The following lemma states when a transfer performed on a graceful tree generates another graceful tree.

Lemma 7 (Hrnčiar and Haviar [7]). *Let f be a graceful labelling of a tree T and let $u, v \in V(T)$ be two distinct vertices. If u is adjacent to (not necessarily distinct) leaves $u_1, u_2 \in V(T)$, such that $u_1 \neq v$, $u_2 \neq v$ and $f(u_1) + f(u_2) = f(u) + f(v)$, then the tree T' obtained from T by moving u_1, u_2 from u to v is also graceful.* □

A $u \to v$ transfer is said to be *of the first type* if the labels of the transferred vertices are the labels in set $\{k, k+1, \ldots, k+p\}$, where $f(u) + f(v) = k + (k+p)$. A transfer of the first type is also denoted by $u \xrightarrow{[k,k+p]} v$. Note that, in a transfer of the first type, the labels of transferred vertices constitute a set of consecutive integers. On the other hand, a $u \to v$ transfer is *of the second type* if the labels of the transferred vertices are the labels in set $\{k, k+1, \ldots, k+p\} \cup \{l, l+1, \ldots, l+p\}$, where $f(u) + f(v) = k + l + p$. A transfer of the second type is also denoted by $u \xrightarrow{[k,k+p],[l,l+p]} v$. In a transfer of the second type, the labels of the transferred vertices can be partitioned into two sets of same cardinality, where each set is composed by consecutive integers. Figure 2 illustrates these concepts.



(a) Graceful tree T. (b) Graceful tree T'. (c) Graceful tree T''.

Figure 2: Tree T' is obtained from T by transfer $11 \xrightarrow{[5,7]} 1$ of the first type; on the other hand, tree T'' is obtained from T by applying transfer $11 \xrightarrow{[3,5],[7,9]} 1$ of the second type.

The next lemma establishes additional conditions under which it is possible to make safe transfers in a graceful tree.

Lemma 8 (Mishra and Panigrahi [9]). *Let T be a tree with a graceful labelling f satisfying the following properties:*

(i) there exist distinct vertices in T with labels $a, \ldots, a + r_1, b - r_2, \ldots, b$ such that $a < b$, $a + r_1 < b - r_2$, and $r_1, r_2 \in \mathbb{Z}_{\geq 0}$;

(ii) the vertex with label a is adjacent to a set of vertices $\mathcal{S}$ with labels $s, \ldots, s + p$, such that:

(a) $p \geq 2$;

(b) $\{s, \ldots, s + p\} \cap \{a, \ldots, a + r_1, b - r_2, \ldots, b\} = \emptyset$; and

(c) for $0 \leq i < \lfloor \frac{p}{2} \rfloor$*, either* $(s+i+1)+(s+p-i) = a+b$ *or* $(s+i)+(s+p-1-i) = a+b$*.*

Then, the following statements are true:

(a) if $|\mathcal{S}|$ *is odd, then it is possible to make a safe transfer* $a \to b$ *of the first type followed by a safe transfer* $b \to (a+1)$ *of the second type, keeping an odd number of vertices at vertex* a *and a positive even number of vertices at* b*, moving the rest of the vertices to* $a+1$*.*

(b) if $|\mathcal{S}|$ *is even, then it is possible to make a sequence of safe transfers of the second type* $a \to b \to a+1 \to b-1 \to a+2 \to b-2 \to \ldots \to z$*, where* $z = a + r_1$ *or* $z = b - r_2$*, keeping a positive even number of vertices of* $\mathcal{S}$ *at each vertex of the sequence.* □

3 Results

In this section, we prove our main results. We start by showing an interesting result which is useful for proving that certain families of trees with a perfect matching are 0-rotatable.

Theorem 9. *Let* T *be a tree with a perfect matching. If the contree of* T *is* 0*-rotatable, then* T *is* 0*-rotatable.*

Proof. Let T be a tree with perfect matching M and $uv \in M$. Let T' be the contree of T and $x \in V(T')$ be the vertex corresponding to edge uv. Suppose T' is 0-rotatable. Hence, T' has a graceful labelling f' such that $f'(x) = 0$. Thus, by Lemma 3, T has two strongly graceful labellings f_1 and f_2 such that: $f_1(u) = 0$ and $f_1(v) = |E(T)|$; $f_2(u) = |E(T)|$ and $f_2(v) = 0$. Therefore, there exist strongly graceful labellings of T which assign label 0 to vertices u and v. Since uv is an arbitrary edge of M, we conclude that T is 0-rotatable. □

Corollary 10. *Every caterpillar with a perfect matching is* 0*-rotatable.*

Proof. The result follows from Theorem 9 and the fact that the contree of a caterpillar with a perfect matching is a path, which is 0-rotatable by Lemma 4. □

Theorem 14 and Theorem 16 prove that two families of caterpillars are 0-rotatable. Before presenting these results, it is necessary to establish some auxiliary lemmas.

Lemma 11. *Let* X, Y, Z *be nonempty sets such that:*

(i) $|Y| \geq \max\{|X|, |Z|\}$*;*

(ii) $X = \{0, \ldots, |X|-1\}$*;*

(iii) $Y = \{|X|, \ldots, |X|+|Y|-1\}$*; and*

(iv) $Z = \{|X|+|Y|, \ldots, |X|+|Y|+|Z|-1\}$*.*

Then, for every $l \in X \cup Z$*, there exists* $t \in Y$ *for which* $|l - t| = |Y|$*.*

Proof. The result follows by letting $t = l + |Y|$ when $l \in X$, and letting $t = l - |Y|$ when $l \in Y$. □

Lemma 12. *Let* T *be either a path* P_n*, with* $n \geq 1$*, or a star* $K_{1,n-1}$*, with* $n \geq 2$*. Let* $v \in V(T)$ *be a leaf of* T*,* t *be a positive integer and* $S = \{t, t+1, \ldots, t+n-1\}$*. Then, for each* $i \in S$*, there exists a labelling* $f\colon V(T) \to S$ *such that* $f(v) = i$ *and* $L^f_{E(T)} = \{1, \ldots, n-1\}$*.* □

Lemma 13. *If a tree T has an α-labelling f, then there exists a bipartition $\{A, B\}$ of T such that $L_A^f = \{0, \dots, |A| - 1\}$ and $L_B^f = \{|A|, \dots, |A| + |B| - 1\}$.* □

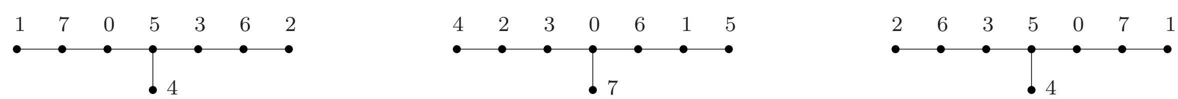
Theorem 14. *Every caterpillar obtained by identifying a vertex of K_2 with a central vertex of P_n is 0-rotatable.*

Proof. Let $P_n = v_1 \cdots v_n$ be a path, with $n \geq 1$. Let T be the caterpillar obtained by identifying a vertex of K_2 with the central vertex $v_{\lceil \frac{n}{2} \rceil}$ of P_n. Let $v_1, \dots, v_n$ be the vertices of the spine of T and let v_{n+1} be the leaf adjacent to $v_{\lceil \frac{n}{2} \rceil}$.

If $diam(T) \in \{1, 2, 3, 4\}$, the result follows from Lemma 6 and Corollary 10. Now, consider $diam(T) \in \{5, 6, 7\}$. By Lemma 6, for $v \in \{v_1, v_2, v_{n-1}, v_n\}$, there exists a graceful labelling f of T such that $f(v) = 0$. Moreover, Figure 3 exhibits two distinct graceful labellings f_5^1, f_5^2 of T with $diam(T) = 5$, such that $f_5^1(v_3) = 0$ and $f_5^2(v_4) = 0$. The complementary labelling of f_5^1 assigns label 0 to v_7. Figure 4 exhibits three distinct graceful labellings f_6^1, f_6^2, f_6^3 of T with $diam(T) = 6$, such that $f_6^1(v_3) = 0$, $f_6^2(v_4) = 0$, and $f_6^3(v_5) = 0$. The complementary labelling of f_6^2 assigns label 0 to v_8. Finally, Figure 5 exhibits three distinct graceful labellings f_7^1, f_7^2, f_7^3 of T with $diam(T) = 7$, such that $f_7^1(v_3) = 0$, $f_7^2(v_4) = 0$, $f_7^3(v_5) = 0$. The complementary labelling of f_7^3 assigns label 0 to v_6, the complementary labelling of f_7^2 assigns label 0 to v_9, and the result follows.

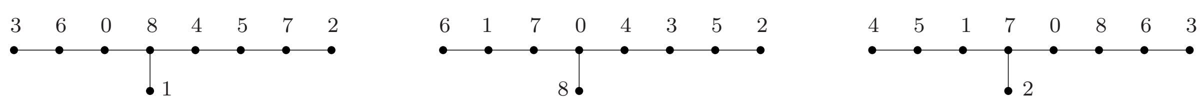
(a) Graceful labelling f_5^1. (b) Graceful labelling f_5^2.

Figure 3: Two graceful labelings of a caterpillar T with $diam(T) = 5$.



(a) Graceful labelling f_6^1. (b) Graceful labelling f_6^2. (c) Graceful labelling f_6^3.

Figure 4: Three graceful labelings of a caterpillar T with $diam(T) = 6$.



(a) Graceful labelling f_7^1. (b) Graceful labelling f_7^2. (c) Graceful labelling f_7^3.

Figure 5: Three graceful labelings of a caterpillar T with $diam(T) = 7$.

Now, we consider the remaining case in which $diam(T) \geq 8$. Let $P \subset T$ be the subgraph induced by vertex set $\{v_1, v_2, \dots, v_{\lceil \frac{n}{2} \rceil}, v_{n+1}\}$ and let $Q \subset T$ be the subgraph induced by vertex set $V(T) \backslash V(P)$. Let n_P and n_Q denote the order of P and Q, respectively, and let m_P and m_Q denote the sizes of P and Q, respectively. Note that both P and Q are paths. Moreover, since $diam(T) \geq 8$, $diam(P) \geq 5$.

First, we prove that, for $v \in V(P)$, there exists a graceful labelling f of T such that $f(v) = 0$. By Lemma 4, P has an α-labelling $g\colon V(P) \to \{0, 1, \dots, m_P\}$ such that $g(v) = 0$. By Lemma 13, there exists a bipartition $\{A, B\}$ of P such that $L_A^g = \{0, 1, \dots, |A| - 1\}$ and $L_B^g = \{|A|, \dots, |A| + |B| - 1\}$.

Using this bipartition, we modify g in order to obtain another labelling f_P of P as follows:

$$f_P(u) = \begin{cases} g(u), & \text{if } u \in A; \\ g(u) + n_Q, & \text{if } u \in B. \end{cases}$$

Therefore, we obtain $f_P \colon V(P) \to A \cup B'$ such that $A = \{0, 1, \ldots, |A| - 1\}$ and $B' = \{|A| + n_Q, |A| + 1 + n_Q, \ldots, |A| + |B| - 1 + n_Q\}$. Since each label in B was increased by n_Q, $L^{f_P}_{E(P)} = \{1 + n_Q, 2 + n_Q, \ldots, m_P + n_Q\}$.

Note that the vertex labels $|A|, |A|+1, \ldots, |A|+n_Q-1$ are missing in f_P, as well as the induced edge labels $1, 2, \ldots, n_Q$. Let $C = \{|A|, |A| + 1, \ldots, |A| + n_Q - 1\}$ and let $l = f_P(v_{\lceil \frac{n}{2} \rceil})$. Next, we show that there exists an integer $t \in C$, such that $|l - t| = |C| = n_Q$.

By the definition of P, we have that $|A| + |B'| = n_P = \lceil \frac{n}{2} \rceil + 1$. Moreover, since P is a path, one of the following holds: (i) $|A| = |B'| = (\lceil \frac{n}{2} \rceil + 1)/2$; (ii) $|A| = \lfloor (\lceil \frac{n}{2} \rceil + 1)/2 \rfloor$ and $|B'| = |A| + 1$; or (iii) $|B'| = \lfloor (\lceil \frac{n}{2} \rceil + 1)/2 \rfloor$ and $|A| = |B'| + 1$. Since $|C| = n_Q = \lfloor \frac{n}{2} \rfloor$ and $\lfloor \frac{n}{2} \rfloor > \lceil (\lceil \frac{n}{2} \rceil + 1)/2 \rceil$ for $n \geq 9$, we obtain that $|C| > |A|$ and $|C| > |B'|$. Thus, considering $X = A$, $Y = C$, $Z = B'$, and l as previously chosen, by Lemma 11, there exists $t \in Y$, such that $|l - t| = |Y| = |C|$, as required.

By Lemma 12, there exists a labelling $f_Q \colon V(Q) \to C$ such that: (i) $f_Q(v_{\lceil \frac{n}{2} \rceil + 1}) = t$; and (ii) $L^{f_Q}_{E(Q)} = \{1, \ldots, n_Q - 1\}$. Define a labelling $f \colon V(T) \to \{0, 1, \ldots, |E(T)|\}$ such that:

$$f(u) = \begin{cases} f_P(u), & \text{if } u \in P; \\ f_Q(u), & \text{if } u \in Q. \end{cases}$$

Labelling f is a graceful labelling of T since: (i) f is an injective function from $V(T)$ to $\{0, 1, \ldots, m_P + m_Q + 1 = |E(T)|\}$; (ii) the induced edge labels of Q are $1, 2, \ldots, n_Q - 1$; (iii) the induced edge labels of P are $n_Q + 1, n_Q + 2, \ldots, |E(T)|$; and (iv) $f(v_{\lceil n/2 \rceil} v_{\lceil n/2 \rceil + 1}) = n_Q$.

In order to conclude the proof, we have to show that there exists a graceful labelling f such that $f(v) = 0$ for each vertex $v \in V(Q)$. It can be done by the previous reasoning, considering $V(P) = \{v_{\lceil \frac{n}{2} \rceil}, \ldots, v_n, v_{n+1}\}$ and $V(Q) = V(T) \backslash V(P)$. □

Theorem 16 proves that every caterpillar obtained by linking one leaf of star $K_{1,s-1}$ to a leaf of path P_n, with $n \geq 3$ and $s \geq \lceil \frac{n}{2} \rceil$, is 0-rotatable. In our proof we use a specific labelling of a caterpillar which is presented in the next lemma.

Lemma 15. *Let T be the caterpillar obtained by linking one leaf of star $K_{1,s-1}$, $s \geq 3$, to a leaf of path P_5. If v is the central vertex of P_5, then there exists a graceful labelling f of T such that $f(v) = 0$.* □

Theorem 16. *Let T be the caterpillar obtained by linking one leaf of the star $K_{1,s-1}$ to a leaf of the path P_n. If $n \geq 3$ and $s \geq \lceil \frac{n}{2} \rceil$, then T is 0-rotatable.*

Proof. Let $P_n = v_1 \cdots v_n$ and $V(K_{1,s-1}) = \{x_1, \ldots, x_s\}$, with x_s its central vertex. Let T be the caterpillar obtained by linking x_1 to v_1. Thus, T has vertex set $V(T) = V(K_{1,s-1}) \cup V(P_n)$ and edge set $E(T) = E(K_{1,s-1}) \cup E(P_n) \cup \{x_1 v_1\}$.

Suppose $n \geq 3$ and $s \geq \lceil \frac{n}{2} \rceil$. In the following, we prove that there exists a graceful labelling f of T such that $f(v) = 0$ for every $v \in V(T)$. We consider two cases depending on which subgraph, $K_{1,s-1}$ or P_n, vertex v belongs to.

Case 1. $v \in V(K_{1,s-1})$.

By Lemma 6, for every $v \in \{x_2, \ldots, x_s\}$, there exists a graceful labelling f of T such that $f(v) = 0$. Therefore, in order to conclude this case, it remains to show that there exists a graceful labelling f of T such that $f(x_1) = 0$.

Let H_1 and H_2 be subgraphs of T induced by the vertex set $\{v_1, x_1, x_2, \ldots, x_s\}$ and $\{v_2, v_3, \ldots, v_n\} = V(T)\backslash V(H_1)$, repectively. Define a graceful labelling $h_1 \colon V(H_1) \to \{0, \ldots, s\}$ as follows: (i) $h_1(x_i) = i - 1$, for $1 \le i \le s$; and (ii) $h_1(v_1) = s$. Since $h_1(x_s) = s - 1$ and its neighbours have labels $0, 1, \ldots, s-2$, the edges incident with x_s have induced labels $1, 2, \ldots, s-1$. Moreover, since $h_1(x_1) = 0$ and $h_1(v_1) = s$, the edge x_1v_1 has label s. Next, we modify h_1 in order to obtain another labelling h_1':

$$h_1'(v) = \begin{cases} h_1(v), & \text{if } v \in \{x_1, x_2, \ldots, x_{s-1}\}; \\ h_1(v) + |V(H_2)|, & \text{if } v \in \{x_s, v_1\}. \end{cases}$$

Since $n \ge 3$ and $|V(H_2)| = n - 1$, $|V(H_2)| \ge 2$. By the definition, $h_1'(x_s) = n + s - 2$. Moreover, the neighbours of x_s have labels $0, 1, \ldots, s-2$ under h_1'. Therefore, $L^{h_1'}_{E(K_{1,s-1})} = \{n, n+1, \ldots, n+s-2\}$. Also, since $h_1'(x_1) = 0$ and $h_1'(v_1) = n + s - 1$, we have that $h_1'(x_1v_1) = n + s - 1$. Thus, we conclude that the vertex labels $s-1, s, \ldots, s+n-3$ are missing, as well as the edge labels $1, 2, \ldots, n-1$.

Since H_2 is a path with $|V(H_2)| \ge 2$, by Lemma 12, H_2 has a labelling $h_2 \colon V(H_2) \to \{s-1, s, \ldots, s+n-3\}$ such that $h_2(v_2) = s$ and $L^{h_2}_{E(H_2)} = \{1, 2, \ldots, n-2\}$. We define labelling $f \colon V(T) \to \{0, 1, \ldots, |E(T)|\}$ as follows:

$$f(v) = \begin{cases} h_1'(v), & \text{if } v \in V(H_1); \\ h_2(v), & \text{if } v \in V(H_2). \end{cases}$$

Labelling f is graceful since: (i) f is an injective function from $V(T)$ to $\{0, \ldots, |E(T)|\}$; and (ii) $L^f_{E(H_2)} = \{1, \ldots, n-2\}$, $L^f_{E(H_1)} = \{n, \ldots, n+s-1\}$, and $|f(v_2) - f(v_1)| = |s - (n+s-1)| = n - 1$. Thus, $L^f_{E(T)} = \{1, \ldots, n+s-1\}$ and the result follows.

Case 2. $v \in V(P_n)$.

If $n = 5$ and v is the central vertex of P_5, the result follows by Lemma 15. Thus, consider $n \ne 5$ or v different from the central vertex of P_5. By Lemma 4, since v is not the central vertex of P_5, path P_n has an α-labelling g such that $g(v) = 0$. By Lemma 13, there exists a bipartition $\{A, B\}$ of P_n such that $L^g_A = \{0, 1, \ldots, |A| - 1\}$ and $L^g_B = \{|A|, \ldots, |A| + |B| - 1\}$. Using this bipartition, we modify g in order to obtain another labelling f_P of P_n. For each $u \in V(P_n)$, define

$$f_P(u) = \begin{cases} g(u), & \text{if } u \in A; \\ g(u) + s, & \text{if } u \in B. \end{cases}$$

Thus, we obtain the labelling $f_P \colon V(P_n) \to A \cup B'$, such that $A = \{0, 1, \ldots, |A| - 1\}$ and $B' = \{|A| + s, |A| + s + 1, \ldots, |A| + s + |B| - 1\}$. Since each label in B was increased by s, $L^{f_P}_{E(P_n)} = \{1+s, 2+s, \ldots, n-1+s = |E(T)|\}$. Note that the vertex labels $|A|, |A|+1, \ldots, |A|+s-1$ are missing in f_P, as well as the induced edge labels $1, 2, \ldots, s$. Let $C = \{|A|, |A|+1, \ldots, |A|+s-1\}$ and let $l = f_P(v_1)$. Next, we show that there exists an integer $t \in C$, such that $|l - t| = |C| = s$.

Consider $X = A$, $Y = C$, $Z = B'$, and l as previously chosen. Since $|C| \ge \lceil \frac{n}{2} \rceil$, by Lemma 11, there exists $t \in Y$, such that $|l - t| = |Y| = |C|$, as required. By Lemma 12, there exists a labelling

$f_K\colon V(K_{1,s-1}) \to C$, such that: (i) $f_K(x_1) = t$; and (ii) $L^{f_K}_{K_{1,s-1}} = \{1, 2, \ldots, s-1\}$. Thus, define labelling $f\colon V(T) \to \{0, 1, \ldots, |E(T)|\}$ as follows:

$$f(u) = \begin{cases} f_P(u), & \text{if } u \in V(P_n); \\ f_K(u), & \text{if } u \in V(K_{1,s-1}). \end{cases}$$

Labelling f is graceful since: (i) f is an injective function from $V(T)$ to $\{0, 1, \ldots, |E(T)|\}$; and (ii) $L^f_{E(K_{1,s-1})} = \{1, \ldots, s-1\}$, $L^f_{E(P_n)} = \{s+1, \ldots, s+n-1\}$ and $f(x_1v_1) = s$. Therefore, $L^f_{E(T)} = \{1, \ldots, |E(T)|\}$ and the result follows. □

3.1 Caterpillars with diameter five

The main result of this section is Theorem 19, which states that every caterpillar T with diameter five is 0-rotatable. In order to prove this result, for each non-leaf vertex $v \in V(T)$, we construct a graceful labelling f of T that assigns label 0 to v and assigns label $|E(T)|$ to any leaf $u \in V(T)$ adjacent to v. Consequently, we use its complementary labelling $\overline{f}$ in order to obtain $\overline{f}(u) = 0$ and $\overline{f}(v) = |E(T)|$. Since $\overline{f}$ is also a graceful labelling and f is constructed considering an arbitrary non-leaf vertex v of T, we conclude that T is 0-rotatable.

The above mentioned labellings are obtained either directly from Lemma 6, or by modifying one of the trees presented in Figure 6. These trees are modified by transfer operations and need some properties presented in Lemma 17.

Given two finite sets of integers A and B, we say that $A < B$ if $\max\{a\colon a \in A\} < \min\{b\colon b \in B\}$. An ordered pair $(\{r, s, t\}, \mathcal{N})$ with $r, s, t \in \mathbb{Z}_{\geq 0}$ and $\mathcal{N} \subseteq \mathbb{Z}_{\geq 0}$ is called a *special pair* if it satisfies the following conditions:

(i) $r \leq s$ and $t = r + s$;

(ii) given the index set $\mathcal{I} = \{0, t-1, t, t+1\}$ and $\mathcal{N} = \{n_i\colon n_i \in \mathbb{Z}_{\geq 0}, i \in \mathcal{I},$ and $n_0 \geq 1\}$, then $\sum_{i \in \mathcal{I}} n_i = s - r + 1$; and

(iii) exactly one of the following conditions holds:

 (a) n_i is even for $i \in \mathcal{I} \backslash \{0\}$;

 (b) $n_{t-1} \equiv n_{t+1} \equiv t \pmod 2$ and $n_t \not\equiv t \pmod 2$.

Lemma 17. *Let $(\{r, s, t\}, \mathcal{N})$ be a special pair. Let T be a tree and let f be a graceful labelling of T satisfying the following properties:*

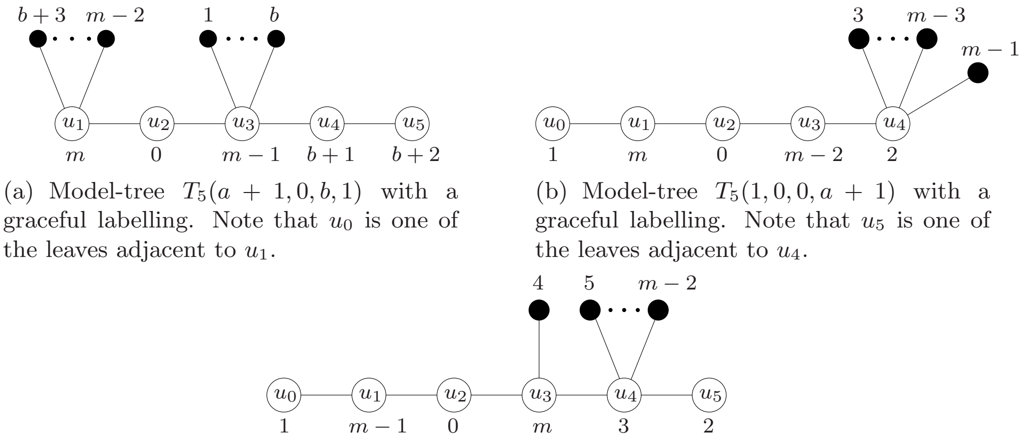
(i) T has a vertex v that is adjacent to a set $\mathcal{S}$ of vertices with labels $r, r+1, \ldots, s$;

(ii) for each $i \in \{t-1, t, t+1\}$, if T has a vertex v_i such that $f(v) + f(v_i) = i$, then $v_i \notin \mathcal{S}$.

Then, for each $i \in \{t-1, t, t+1\}$, it is possible to safely transfer n_i vertices of $\mathcal{S}$ from v to v_i. □

The main result of this section is Lemma 18. In order to present it, we need an additional definition: the *model-tree* $T_d(c_1, c_2, \ldots, c_{d-1})$ is the caterpillar with diameter d and spine $P = u_0 \cdots u_d$ such that, for $i \in \{1, \ldots, d-1\}$, vertex u_i is adjacent to exactly c_i leaves. Figure 6 shows three model-trees with special graceful labellings.

Lemma 18. *Let T be a caterpillar with diameter five. Let $v \in V(T)$ be a central vertex of T and $w \in \{v\} \cup N_1(v)$. Then, T has a graceful labelling f such that $f(w) = 0$.*



(a) Model-tree $T_5(a+1,0,b,1)$ with a graceful labelling. Note that u_0 is one of the leaves adjacent to u_1.

(b) Model-tree $T_5(1,0,0,a+1)$ with a graceful labelling. Note that u_5 is one of the leaves adjacent to u_4.

(c) Model-tree $T_5(1,0,1,a+1)$ with a graceful labelling.

Figure 6: Scheme of three model-trees with a graceful labelling.

Proof. Let T be a caterpillar with $diam(T) = 5$ and spine $P = u_0u_1u_2u_3u_4u_5$. For each $i \in \{1,2,3,4\}$, define U_i as the set of leaves from $V(T)\backslash\{u_0,u_5\}$ that are adjacent to u_i. The central vertices of T are u_2 and u_3. We prove the result for u_2; the result for u_3 is analogous.

Let $T' \subseteq T$ be the subtree induced by vertex set $V(T)\backslash U_2$. Note that, if T' has a graceful labelling $f'\colon V(T') \to \{0,\dots,|E(T')|\}$ such that $f'(u_2) = 0$, then it is possible to expand T' by adding $|U_2|$ leaves to vertex u_2 and label these leaves with consecutive integers $|E(T')|+1,\dots,|E(T')|+|U_2|$, obtaining a graceful labelling f of T with $f(u_2) = 0$. Furthermore, by applying the complementary labelling $\overline{f}$ of f, we obtain a graceful labelling $\overline{f}$ of T with label 0 assigned to a leaf adjacent to u_2. Therefore, we can assume that $U_2 = \emptyset$ and prove that T has a graceful labelling f such that $f(u_2) = 0$. We consider three cases depending on the parities of $|U_1|$, $|U_3|$ and $|U_4|$.

Case 1. ($|U_1| \equiv |U_4| \pmod 2$) or ($|U_1| \equiv 1 \pmod 2$ and $|U_4| \equiv 0 \pmod 2$).

Initially, we modify T in order to obtain a model-tree $T' = T_5(a+1,0,b,1)$, with $a = |U_1|+|U_4|$, $b = |U_3|$, and $V(T') = V(T)$, changing the edge set of T as follows: $E(T') = (E(T)\backslash\{u_4w\colon w \in U_4\}) \cup \{u_1w\colon w \in U_4\}$. Tree T' has a graceful labelling f such that $f(u_2) = 0$, as illustrated in Figure 6(a). Next, we show that it is possible to safely transfer $|U_4|$ leaves from u_1 to u_4, obtaining a graceful labelling for T.

Let $m = |E(T')| = |U_1| + |U_3| + |U_4| + 5 = a + b + 5$. Let r, s, t be three positive integers such that $r = b+3$, $s = m-2$, and $t = r+s = m+b+1$. Let $\mathcal{I} = \{0, t-1, t, t+1\}$ be an index set and let $\mathcal{N} = \{n_i\colon i \in \mathcal{I}\}$ such that $n_0 = |U_1|+1$, $n_t = |U_4|$, $n_{t-1} = n_{t+1} = 0$. Note that the ordered pair $(\{r,s,t\},\mathcal{N})$ is a special pair since: (i) $r \le s$ and $t = r+s$; (ii) $n_0 \ge 1$ and $\sum_{i\in\mathcal{I}} n_i = a+1 = s-r+1$; (iii) when $|U_4| \equiv 0 \pmod 2$, n_i is even, for $i \in \mathcal{I}\backslash\{0\}$; and (iv) when $|U_1| \equiv |U_4| \equiv 1 \pmod 2$, we have that $n_{t-1} \equiv n_{t+1} \equiv t \pmod 2$ and $n_t \not\equiv t \pmod 2$. Moreover, note that: (i) the vertex $u_1 \in V(T')$ is adjacent to a set of leaves $\mathcal{S}$ with labels $b+3,\dots,m-2$; (ii) vertex $u_4 \notin \mathcal{S}$ and $f(u_1)+f(u_4) = m+b+1 = t$. Therefore, by Lemma 17, considering $u_1 = v$ and $u_4 = v_t$, we can safely transfer $n_t = |U_4|$ leaves of set S from vertex u_1 to vertex u_4.

Case 2. $|U_1| \equiv |U_3| \equiv 0 \pmod 2$ and $|U_4| \equiv 1 \pmod 2$.

If $|E(T)| = 6$, the result follows from the graceful labelling of T depicted in Figure 6(b). Thus, suppose $|E(T)| \ge 7$. By the same reasoning of the previous case, we construct $T' = T_5(1,0,0,a+1)$, with $a = |U_1| + |U_3| + |U_4|$ and $V(T') = V(T)$, changing the edge set of T as follows: $E(T') =$

$(E(T)\backslash(\{u_1w\colon w\in U_1\}\cup\{u_3w\colon w\in U_3\}))\cup\{u_4w\colon w\in U_1\cup U_3\}$. Tree T' has a graceful labelling f such that $f(u_2)=0$, illustrated in Figure 6(b). Next, we show that it is possible to safely transfer $|U_i|$ leaves from u_4 to u_i, for $i\in\{1,3\}$. Let $m=|E(T')|=|U_1|+|U_3|+|U_4|+5=a+5$. We consider two subcases.

Subcase 1. $|U_1|>0$.

Since $f(u_1)+f(u_4)=m+2$ and u_4 has two leaves with labels $m-1$ and 3, by Lemma 7, we can safely transfer this pair of leaves from vertex u_4 to vertex u_1. Additionally, let r,s,t be three positive integers such that $r=4$, $s=m-3$, and $t=r+s=m+1$. Let $\mathcal{I}=\{0,t-1,t,t+1\}$ be an index set and let $\mathcal{N}=\{n_i\colon i\in\mathcal{I}\}$ such that $n_0=|U_4|+1$, $n_{t-1}=|U_3|$, $n_t=0$ and $n_{t+1}=|U_1|-2$. Note that the ordered pair $(\{r,s,t\},\mathcal{N})$ is a special pair since: (i) $r\le s$ and $t=r+s$; (ii) $n_0\ge 1$, $\sum_{i\in\mathcal{I}}n_i=a-1=s-r+1$; (iii) $n_i\equiv 0 \pmod 2$ for $i\in\mathcal{I}\backslash\{0\}$. Moreover, note that: (i) vertex $u_4\in V(T')$ is adjacent to a set of leaves $\mathcal{S}$ with labels $4,\dots,m-3$; (ii) vertices $u_1,u_3\notin\mathcal{S}$, $f(u_3)+f(u_4)=t-1$, and $f(u_1)+f(u_4)=t+1$. Therefore, by Lemma 17, we can safely transfer $n_{t-1}=|U_3|$ leaves from u_4 to u_3 and $n_{t+1}=|U_1|-2$ leaves from u_4 to u_1.

Subcase 2. $|U_1|=0$.

In this subcase, it is sufficient to safely transfer $|U_3|$ leaves from u_4 to u_3. Since $f(u_3)+f(u_4)=(m-2)+2=m$, we move the pairs of leaves with labels in the set $\{3+i,m-3-i\colon 0\le i<|U_3|/2\}$ from u_4 to u_3. Since $(3+i)+(m-3-i)=m$, by Lemma 7, the tree obtained after these transfers is graceful.

Case 3. $|U_1|\equiv 0 \pmod 2$ and $|U_3|\equiv|U_4|\equiv 1 \pmod 2$.

Let $u\in U_3$. As in previous cases, modify T so as to obtain a model-tree $T'=T_5(1,0,1,a+1)$, with $a=|U_1|+|U_3|+|U_4|-1$ and $V(T')=V(T)$, changing the edge set of T as follows: $E(T')=(E(T)\backslash(\{u_1w\colon w\in U_1\}\cup\{u_3w\colon w\in U_3\backslash u\}))\cup\{u_4w\colon w\in U_1\cup U_3\backslash u\}$. Tree T' has a graceful labelling f such that $f(u_2)=0$, as illustrated in Figure 6(c).

Next, we show how to safely transfer $|U_1|$ leaves from u_4 to u_1. Moreover, by construction, exactly one leaf of U_3 is adjacent to vertex u_3 in T'. Hence, we also show how to safely transfer $|U_3|-1$ leaves from u_4 to u_3. Let $m=|E(T')|=|U_1|+|U_3|+|U_4|+5=a+6$. Let r,s,t be three positive integers such that $r=5$, $s=m-2$, and $t=r+s=m+3$. Let $\mathcal{I}=\{0,t-1,t,t+1\}$ be an index set and let $\mathcal{N}=\{n_i\colon i\in\mathcal{I}\}$ such that $n_0=|U_4|$, $n_{t-1}=|U_1|$, $n_t=|U_3|-1$ and $n_{t+1}=0$. The ordered pair $(\{r,s,t\},\mathcal{N})$ is a special pair since: (i) $r\le s$ and $t=r+s$; (ii) $n_0\ge 1$, $\sum_{i\in\mathcal{I}}n_i=a=s-r+1$; and (iii) n_i is even for $i\in\mathcal{I}\backslash\{0\}$. Moreover, note that: (i) the vertex $u_4\in V(T')$ is adjacent to a set of leaves $\mathcal{S}$ with labels $5,\dots,m-2$; (ii) vertices $u_1,u_3\notin\mathcal{S}$, $f(u_1)+f(u_4)=t-1$, and $f(u_3)+f(u_4)=t$. Therefore, by Lemma 17, we can safely transfer $n_{t-1}=|U_1|$ leaves of S from u_4 to u_1 and $n_t=|U_3|-1$ leaves of S from u_4 to u_3. This concludes the proof. □

Theorem 19. *If T is a caterpillar with diameter five, then T is 0-rotatable.*

Proof. The result follows from Lemma 6 and Lemma 18. □

3.2 Caterpillars with diameter six

The main result of this section is Theorem 22, which states that every caterpillar with diameter six is 0-rotatable. The technique used to prove this result is the same used to prove Theorem 19. Accordingly, Lemma 20 and Lemma 21 present auxiliary results needed in the proof of Theorem 22. Furthermore, Figure 7 shows four model-trees of diameter six with graceful labellings f such that $f(u_3)=0$, that are used in Lemma 20.

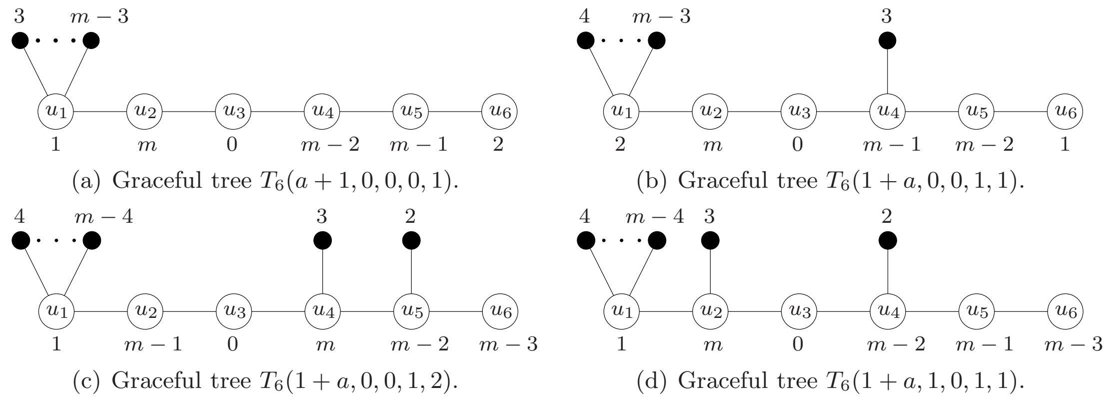


Figure 7: Scheme of four model-trees of diameter six with graceful labellings.

Lemma 20. *Let T be a caterpillar with diameter six, let $v \in V(T)$ be the central vertex of T, and let $w \in \{v\} \cup N_1(v)$. Then, T has a graceful labelling f such that $f(w) = 0$.*

Proof. Let T be a caterpillar with diameter six and spine $P = u_0u_1u_2u_3u_4u_5u_6$. For each $i \in \{1, 2, 3, 4, 5\}$, define U_i as the set of leaves from $V(T)\backslash\{u_0, u_6\}$ that are adjacent to u_i. Note that u_3 is the unique central vertex of T. As shown in the proof of Lemma 18, we can assume $|U_3| = 0$.

In our proof, we consider five cases depending on the parities of the $|U_i|$s. In order to do this, we introduce the following definition: given tree T, we assign T a *5-tuple* $(p_1, p_2, -, p_4, p_5)$ such that, for each $i \in \{1, 2, 4, 5\}$, p_i is the parity of $|U_i|$. Since $p_i \in \{0, 1\}$, there exist 16 distinct 5-tuples.

Case 1. Tree T is assigned one of the following 5-tuples: $(0, 0, -, 0, 0)$, $(1, 0, -, 0, 1)$, $(1, 0, -, 0, 0)$, $(1, 1, -, 1, 0)$.

Let $T' = T_6(a + 1, 0, 0, 0, 1)$, with $a = |U_1| + |U_2| + |U_4| + |U_5|$, $V(T') = V(T)$, and $E(T') = (E(T)\backslash(\{u_2w\colon\ w \in U_2\} \cup \{u_4w\colon w \in U_4\} \cup \{u_5w\colon w \in U_5\})) \cup \{u_1w\colon w \in U_2 \cup U_4 \cup U_5\}$. Thus, T' has a graceful labelling f such that $f(u_3) = 0$, as illustrated in Figure 7(a).

Let $m = |E(T')| = a + 6$, $r = 3$, $s = m - 3$, and $t = r + s = m$. Also, let $\mathcal{I} = \{0, t - 1, t, t + 1\}$ and $\mathcal{N} = \{n_i\colon i \in \mathcal{I}\}$ with $n_0 = |U_1| + 1$, $n_{t-1} = |U_4|$, $n_t = |U_5|$, and $n_{t+1} = |U_2|$. Note that ordered pair $(\{r, s, t\}, \mathcal{N})$ is a special pair since: (i) $r \le s$ and $t = r + s$; (ii) $n_0 \ge 1$ and $\sum_{i\in\mathcal{I}} n_i = a + 1 = s - r + 1$; (iii) if T' is assigned $(0, 0, -, 0, 0)$ or $(1, 0, -, 0, 0)$, then n_i is even for $i \in \mathcal{I}\backslash\{0\}$; and (iv) if T' is assigned $(1, 0, -, 0, 1)$ or $(1, 1, -, 1, 0)$, then $n_{t-1} \equiv n_{t+1} \equiv t \pmod 2$ and $n_t \not\equiv t \pmod 2$. Moreover: (i) vertex $u_1 \in V(T')$ is adjacent to a set $\mathcal{S}$ of leaves with labels $3, \ldots, m-3$; and (ii) $u_2, u_4, u_5 \notin \mathcal{S}$, $f(u_1)+f(u_2) = t+1$, $f(u_1)+f(u_4) = t-1$, and $f(u_1)+f(u_5) = t$. Therefore, by Lemma 17, for $i \in \{2, 4, 5\}$, we can safely transfer $|U_i|$ leaves from u_1 to u_i.

Case 2. Tree T is assigned one of the following 5-tuples: $(1, 0, -, 1, 0)$, $(0, 0, -, 1, 0)$.

Let $v \in U_4$; we modify T so as to obtain another tree $T' = T_6(a + 1, 0, 0, 1, 1)$, with $a = |U_1| + |U_2| + |U_4| + |U_5| - 1$, $V(T') = V(T)$, and $E(T') = (E(T)\backslash(\{u_2w\colon w \in U_2\} \cup \{u_5w\colon w \in U_5\} \cup \{u_4w\colon w \in U_4\backslash v\})) \cup \{u_1w\colon w \in U_2 \cup U_5 \cup U_4\backslash v\}$. Thus, T' has a graceful labelling f such that $f(u_3) = 0$, as illustrated in Figure 7(b).

Let $m = |E(T')| = a + 7$ and r, s, t be three positive integers such that $r = 4$, $s = m - 3$, and $t = r + s = m + 1$. Let $\mathcal{I} = \{0, t - 1, t, t + 1\}$ be an index set and $\mathcal{N} = \{n_i\colon i \in \mathcal{I}\}$ be such that $n_0 = |U_1| + 1$, $n_{t-1} = |U_5|$, $n_t = |U_4| - 1$, and $n_{t+1} = |U_2|$. Note that ordered pair $(\{r, s, t\}, \mathcal{N})$ is a special pair. Moreover: (i) vertex $u_1 \in V(T')$ is adjacent to a set $\mathcal{S}$ of leaves with labels $4, \ldots, m-3$; (ii) $u_2, u_4, u_5 \notin \mathcal{S}$, $f(u_1) + f(u_2) = t+1$, $f(u_1) + f(u_4) = t$, and $f(u_1) + f(u_5) = t - 1$.

Therefore, by Lemma 17, for $i \in \{2,5\}$, we can safely transfer $|U_i|$ leaves from u_1 to u_i and we can also safely transfer $|U_4|-1$ leaves from u_1 to u_4.

Case 3. Tree T is assigned one of the following 5-tuples: $(0,0,-,1,1)$, $(1,1,-,1,1)$, $(1,0,-,1,1)$.

Let $v_4 \in U_4$ and $v_5 \in U_5$. Let $T' = T_6(a+1,0,0,1,2)$, with $a = |U_1|+|U_2|+|U_4|+|U_5|-2$, $V(T') = V(T)$, and $E(T') = (E(T)\backslash(\{u_2w\colon w \in U_2\} \cup \{u_4w\colon w \in U_4\backslash v_4\} \cup \{u_5w\colon w \in U_5\backslash v_5\})) \cup \{u_1w\colon w \in U_2 \cup (U_4\backslash v_4) \cup (U_5\backslash v_5)\}$. Thus, T' has a graceful labelling f such that $f(u_3) = 0$, as illustrated in Figure 7(c).

Let $m = |E(T')| = a+8$, $r = 4$, $s = m-4$, and $t = r+s = m$. Let $\mathcal{I} = \{0, t-1, t, t+1\}$ and $\mathcal{N} = \{n_i\colon i \in \mathcal{I}\}$ with $n_0 = |U_1|+1$, $n_{t-1} = |U_5|-1$, $n_t = |U_2|$, and $n_{t+1} = |U_4|-1$. Then, using the same reasoning of the previous case, one can see that $(\{r,s,t\},\mathcal{N})$ is a special pair. Moreover, vertex $u_1 \in V(T')$ is adjacent to a set $\mathcal{S}$ of leaves with labels $4,\dots,m-4$; $u_2,u_4,u_5 \notin \mathcal{S}$, $f(u_1)+f(u_2) = t$, $f(u_1)+f(u_4) = t+1$, and $f(u_1)+f(u_5) = t-1$. Therefore, by Lemma 17, we can safely transfer $|U_2|$ leaves from u_1 to u_2 and, for $i \in \{4,5\}$, we can safely transfer $|U_i|-1$ leaves from u_1 to u_i.

Case 4. Tree T is assigned the 5-tuple $(0,1,-,1,0)$.

Let $v_2 \in U_2$ and $v_4 \in U_4$. Let $T' = T_6(a+1,1,0,1,1)$, with $a = |U_1|+|U_2|+|U_4|+|U_5|-2$, $V(T') = V(T)$, and $E(T') = (E(T)\backslash(\{u_5w\colon w \in U_5\} \cup \{u_2w\colon w \in U_2\backslash v_2\} \cup \{u_4w\colon w \in U_4\backslash v_4\})) \cup \{u_1w\colon w \in U_5 \cup (U_2\backslash v_2) \cup (U_4\backslash v_4)\}$. Thus, T' has a graceful labelling f such that $f(u_3) = 0$, as illustrated in Figure 7(d).

Let $m = |E(T')| = a+8$, $r = 4$, $s = m-4$, and $t = r+s = m$. Let $\mathcal{I} = \{0, t-1, t, t+1\}$ and $\mathcal{N} = \{n_i\colon i \in \mathcal{I}\}$, with $n_0 = |U_1|+1$, $n_{t-1} = |U_4|-1$, $n_t = |U_5|$, and $n_{t+1} = |U_2|-1$. Note that $(\{r,s,t\},\mathcal{N})$ is a special pair. Moreover, vertex $u_1 \in V(T')$ is adjacent to a set $\mathcal{S}$ of leaves with labels $4,\dots,m-4$; and $u_2,u_4,u_5 \notin \mathcal{S}$, $f(u_1)+f(u_2) = t+1$, $f(u_1)+f(u_4) = t-1$, and $f(u_1)+f(u_5) = t$. Therefore, by Lemma 17, we can safely transfer $|U_5|$ leaves from u_1 to u_5 and, for $i \in \{2,4\}$, we can safely transfer $|U_i|-1$ leaves from u_1 to u_i.

Case 5. Tree T is assigned one of the following 5-tuples: $(0,0,-,0,1)$, $(0,1,-,1,1)$, $(0,1,-,0,0)$, $(0,1,-,0,1)$, $(1,1,-,0,0)$, $(1,1,-,0,1)$.

For a,b,c,d non-negative integers, tree $T_6(a,b,0,c,d)$ is isomorphic to $T_6(d,c,0,b,a)$. Thus, the trees in this case are isomorphic to trees treated in Case 1, Case 2, and Case 3, and the result follows, concluding the proof. □

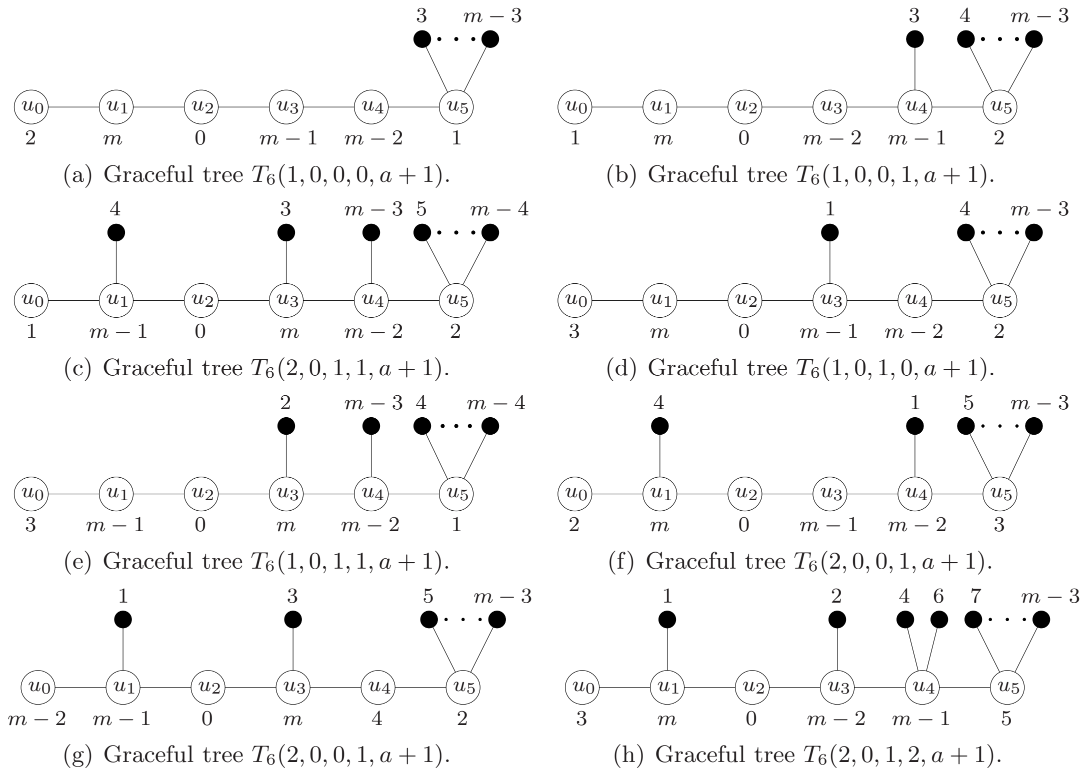
For the next lemma, consider the eight model-trees exhibited in Figure 8, each of which with a graceful labelling f with $f(u_2) = 0$.

Lemma 21. *Let T be a caterpillar with diameter six and spine $P = u_0u_1u_2u_3u_4u_5u_6$. Also, let $w \in \{u_2,u_4\} \cup N_1(u_2) \cup N_1(u_4)$. Then, T has a graceful labelling f such that $f(w) = 0$.*

Proof. Let T be a caterpillar with diameter six and let $P = u_0u_1u_2u_3u_4u_5u_6$ be its spine. For each $i \in \{1,2,3,4,5\}$, define U_i as the set of leaves from $V(T)\backslash\{u_0,u_6\}$ that are adjacent to u_i. We prove the result for u_2 and the proof for u_4 is analogous. As shown in the proof of Lemma 18, we can assume $U_2 = \emptyset$.

In our proof, we consider eight cases depending on the parities of the $|U_i|$s. In order to do this, we assign T a *5-tuple* $(p_1,-,p_3,p_4,p_5)$ such that, for each $i \in \{1,3,4,5\}$, p_i is the parity of $|U_i|$.

Case 1. Tree T is assigned one of the following 5-tuples: $(0,-,0,0,0)$, $(0,-,0,0,1)$, $(0,-,1,0,1)$, $(1,-,0,1,1)$.



(a) Graceful tree $T_6(1,0,0,0,a+1)$. (b) Graceful tree $T_6(1,0,0,1,a+1)$. (c) Graceful tree $T_6(2,0,1,1,a+1)$. (d) Graceful tree $T_6(1,0,1,0,a+1)$. (e) Graceful tree $T_6(1,0,1,1,a+1)$. (f) Graceful tree $T_6(2,0,0,1,a+1)$. (g) Graceful tree $T_6(2,0,0,1,a+1)$. (h) Graceful tree $T_6(2,0,1,2,a+1)$.

Figure 8: Scheme of eight model-trees with a graceful labelling.

In this case, modify T so as to obtain another tree $T' = T_6(1,0,0,0,a+1)$, with $a = |U_1| + |U_3| + |U_4| + |U_5|$, $V(T') = V(T)$, and $E(T') = (E(T)\backslash\{u_i w\colon i \in \{1,3,4\}, w \in U_i\}) \cup \{u_5 w\colon w \in U_1 \cup U_3 \cup U_4\}$. Thus, as illustrated in Figure 8(a), T' has a graceful labelling f such that $f(u_2) = 0$. Next, we show how to safely transfer $|U_i|$ leaves from u_5 to u_i for $i \in \{1,3,4\}$.

Let $m = |E(T')| = a+6$, $r = 3$, $s = m-3$, and $t = r+s = m$. Let $\mathcal{I} = \{0, t-1, t, t+1\}$ and $\mathcal{N} = \{n_i\colon i \in \mathcal{I}\}$ with $n_0 = |U_5|+1$, $n_{t-1} = |U_4|$, $n_t = |U_3|$, and $n_{t+1} = |U_1|$. Note that $(\{r,s,t\}, \mathcal{N})$ is a special pair since: (i) $r \leq s$ and $t = r+s$; (ii) $n_0 \geq 1$ and $\sum_{i\in\mathcal{I}} n_i = a+1 = s-r+1$; (iii) if T' is assigned $(0,-,0,0,0)$ or $(0,-,0,0,1)$, then n_i is even for $i \in \mathcal{I}\backslash\{0\}$; and (iv) if T' is assigned $(0,-,1,0,1)$ or $(1,-,0,1,1)$, then $n_{t-1} \equiv n_{t+1} \equiv t \pmod 2$ and $n_t \not\equiv t \pmod 2$. Moreover, vertex $u_5 \in V(T')$ is adjacent to a set of leaves $\mathcal{S}$ with labels $3, \ldots, m-3$, $u_1, u_3, u_4 \notin \mathcal{S}$, $f(u_1) + f(u_5) = t+1$, $f(u_3) + f(u_5) = t$, and $f(u_4) + f(u_5) = t-1$. Therefore, by Lemma 17, we can safely transfer $|U_i|$ leaves from u_5 to u_i, for $i \in \{1,3,4\}$.

Case 2. Tree T is assigned one of the following 5-tuples: $(0,-,0,1,0)$, $(0,-,0,1,1)$, $(1,-,1,1,1)$.

Let $v_4 \in U_4$ and $T' = T_6(1,0,0,1,a+1)$, with $a = |U_1| + |U_3| + |U_4| + |U_5| - 1$, $V(T') = V(T)$, and $E(T') = (E(T)\backslash\{u_i w\colon i \in \{1,3\}, w \in U_i\} \cup \{u_4 w\colon w \in U_4\backslash v_4\}) \cup \{u_5 w\colon w \in U_1 \cup U_3 \cup U_4\backslash v_4\}$. Thus, T' has a graceful labelling f such that $f(u_2) = 0$, as illustrated in Figure 8(b).

Let $m = |E(T')| = a+7$, $r = 4$, $s = m-3$, and $t = r+s = m+1$. Let $\mathcal{I} = \{0, t-1, t, t+1\}$ and $\mathcal{N} = \{n_i\colon i \in \mathcal{I}\}$ with $n_0 = |U_5|+1$, $n_{t-1} = |U_3|$, $n_t = |U_4|-1$, and $n_{t+1} = |U_1|$. Note that $(\{r,s,t\}, \mathcal{N})$ is a special pair since: (i) $r \leq s$ and $t = r+s$; (ii) $n_0 \geq 1$ and $\sum_{i\in\mathcal{I}} n_i = a+1 = s-r+1$; (iii) if T' is assigned $(0,-,0,1,0)$ or $(0,-,0,1,1)$, then n_i is even for $i \in \mathcal{I}\backslash\{0\}$; and (iv) if T' is assigned $(1,-,1,1,1)$, then $n_{t-1} \equiv n_{t+1} \equiv t \pmod 2$ and $n_t \not\equiv t \pmod 2$. Moreover, vertex $u_5 \in V(T')$ is adjacent to a set of leaves $\mathcal{S}$ with labels $4, \ldots, m-3$, $u_1, u_3, u_4 \notin \mathcal{S}$, $f(u_1) + f(u_5) = t+1$,

$f(u_3)+f(u_5)=t-1$, and $f(u_4)+f(u_5)=t$. Therefore, by Lemma 17, for $i\in\{1,3\}$, we can safely transfer $|U_i|$ leaves from u_5 to u_i and we can also safely transfer $|U_4|-1$ leaves from u_5 to u_4.

Case 3. Tree T is assigned the 5-tuple $(1,-,1,1,0)$.

Let $v_1\in U_1$, $v_3\in U_3$, and $v_4\in U_4$. Modify T so as to obtain $T'=T_6(2,0,1,1,a+1)$, with $a=|U_1|+|U_3|+|U_4|+|U_5|-3$, $V(T')=V(T)$, and $E(T')=(E(T)\backslash(\{u_1w\colon w\in U_1\backslash v_1\}\cup\{u_3w\colon w\in U_3\backslash v_3\}\cup\{u_4w\colon w\in U_4\backslash v_4\}))\cup\{u_5w\colon w\in(U_1\backslash v_1)\cup(U_3\backslash v_3)\cup(U_4\backslash v_4)\}$. Thus, T' has a graceful labelling f such that $f(u_2)=0$, as illustrated in Figure 8(c).

Let $m=|E(T')|=a+9$, $r=5$, $s=m-4$, and $t=r+s=m+1$. Let $\mathcal{I}=\{0,t-1,t,t+1\}$ and $\mathcal{N}=\{n_i\colon i\in\mathcal{I}\}$ with $n_0=|U_5|+1$, $n_{t-1}=|U_4|-1$, $n_t=|U_1|-1$, and $n_{t+1}=|U_3|-1$. Note that $(\{r,s,t\},\mathcal{N})$ is a special pair. Moreover, vertex $u_5\in V(T')$ is adjacent to a set of leaves $\mathcal{S}$ with labels $5,\ldots,m-4$, $u_1,u_3,u_4\notin\mathcal{S}$, $f(u_1)+f(u_5)=t$, $f(u_3)+f(u_5)=t+1$, and $f(u_4)+f(u_5)=t-1$. Therefore, by Lemma 17, we can safely transfer $|U_i|-1$ leaves from u_5 to u_i, $i\in\{1,3,4\}$.

Case 4. Tree T is assigned the 5-tuple $(0,-,1,0,0)$.

Let $v_3\in U_3$. Consider $T'=T_6(1,0,1,0,a+1)$, with $a=|U_1|+|U_3|+|U_4|+|U_5|-1$, $V(T')=V(T)$, and $E(T')=(E(T)\backslash(\{u_1w\colon w\in U_1\}\cup\{u_3w\colon w\in U_3\backslash v_3\}\cup\{u_4w\colon w\in U_4\}))\cup\{u_5w\colon w\in U_1\cup(U_3\backslash v_3)\cup U_4\}$. Thus, T' has a graceful labelling f such that $f(u_2)=0$, as illustrated in Figure 8(d).

Let $m=|E(T')|=a+7$, $r=4$, $s=m-3$, and $t=r+s=m+1$. Let $\mathcal{I}=\{0,t-1,t,t+1\}$ and $\mathcal{N}=\{n_i\colon i\in\mathcal{I}\}$ with $n_0=|U_5|+1$, $n_{t-1}=|U_4|$, $n_t=|U_3|-1$, and $n_{t+1}=|U_1|$. Note that $(\{r,s,t\},\mathcal{N})$ is a special pair. Moreover, vertex $u_5\in V(T')$ is adjacent to a set of leaves $\mathcal{S}$ with labels $4,\ldots,m-3$, $u_1,u_3,u_4\notin\mathcal{S}$, $f(u_1)+f(u_5)=t+1$, $f(u_3)+f(u_5)=t$, and $f(u_4)+f(u_5)=t-1$. Therefore, by Lemma 17, for $i\in\{1,4\}$, we can safely transfer $|U_i|$ leaves from u_5 to u_i and we can also safely transfer $|U_3|-1$ leaves from u_5 to u_3.

Case 5. Tree T is assigned one of the following 5-tuples: $(0,-,1,1,0)$, $(0,-,1,1,1)$.

Let $v_3\in U_3$ and $v_4\in U_4$. Consider $T'=T_6(1,0,1,1,a+1)$, with $a=|U_1|+|U_3|+|U_4|+|U_5|-2$, $V(T')=V(T)$, and $E(T')=(E(T)\backslash(\{u_1w\colon w\in U_1\}\cup\{u_3w\colon w\in U_3\backslash v_3\}\cup\{u_4w\colon w\in U_4\backslash v_4\}))\cup\{u_5w\colon w\in U_1\cup(U_3\backslash v_3)\cup(U_4\backslash v_4)\}$. Thus, T' has a graceful labelling f such that $f(u_2)=0$, as illustrated in Figure 8(e).

Let $m=|E(T')|=a+8$, $r=4$, $s=m-4$, and $t=r+s=m$. Let $\mathcal{I}=\{0,t-1,t,t+1\}$ and $\mathcal{N}=\{n_i\colon i\in\mathcal{I}\}$ with $n_0=|U_5|+1$, $n_{t-1}=|U_4|-1$, $n_t=|U_1|$, and $n_{t+1}=|U_3|-1$. Note that $(\{r,s,t\},\mathcal{N})$ is a special pair. Moreover, vertex $u_5\in V(T')$ is adjacent to a set of leaves $\mathcal{S}$ with labels $4,\ldots,m-4$, $u_1,u_3,u_4\notin\mathcal{S}$, $f(u_1)+f(u_5)=t$, $f(u_3)+f(u_5)=t+1$, and $f(u_4)+f(u_5)=t-1$. Therefore, by Lemma 17, for $i\in\{3,4\}$, we can safely transfer $|U_i|-1$ leaves from u_5 to u_i, and we can also safely transfer $|U_1|$ leaves from u_5 to u_1.

Case 6. Tree T is assigned the 5-tuple $(1,-,0,1,0)$.

Let $v_1\in U_1$ and $v_4\in U_4$. Let $T'=T_6(2,0,0,1,a+1)$, with $a=|U_1|+|U_3|+|U_4|+|U_5|-2$, $V(T')=V(T)$, and $E(T')=(E(T)\backslash(\{u_1w\colon w\in U_1\backslash v_1\}\cup\{u_3w\colon w\in U_3\}\cup\{u_4w\colon w\in U_4\backslash v_4\}))\cup\{u_5w\colon w\in(U_1\backslash v_1)\cup U_3\cup(U_4\backslash v_4)\}$. Thus, T' has a graceful labelling f such that $f(u_2)=0$, as illustrated in Figure 8(f).

Let $m=|E(T')|=a+8$, $r=5$, $s=m-3$, and $t=r+s=m+2$. Let $\mathcal{I}=\{0,t-1,t,t+1\}$ and $\mathcal{N}=\{n_i\colon i\in\mathcal{I}\}$ with $n_0=|U_5|+1$, $n_{t-1}=|U_4|-1$, $n_t=|U_3|$, and $n_{t+1}=|U_1|-1$. Note that $(\{r,s,t\},\mathcal{N})$ is a special pair. Moreover, vertex $u_5\in V(T')$ is adjacent to a set of leaves $\mathcal{S}$ with labels $5,\ldots,m-3$, $u_1,u_3,u_4\notin\mathcal{S}$, $f(u_1)+f(u_5)=t+1$, $f(u_3)+f(u_5)=t$, and $f(u_4)+f(u_5)=t-1$. Therefore, by Lemma 17, for $i\in\{1,4\}$, we can safely transfer $|U_i|-1$ leaves from u_5 to u_i and we can also safely transfer $|U_3|$ leaves from u_5 to u_3.

Case 7. Tree T is assigned one of the following 5-tuples: $(1, -, 1, 0, 0)$, $(1, -, 1, 0, 1)$.

Subcase 1. $|U_4| = 0$.

Let $v_1 \in U_1$ and $v_3 \in U_3$. Modify T so as to obtain $T' = T_6(2, 0, 1, 0, a+1)$, with $a = |U_1| + |U_3| + |U_4| + |U_5| - 2$, $V(T') = V(T)$, and $E(T') = (E(T) \backslash (\{u_1 w \colon w \in U_1 \backslash v_1\} \cup \{u_3 w \colon w \in U_3 \backslash v_3\})) \cup \{u_5 w \colon w \in (U_1 \backslash v_1) \cup (U_3 \backslash v_3)\}$. Thus, T' has a graceful labelling f such that $f(u_2) = 0$, as illustrated in Figure 8(g).

Let $m = |E(T')| = a + 8$, $r = 5$, $s = m - 3$, and $t = r + s = m + 2$. Let $\mathcal{I} = \{0, t-1, t, t+1\}$ and $\mathcal{N} = \{n_i \colon i \in \mathcal{I}\}$ with $n_0 = |U_5| + 1$, $n_{t-1} = |U_1| - 1$, $n_t = |U_3| - 1$, and $n_{t+1} = 0$. Note that $(\{r, s, t\}, \mathcal{N})$ is a special pair. Moreover, vertex $u_5 \in V(T')$ is adjacent to a set of leaves $\mathcal{S}$ with labels $5, \ldots, m-3$, $u_1, u_3 \notin \mathcal{S}$, $f(u_1) + f(u_5) = t - 1$ and $f(u_3) + f(u_5) = t$. Therefore, by Lemma 17, we can safely transfer $|U_i| - 1$ leaves from u_5 to u_i, $i \in \{1, 3\}$.

Subcase 2. $|U_4| \geq 2$.

Let $v_1 \in U_1$, $v_3 \in U_3$, and $v_4^1, v_4^2 \in U_4$. Consider $T' = T_6(2, 0, 1, 2, a+1)$, with $a = |U_1| + |U_3| + |U_4| + |U_5| - 4$, $V(T') = V(T)$, and $E(T') = (E(T) \backslash (\{u_1 w \colon w \in U_1 \backslash v_1\} \cup \{u_3 w \colon w \in U_3 \backslash v_3\} \cup \{u_4 w \colon w \in U_4 \backslash \{v_4^1, v_4^2\}\})) \cup \{u_5 w \colon w \in (U_1 \backslash v_1) \cup (U_3 \backslash v_3) \cup (U_4 \backslash \{v_4^1, v_4^2\})\}$. Thus, T' has a graceful labelling f such that $f(u_2) = 0$, as illustrated in Figure 8(h).

Let $m = |E(T')|$, $r = 7$, $s = m - 3$, and $t = r + s = m + 4$. Let $\mathcal{I} = \{0, t-1, t, t+1\}$ and $\mathcal{N} = \{n_i \colon i \in \mathcal{I}\}$ with $n_0 = |U_5| + 1$, $n_{t-1} = |U_3| - 1$, $n_t = |U_4| - 2$, and $n_{t+1} = |U_1| - 1$. Note that $(\{r, s, t\}, \mathcal{N})$ is a special pair. Moreover, vertex $u_5 \in V(T')$ is adjacent to a set of leaves $\mathcal{S}$ with labels $7, \ldots, m-3$, $u_1, u_3, u_4 \notin \mathcal{S}$, $f(u_1) + f(u_5) = t+1$, $f(u_3) + f(u_5) = t-1$, and $f(u_4) + f(u_5) = t$. Therefore, by Lemma 17, for $i \in \{1, 3\}$, we can safely transfer $|U_i| - 1$ leaves from u_5 to u_i and we can also safely transfer $|U_4| - 2$ leaves from u_5 to u_4.

Case 8. Tree T is assigned one of the following 5-tuples: $(1, -, 0, 0, 0)$, $(1, -, 0, 0, 1)$.

Let T be as in the hypothesis. We modify T in order to obtain another tree T', with $V(T') = V(T)$ and $E(T') = (E(T) \backslash (\{u_i w \colon i \in \{1, 4, 5\}, w \in U_i \cup \{u_0, u_6\}\})) \cup (u_3 w \colon w \in U_1 \cup U_4 \cup U_5 \cup \{u_0, u_6\})$. Figure 9 shows a scheme of T' with a graceful labelling f such that $f(u_2) = 0$. Note that u_3 is adjacent to exactly $|U_1| + |U_3| + |U_4| + |U_5| + 2 = m - 4$ leaves. Next, we show how to perform a sequence of transfers in T' so as to obtain a graceful labelling for T.

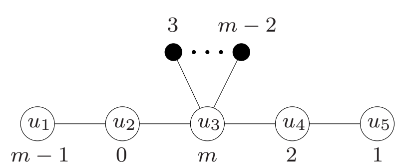


Figure 9: Scheme of tree T' with a graceful labelling.

Since $f(u_3) + f(u_5) = m + 1$ and vertex u_3 is adjacent to vertices with labels $3, \ldots, m-2$, by Lemma 7, we can safely transfer all the $m - 4 - |U_3|$ leaves with labels in the interval $[3 + \frac{|U_3|}{2}, m - 2 - \frac{|U_3|}{2}]$ from vertex u_3 to vertex u_5. After this transfer, u_3 is adjacent to $|U_3|$ leaves and u_5 is adjacent to exactly $|U_1| + |U_4| + |U_5| + 2$ leaves.

Now, consider $a = f(u_5) = 1$, $b = f(u_1) = m - 1$, $r_1 = 1$, $r_2 = 0$, and the set $\mathcal{S}$ of leaves adjacent to u_5 with labels $3 + \frac{|U_3|}{2}, \ldots, m - 2 - \frac{|U_3|}{2}$. By Lemma 8, it is possible to perform a sequence of safe transfers $u_5 \to u_1 \to u_4$, such that the resulting tree has $|U_5| + 1$ leaves at vertex u_5, $|U_1| + 1$ leaves at vertex u_1, and $|U_4|$ leaves at vertex u_4. This concludes the proof. $\square$

Theorem 22. *If T is a caterpillar with diameter six, then T is 0-rotatable.*

Proof. The result follows from Lemma 6, Lemma 20 and Lemma 21. □

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