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Abstract

In this paper we introduce a superclass of split digraphs, which we call **spine digraphs**. Those are the digraphs D whose vertex set can be partitioned into two sets X and Y such that the subdigraph induced by X is traceable and Y is a stable set. We also show that Linial's Conjecture holds for spine digraphs.

1 Introduction

The digraphs considered in this text do not contain loops or parallel arcs (but may contain cycles of length two). Let D be a digraph. We denote the set of vertices of D by $V(D)$ and the set of arcs of D by $A(D)$. We use (u, v) to denote an arc with **head** v and **tail** u. We say that u and v are **adjacent** if $(u, v) \in A(D)$ or $(v, u) \in A(D)$. By a path of D, we mean a directed path of D and by a stable set of D, we mean a stable set of the underlying graph of D. We denote by $V(P)$ the set of vertices of a path P and the **size** of a path P, denoted by $|P|$, is $|V(P)|$[1]. We denote by $\lambda(D)$ the size of the longest path in D and by $\alpha(D)$ the size of a maximum stable set. A **path partition** of D is a set of vertex-disjoint paths of D that cover $V(D)$. We say that $\mathcal{P}$ is an **optimal** path partition of D if there is no path partition $\mathcal{P}'$ of D such that $|\mathcal{P}'| < |\mathcal{P}|$. We denote by $\pi(D)$ the size of an optimal path partition of a digraph D.

Dilworth [1] showed that for every transitive acyclic digraph D we have $\pi(D) = \alpha(D)$. Note that this equality is not valid for every digraph; for example, if D is a directed cycle with 5 vertices, then $\pi(D) = 1$ and $\alpha(D) = 2$. However, Gallai and Milgram [2] have shown that $\pi(D) \leq \alpha(D)$ for every digraph D.

Greene and Kleitman [3] proved a generalization of Dilworth's Theorem, which we describe next. Let k be a positive integer. The ***k*-norm** of a path partition $\mathcal{P}$, denoted by $|\mathcal{P}|_k$, is defined as $|\mathcal{P}|_k = \sum_{P \in \mathcal{P}} \min\{|P|, k\}$. We say that $\mathcal{P}$ is a ***k*-optimal path partition** of D if there is no path partition $\mathcal{P}'$ such that $|\mathcal{P}'|_k < |\mathcal{P}|_k$. We denote by $\pi_k(D)$ the k-norm of a k-optimal path partition of D. A ***k*-partial coloring** $\mathcal{C}^k$ is a set of k

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[1] Usually $|P|$ denotes the length of a path (number of arcs), but here it denotes the number of vertices.

disjoint stable sets called **color classes** (empty color classes are allowed). The **weight** of a k-partial coloring $\mathcal{C}^k$, denoted by $||\mathcal{C}^k||$, is defined as $||\mathcal{C}^k|| = \sum_{C \in \mathcal{C}^k} |C|$. We say that $\mathcal{C}^k$ is an **optimal *k*-partial coloring** of D if there is no k-partial coloring $\mathcal{B}^k$ such that $||\mathcal{B}^k|| > ||\mathcal{C}^k||$. We denote by $\alpha_k(D)$ the weight of an optimal k-partial coloring of D. Given these definitions, what Greene and Kleitman [3] showed was that for every transitive acyclic digraph D, we have $\pi_k(D) = \alpha_k(D)$. Note that $\pi(D) = \pi_1(D)$ and $\alpha(D) = \alpha_1(D)$. Thus, Dilworth's Theorem is a particular case of Greene-Kleitman's Theorem in which $k = 1$.

As Gallai-Milgram's Theorem extends Dilworth's Theorem, it is a natural question whether Greene-Kleitman's Theorem can be extended to digraphs in general. More precisely, is it true that for every digraph D we have that $\pi_k(D) \leq \alpha_k(D)$? Linial [4] conjectured that the answer for this question is positive.

Linial's Conjecture [4]. *Let D be a digraph and k be a positive integer. Then, $\pi_k(D) \leq \alpha_k(D)$.*

Linial's Conjecture remains open, but we know it holds for acyclic digraphs [4], bipartite digraphs [5], digraphs which contain a Hamiltonian path [5], $k = 1$ [6], $k = 2$ [7] and $k \geq \lambda(D) - 3$ [8]. For more about this problem, we refer you to the survey presented by Hartman [9].

Linial also introduced a somewhat dual problem, which we are going to call as **Linial's Dual Conjecture**, in which the roles of paths and stable sets are exchanged. To properly state that, we need a few definitions first. Let D be a digraph and k a positive integer. A **k-path** in D is a set of k disjoint paths of D (we allow empty paths). The **weight** of a k-path $\mathcal{P}^k$, denoted by $||\mathcal{P}^k||$, is defined as $||\mathcal{P}^k|| = \sum_{P \in \mathcal{P}^k} |P|$. We say that $\mathcal{P}^k$ is an **optimal k-path** of D if there is no k-path $\mathcal{Q}^k$ of D such that $||\mathcal{Q}^k|| > ||\mathcal{P}^k||$. We denote by $\lambda_k(D)$ the weight of an optimal k-path of D. A **coloring** of D is a partition of $V(D)$ into stable sets. The **k-norm** of a coloring $\mathcal{C} = \{C_1, \ldots, C_t\}$, denoted by $|\mathcal{C}|_k$, is defined as $|\mathcal{C}|_k = \sum_{C \in \mathcal{C}} \min\{|C|, k\}$. We say that $\mathcal{C}$ is a **k-optimal coloring** of D if there is no coloring $\mathcal{C}'$ of D such that $|\mathcal{C}'|_k < |\mathcal{C}|_k$. We denote by $\chi_k(D)$ the k-norm of a k-optimal coloring of D.

Linial's Dual Conjecture [4]. *Let D be a digraph and k be a positive integer. Then, $\chi_k(D) \leq \lambda_k(D)$.*

This conjecture also remains open and, like Linial's Conjecture, we know it holds for some particular cases, such as acyclic digraphs [10], bipartite digraphs [11], $k = 1$ [12, 13], $k \geq \pi(D)$ (trivial, since $\lambda_k(D) = |V(D)|$), and split digraphs [11], which we define next.

Recall that our digraphs may have no loops nor parallel arcs. A **semi-complete digraph** is a digraph D such that for every pair of distinct vertices u, v, $(u, v) \in A(D)$ or $(v, u) \in A(D)$ or both. A **tournament** is a digraph D such that for every pair of distinct vertices u, v, either $(u, v) \in A(D)$ or $(v, u) \in A(D)$. Rédei [14] proved that every tournament (and hence, every semi-complete digraph) is **traceable** (i. e. contains a Hamiltonian path).

For a digraph D and $X \subseteq V(D)$, we denote by $D[X]$ the subdigraph of D induced by X. A digraph D is a **split digraph** if there exists a partition $\{X, Y\}$ of D such that $D[X]$ is a semi-complete digraph and Y is a stable set of D.

Hartman, Saleh and Hershkowitz [11] proved that $\chi_k(D) \leq \lambda_k(D)$ (Linial's Dual Conjecture) for every split digraph. In fact, their proof can be extended to a superclass of split digraphs which we introduce next. We say that D is a **spine digraph** if there exists a partition $\{X, Y\}$ of $V(D)$ such that $D[X]$ is traceable and Y is a stable set in D. In this paper we prove Linial's Conjecture for spine digraphs. We shall use the notation $D[X, Y]$ to indicate that D is a spine digraph with such partition $\{X, Y\}$.

2 Linial's conjecture for spine digraphs

First let us discuss the general idea of the proof of Hartman, Saleh and Hershkowitz [11] that $\chi_k(D) \leq \lambda_k(D)$ for every spine digraph $D[X, Y]$. They first showed that $\chi_k(D) \leq |X| + k$ and $\lambda_k(D) \geq |X| + k - 1$ by exhibiting appropriate coloring and k-path. If $\chi_k(D) \leq |X| + k - 1$, then the result follows. Therefore, the critical case is when $\chi_k(D) = |X| + k$. In this case, they showed that $\lambda_k(D) \geq |X| + k$ by constructing a k-path with such weight.

We follow the same strategy. However, here the critical case (described later) is more complicated. We begin by presenting simple bounds for $\pi_k(D)$ and $\alpha_k(D)$.

Lemma 1. *Let $D[X, Y]$ be a spine digraph. Then, $\pi_k(D) \leq |Y| + \min\{|X|, k\}$.*

Proof. Let P be a Hamiltonian path in $D[X]$ and $\mathcal{P} = \{P\} \cup \{(y) : y \in Y\}$. Clearly, $\mathcal{P}$ is a path partition of D for which $|\mathcal{P}|_k = \min\{|X|, k\} + |Y|$. Therefore, $\pi_k(D) \leq |\mathcal{P}|_k = \min\{|X|, k\} + |Y|$. ■

Lemma 2. *Let $D[X, Y]$ be a spine digraph. Then, $\alpha_k(D) \geq |Y| + \min\{|X|, k-1\}$. Moreover, if $|X| < k$, then $\alpha_k(D) = |V(D)|$.*

Proof. First, suppose that $|X| < k$. Let $\mathcal{C}^k = \{Y\} \cup \{\{x\} : x \in X\}$. Note that $\mathcal{C}^k$ is a k-partial coloring of D with $||\mathcal{C}^k|| = |V(D)|$. Therefore, $\alpha_k(D) = ||\mathcal{C}^k|| = |Y| + |X| = |Y| + \min\{|X|, k-1\}$ and the result follows. Thus assume that $|X| \geq k$. Take $S \subset X$ such that $|S| = k - 1$, and let $\mathcal{C}^k = \{Y\} \cup \{\{x\} : x \in S\}$. Clearly, $\mathcal{C}^k$ is a k-partial coloring for which $||\mathcal{C}^k|| = |Y| + k - 1$. Therefore, $\alpha_k(D) \geq ||\mathcal{C}^k|| = |Y| + k - 1 = |Y| + \min\{|X|, k-1\}$. ■

A spine digraph $D[X, Y]$ is ***k*-loose** if either $|X| < k$ or there is a set $S \subseteq X$ such that $|S| = k$ and no vertex $y \in Y$ is adjacent to every vertex in S. A spine digraph $D[X, Y]$ that is not k-loose is called ***k*-tight**.

Lemma 3. *If $D[X, Y]$ is a k-loose spine digraph, then $\alpha_k(D) \geq |Y| + \min\{|X|, k\}$.*

Proof. If $|X| < k$, then by Lemma 2, $\alpha_k(D) = |V(D)| = |Y| + |X| = |Y| + \min\{|X|, k\}$. We may thus assume that $|X| \geq k$. So, there exists $S \subseteq X$ such that $|S| = k$ and no vertex $y \in Y$ is adjacent to every vertex in S. Suppose that $S = \{x_1, x_2, \ldots, x_k\}$ and let $\mathcal{C}_0^k = \{C_1, C_2, \ldots, C_k\}$ be a k-partial coloring in which $C_i = \{x_i\}$ for $i = 1, 2, \ldots, k$. For each $y \in Y$, choose some vertex x_i not adjacent to y (which exists by the choice of S) and add y in color class C_i. The k-partial coloring $\mathcal{C}^k$ thus obtained has weight $|Y| + k = |Y| + \min\{|X|, k\}$. Therefore, $\alpha_k(D) \geq ||\mathcal{C}^k|| = |Y| + \min\{|X|, k\}$. ■

Theorem 1. *Let $D[X,Y]$ be a k-loose spine digraph. Then, $\pi_k(D) \leq \alpha_k(D)$.*

Proof. By Lemma 3, $\alpha_k(D) \geq |Y| + \min\{|X|, k\}$. On the other hand, by Lemma 1, $\pi_k(D) \leq |Y| + \min\{|X|, k\}$ and the result follows. ■

Lemma 4. *Let $D[X,Y]$ be a spine digraph such that $\lambda(D) > |X|$. Then, $\pi_k(D) \leq \alpha_k(D)$.*

Proof. If $\alpha_k(D) = |V(D)|$, then the result follows trivially. Thus, we may assume that $\alpha_k(D) < |V(D)|$. By Lemma 2, we have that $|X| \geq k$ and also that $\alpha_k(D) \geq |Y| + \min\{|X|, k-1\} = |Y| + k - 1$. Since $\lambda(D) > |X|$, there exists a path P in D such that $|P| = |X| + 1$. Let $\mathcal{P} = \{P\} \cup \{(v) : v \notin V(P)\}$. Clearly, $\mathcal{P}$ is a path partition of D and $|\mathcal{P}|_k = \min\{|P|, k\} + |Y| - 1 = |Y| + k - 1$. Therefore, $\pi_k(D) \leq |\mathcal{P}|_k = |Y| + k - 1 \leq \alpha_k(D)$. ■

In view of the two preceding results, in order to complete the proof of Linial's Conjecture for spine digraphs, we must deal with the case in which D is k-tight and $\lambda(D) \leq |X|$. To do so, we present two auxiliary lemmas; but first, we need some definitions.

Given a path $P = (x_1, x_2, \ldots, x_\ell)$, we denote by $ter(P)$ the terminal vertex of P, namely x_ℓ. The subpath $(x_1, x_2, \ldots, x_i)$ is denoted by Px_i and the subpath $(x_i, x_{i+1}, \ldots, x_\ell)$ is denoted by x_iP. We denote by $W \circ Q$ the concatenation of two paths W and Q.

Let $D[X,Y]$ be a spine digraph and let $P = (x_1, x_2, \ldots, x_\ell)$ be a Hamiltonian path of $D[X]$. We say that the Hamiltonian path P is **zigzag-free** in D if there is no vertex $y \in Y$ such that $(y, x_1) \in A(D)$, or $(x_\ell, y) \in A(D)$, or $(x_i, y) \in A(D)$ and $(y, x_{i+1}) \in A(D)$.

Lemma 5. *Let $D[X,Y]$ be a spine digraph, let $P = (x_1, x_2, \ldots, x_\ell)$ be a Hamiltonian zigzag-free path of $D[X]$ and let $y \in Y$ be a vertex adjacent to the first t vertices of P. Then $(x_i, y) \in A(D)$ for $i = 1, 2, \ldots, t$.*

Proof. The proof is by induction on t. If $t = 1$, then the result is obvious. Now, suppose that $t > 1$. By induction hypothesis, we have that $(x_i, y) \in A(D)$ for $i = 1, 2 \ldots, t-1$. If $(y, x_t) \in A(D)$, then P is not zigzag-free in D. Hence, $(x_t, y) \in A(D)$ and the result follows. ■

Lemma 6. *Let $D[X,Y]$ be a k-tight spine digraph and let $P = (x_1, x_2, \ldots, x_\ell)$ be a Hamiltonian zigzag-free path of $D[X]$. Then, there exist paths P_1 and P_2 such that:*

(i) $V(P_1) \cap V(P_2) = \varnothing$;

(ii) $|P_1| + |P_2| = |X| + k + 1$;

(iii) $ter(P_1) \cup ter(P_2) = \{x_\ell, y\}$, *for some* $y \in Y$;

(iv) $X \subseteq V(P_1) \cup V(P_2)$.

Proof. The proof is by induction on k. Suppose that $k = 1$. Since D is 1-tight, we know that every $x_i \in X$ is adjacent to at least one vertex in Y. Let $y' \in Y$ be a vertex adjacent to x_1. Since P is zigzag-free in D, we have that $(x_1, y') \in A(D)$. Among all arcs $(x_i, y) \in A(D)$ with $y \in Y$ and $1 \leq i \leq \ell$, choose an arc a such that i is maximum. Since $(x_1, y') \in A(D)$, one such arc exists. As P is zigzag-free in D, we have that $i < \ell$ and so the vertex

x_{i+1} exists. Let $y'' \in Y$ be a vertex adjacent to x_{i+1}. By the choice of a, we have that $(y'', x_{i+1}) \in A(D)$. Since P is zigzag-free in D, we conclude that $y'' \neq y$. Therefore, we have that $P_1 = Px_i \circ (x_i, y)$ and $P_2 = (y'', x_{i+1}) \circ x_{i+1}P$ meet the conditions (i) through (iv) above. This concludes the base case.

Now, suppose that $k > 1$. Since D is k-tight, then $|X| \geq k$ and there exists a vertex $y^* \in Y$ which is adjacent to every vertex of $S = \{x_1, x_2, \dots, x_k\}$, the set of the k first vertices of P. By Lemma 5, we have that $(x_i, y^*) \in A(D)$ for every vertex $x_i \in S$. In particular, $(x_k, y^*) \in A(D)$. Among all arcs $(x_i, y) \in A(D)$ with $y \in Y$ and $1 \leq i \leq \ell$, choose an arc a such that i is maximum. Note that such arc a exists and that $i \geq k$, since $(x_k, y^*) \in A(D)$. As P is zigzag-free in D, we have that $i < \ell$ and so the vertex x_{i+1} exists. Note that by the choice of i, if some vertex $y' \in Y$ is adjacent to x_{i+1} then $(y', x_{i+1}) \in A(D)$.

Let $X' = V(Px_i)$ and let

$$Y' = \{y' : y' \in Y \text{ and } y' \text{ is adjacent to } x_{i+1}\}.$$

Let $D' = D[X' \cup Y']$. Clearly, D' is a spine digraph. Let $P' = Px_i$. To show that P' is zigzag-free in D', suppose the contrary. Since P is zigzag-free in D, there must exist some arc $(x_i, y') \in A(D')$ with $y' \in Y'$. However, by the definition of Y', we have that $(y', x_{i+1}) \in A(D)$ which contradicts the fact that P is zigzag-free in D.

We now claim that D' is $(k-1)$-tight. Let $S' \subset X'$ with $|S'| = k - 1$. We need to show that there exists $y' \in Y'$ such that y' is adjacent to every $x \in S'$. Let $S = S' \cup \{x_{i+1}\}$. Since D is k-tight, there exists $y' \in Y$ such that y' is adjacent to every $x \in S$. By the definition of Y', it follows that $y' \in Y'$. Therefore, D' is $(k-1)$-tight.

By the induction hypothesis applied to D' and P', there exist paths P'_1 and P'_2 in D' which satisfy conditions (i) through (iv). Without loss of generality, assume that $ter(P'_1) = x_i$ and $ter(P'_2) = y'$, for some $y' \in Y'$. Let $P_1 = P'_1 \circ (x_i, y)$ and $P_2 = P'_2 \circ (y', x_{i+1}) \circ x_{i+1}P$. We claim that P_1 and P_2 meet conditions (i) through (iv). Conditions (iii) and (iv) obviously hold. Condition (i) holds because P'_1 and P'_2 are disjoint by induction hypothesis and neither vertex y nor any vertex of $x_{i+1}P$ are vertices of D'. Condition (ii) holds because $|P'_1| + |P'_2| = i + k$ by induction hypothesis. Therefore

$$|P_1| + |P_2| = |P'_1| + |P'_2| + |X| - i + 1 = |X| + k + 1$$

and the proof is complete. ■

Theorem 2. *Let $D[X, Y]$ be a spine digraph. Then, $\pi_k(D) \leq \alpha_k(D)$.*

Proof. We may assume that D is k-tight, otherwise the result follows by Theorem 1. We may also assume that $\lambda(D) \leq |X|$, otherwise the result follows by Lemma 4. Let $P = (x_1, x_2, \dots, x_\ell)$ be a Hamiltonian path in $D[X]$. Clearly P is zigzag-free in D. By Lemma 6, there exists disjoint paths P_1 and P_2 in D' such that $|P_1| + |P_2| = |X| + k + 1$. Note that $|P_i| > k$, for $i = \{1, 2\}$, otherwise P_{3-i} would be larger than $|X|$. Let $\mathcal{P} = \{P_1, P_2\} \cup \{(y) : y \notin V(P_1) \cup V(P_2)\}$. It is easy to see that $\mathcal{P}$ is a path partition in D. The k-norm of $\mathcal{P}$ is $|\mathcal{P}|_k = \min\{|P_1|, k\} + \min\{|P_2|, k\} + |Y| - k - 1 = |Y| + k - 1$. So, $\pi_k(D) \leq |Y| + k - 1$. By Lemma 2, we know that $\alpha_k(D) \geq |Y| + \min\{|X|, k - 1\} = |Y| + k - 1$ and the result follows. ■

Corollary 1. *If D is a split digraph, then $\pi_k(D) \leq \alpha_k(D)$.*

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References

[1] R. P. Dilworth, A decomposition theorem for partially ordered sets, Annals of Mathematics 51 (1) (1950) 161–166.

[2] T. Gallai, A. N. Milgram, Verallgemeinerung eines graphentheoretischen satzes von rédei, Acta Sci Math 21 (1960) 181–186.

[3] C. Greene, D. J. Kleitman, The structure of Sperner k-families, Journal of Combinatorial Theory, Series A 20 (1) (1976) 41–68.

[4] N. Linial, Extending the Greene-Kleitman theorem to directed graphs, Journal of Combinatorial Theory, Series A 30 (3) (1981) 331–334.

[5] C. Berge, k-optimal partitions of a directed graph, European Journal of Combinatorics 3 (2) (1982) 97–101.

[6] N. Linial, Covering digraphs by paths, Discrete Mathematics 23 (3) (1978) 257–272.

[7] E. Berger, I. B.-A. Hartman, Proof of Berge's strong path partition conjecture for $k = 2$, European Journal of Combinatorics 29 (1) (2008) 179–192.

[8] D. Herskovics, Proof of Berge's path partition conjecture for $k \geq \lambda - 3$, Tech. Rep. TR-2013-08, Egerváry Research Group, Budapest (2013).

[9] I. B.-A. Hartman, Berge's conjecture on directed path partitions—a survey, Discrete Mathematics 306 (19–20) (2006) 2498–2514.

[10] R. Aharoni, I. B.-A. Hartman, A. J. Hoffman, Path partitions and packs of acyclic digraphs, Pacific Journal of Mathematics 2 (118) (1985) 249–259.

[11] I. B.-A. Hartman, F. Saleh, D. Hershkowitz, On Greene's theorem for digraphs, Journal of Graph Theory 18 (2) (1994) 169–175.

[12] T. Gallai, On directed paths and circuits, Theory of graphs (1968) 115–118.

[13] B. Roy, Nombre chromatique et plus longs chemins d'un graphe, Revue française d'informatique et de recherche opérationnelle 1 (5) (1967) 129–132.

[14] L. Rédei, Ein kombinatorischer satz, Acta Litt. Szeged 7 (39-43) (1934) 97.