INSTITUTO DE COMPUTAÇÃO
UNIVERSIDADE ESTADUAL DE CAMPINAS

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Patrícia F. Hongo *C. N. Campos*

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Abstract

A dominating set of a graph G is a subset $D \subseteq V(G)$ such that each vertex of G is in D or is adjacent to a vertex in D. The cardinality of a minimum size dominating set for G is denoted by $\gamma(G)$. In 1996, Tarjan and Matheson proved that $\gamma(G) \leq n/3$ for triangulated discs and conjectured that $\gamma(G) \leq n/4$ for triangulated planar graphs with sufficiently large n. In the present work, we verify the conjecture for two simple classes of triangulated planar graphs.

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. In this work, we let n denote the cardinality of the vertex set and assume that G is simple, finite and undirected. The open neighborhood of a vertex v, denoted by $N_G(v)$, is the set of vertices adjacent to v. The closed neighborhood of v is defined as the set $N_G[v] := N_G(v) \cup \{v\}$.

A *dominating set* of G is a subset $D \subseteq V(G)$ such that $\forall v \in V(G)$, there exists $w \in D, v \in N_G[w]$. The vertex set $V(G)$ is a trivial dominating set. Therefore, our interest lies in determining the size of a smallest dominating set for a given graph. The cardinality of one such set is called *domination number* and is denoted by $\gamma(G)$. The problem of deciding whether $\gamma(G) \leq k$ for a given graph G and integer k is NP-complete, as shown by Garey and Johnson [1].

A graph G is said to be *planar* if it can be embedded in the plane so that its edges intersect only at their end points. A drawing of G satisfying this condition is called a *planar embedding* of the graph, also referred to as a *plane graph.* A planar embedding can be seen as a graph isomorphic to G, with vertex set and edge set defined as the set of points and lines that represent the vertices and the edges of the plane graph. Figure 1 shows a planar embedding of K_4, the complete graph on four vertices.

A plane graph divides the plane into arc-wise connected open sets, defined as the *faces* of the graph. The *boundary* of a face f, denoted by $\partial(f)$, is the subgraph whose vertex set and edge set are the sets of vertices and edges incident with f. We call a *triangle* a face bounded by K_3. A *triangulated planar graph* is a graph that has an embedding in which all faces are triangles. The addition of an edge to a triangulated planar graph results in a nonplanar graph if we require the resulting graph to be simple. For this reason,

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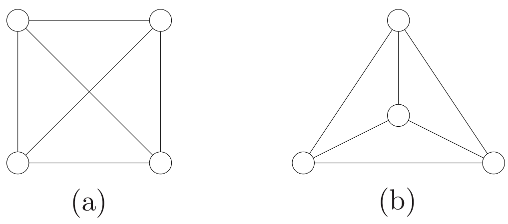


Figure 1: (a) The planar graph K_4 and (b) a planar embedding of K_4.

triangulated planar graphs are also referred to as *maximal planar graphs*. A triangulated disc is a plane graph with the property that all of its faces, except possibly one, are triangles.

In 1996, Matheson and Tarjan [3] proved that $\gamma(G) \leq n/3$ for triangulated discs, and showed that this bound is sharp by presenting an infinite class of outerplanar triangulated discs for which $\gamma(G) = n/3$. Furthermore, they conjectured that $\gamma(G) \leq n/4$ for triangulated planar graphs with $n \geq n_0$, where n_0 is a sufficiently large constant. In 2010, King and Pelsmajer verified this conjecture for triangulated planar graphs with maximum degree at most six, obtaining $n_0 = 4500000$ [2].

In this work, we prove the conjecture for two simple classes of triangulated planar graphs, defined in Sections 2 and 3.

2 Dominating sets for $\mathcal{G}_k^{\Delta}$

In the following, we define the graph G_k^{Δ} and the $\mathcal{G}_k^{\Delta}$-class.

Let P_j^i be a path of j vertices. In this text, we define $V(P_j^i) := \{v_{i0}, ..., v_{i(j-1)}\}$ and $E(P_j^i) := \{v_{il}v_{il+1} : 0 \leq l < j-1\}$. Graph G_0^{Δ} is the simple graph with v_{00} as its only vertex. For $k \geq 1$, $G_k^{\Delta} := G_{k-1}^{\Delta} \cup P_{k+1}^k \cup G[E']$, where $E' := \{v_{(k-1)i}v_{ki} : 0 \leq i < k\} \cup \{v_{(k-1)i}v_{k(i+1)} : 0 \leq i < k\}$. Figure 2 presents G_k^{Δ} for $0 \leq k \leq 3$. Figure 3 exemplifies the labeling for G_3^{Δ}.

We let $\mathcal{G}_k^{\Delta}$ denote the class of graphs obtained by triangulating the outer face of G_k^{Δ}.

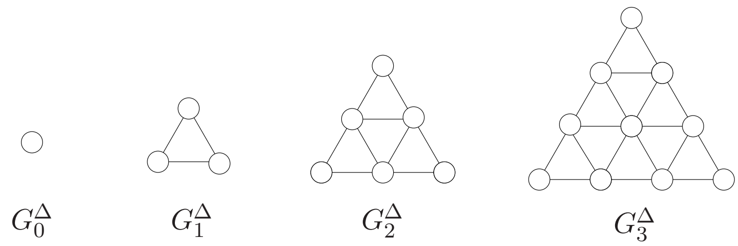


Figure 2: G_k^{Δ}, for $0 \leq k \leq 3$.

Let G be a graph in $\mathcal{G}_k^{\Delta}$-class. If v is a degree two vertex in G_k^{Δ}, then v is called a *corner vertex*. We show that G has a dominating set of size at most $n/4$ for $k \geq 3$. We start by showing that $\gamma(G_k^{\Delta}) > n/4$ for $0 \leq k \leq 3$, and $\gamma(G_k^{\Delta}) \leq n/4$ for $k \geq 4$. Finally, we use these results to find bounds for the domination number of G.

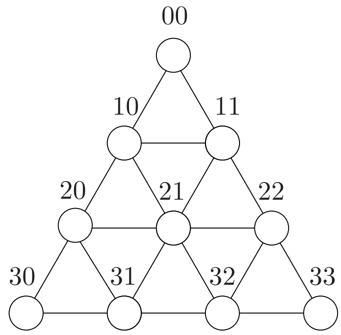


Figure 3: Labeling of G_3^Δ. The i index increases vertically, while the j index increases horizontally.

Graphs G_k^Δ for $0 \leq k \leq 3$ are depicted in Figure 2. For $k = 0$ and $k = 1$, $n < 4$, so we cannot dominate them with at most $n/4$ vertices. Case $k = 2$ requires at least two vertices to be in any dominating set, since it has no universal vertex. Thus, $\gamma(G_2^\Delta) \geq 2 > n/4$. Case $k = 3$ has 10 vertices. Since the three corner vertices have disjoint neighborhoods, this case requires at least three vertices to be in any dominating set. Therefore, $\gamma(G_3^\Delta) \geq 3 > n/4$.

Now, we find dominating sets for G_k^Δ, $k \geq 4$.

Theorem 1 *Let G be isomorphic to G_k^Δ, $k \geq 4$. Then, $\gamma(G) \leq n/4$.*

Proof. We prove the theorem for $k \geq 4$ by induction on k. The base cases $k = 4, 5, 6$ are depicted in Figure 4: the sets of black vertices are dominating sets of size at most $n/4$.

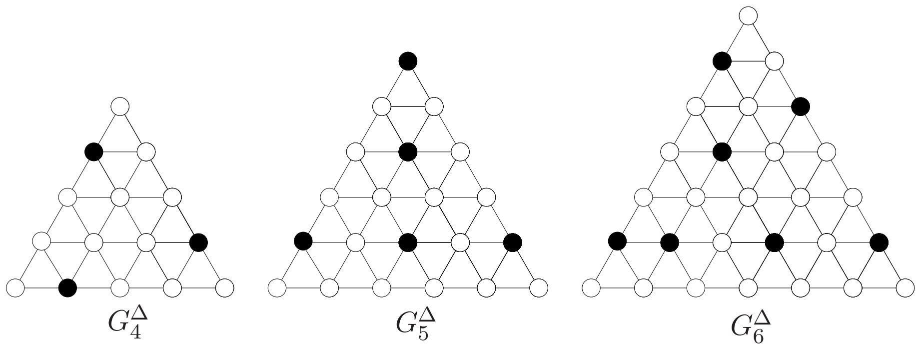


Figure 4: Dominating sets for G_k^Δ, $4 \leq k \leq 6$.

Let G be isomorphic to G_k^Δ, $k > 6$. Let H_1 be the subgraph induced by $S := \{v_{ij} : k-2 \leq i \leq k \text{ and } 0 \leq j \leq i\}$ and H_2 be the subgraph induced by $V(G) \setminus S$. Then, H_2 is isomorphic to G_{k-3}^Δ. By inductive hypothesis, H_2 can be dominated with at most $|V(H_2)|/4$ vertices. Thus, it suffices to show that $\gamma(H_1) \leq |V(H_1)|/4$. We note that H_1 is the subgraph induced by the vertices in the paths P_{k-1}^{k-2}, P_k^{k-1} and P_{k+1}^k, therefore $|V(H_1)| = 3k$. The proof is divided into two cases:

Case 1 $k \equiv 1 \pmod 2$.

Let $D := \{v_{(k-1)j} : j \equiv 0 \pmod 2\}$. Then, D dominates every vertex in the P_k^{k-1} path. Also, since every vertex in P_{k-1}^{k-2} and P_{k+1}^{k} is adjacent to a vertex $v_{(k-1)j}$, j even, D dominates P_{k-1}^{k-2} and P_{k+1}^{k}. Therefore, D is a dominating set for H_1 of size $(k+1)/2$. Since $|V(H_1)| = 3k$ and $(k+1)/2 \leq 3k/4$ for $k \geq 2$, H_1 admits a dominating set with at most $|V(H_1)|/4$ vertices.

Case 2 $k \equiv 0 \pmod 2$.

Let $D := \{v_{(k-1)j} : j \equiv 1 \pmod 2\} \cup \{v_{(k-1)0}\}$. Then, D dominates every vertex in the P_k^{k-1} path. Also, every vertex in P_{k-1}^{k-2} and P_{k+1}^{k} except for v_{k0} is dominated by a vertex $v_{(k-1)j}$, j odd. Since v_{k0} is dominated by $v_{(k-1)0}$, D dominates P_{k-1}^{k-2} and P_{k+1}^{k}. Therefore, D is a dominating set for H_1 of size $k/2+1$. Since $|V(H_1)| = 3k$ and $k/2+1 \leq 3k/4$ for $k \geq 4$, H_1 admits a dominating set with at most $|V(H_1)|/4$ vertices. □

As a corollary of Theorem 1, a triangulated planar graph G resulting from triangulating the outer face of G_k^{Δ}, $k \geq 4$, satisfies Matheson and Tarjan's Conjecture. The cases $k \leq 2$ do not satisfy the conjecture: when $k = 0$ or $k = 1$, G has less than four vertices; when $k = 2$, there is a triangulation, exhibited in Figure 5, which requires at least two vertices to be dominated, since it still does not have a universal vertex.

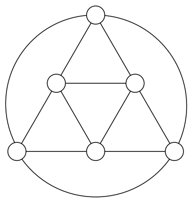


Figure 5: The octahedron cannot be dominated with less than two vertices.

A graph $G \in \mathcal{G}_3^{\Delta}$, however, can be dominated with at most $n/4$ vertices, as shown in the following.

Theorem 2 *Let $G \in \mathcal{G}_3^{\Delta}$. Then, $\gamma(G) \leq 2 < n/4$.*

Proof. Let C be the cycle bounding the outer face of G_3^{Δ}. Let $E := E(G) \backslash E(G_3^{\Delta})$ and let $uv \in E$ such that $d_C(u,v)$, the length of the shortest path $P_{u,v}$ in C connecting u to v, is minimum. Suppose that $d_C(u,v) > 2$. Then, $P_{u,v} + \{uv\}$ is a cycle of length at least four. Since G is triangulated, $\exists xy \in E$, $x \neq u, v$ connecting two vertices x and y in $P_{u,v}$. Since $xy \in E$ and $d_C(x,y) < d_C(u,v)$, this contradics our choice of uv. Therefore, $d_C(u,v) = 2$.

If $d_{G_3^{\Delta}}(u) = d_{G_3^{\Delta}}(v) = 4$, then uv is a parallel edge. We conclude that uv connects a corner vertex to a non-corner vertex. Figure 6 shows that the existence of this edge is all we require in order to find a dominating set of size $2 < n/4$ for G. □.

Corollary 3 *Let $G \in \mathcal{G}_k^{\Delta}$, $k \geq 3$. Then, $\gamma(G) \leq n/4$.*

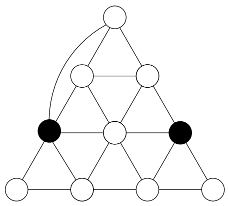


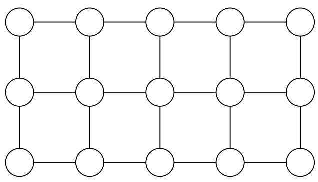
Figure 6: Black vertices form a dominating set.

3 Dominating sets for the $\mathcal{G}^{\Delta}_{p\times q}$-class

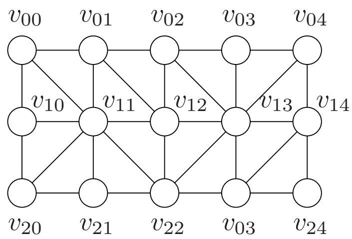
Given two paths P_m and P_n with m and n vertices respectively, we define a *grid* as the graph $G_{m\times n}$ resulting from the cartesian product $P_m \square P_n$. Grids admit a planar embedding with its vertices arranged in m lines and n columns. Figure 7(a) shows a planar embedding of $G_{3\times 5}$.

In this work, we define the $G^{\Delta}_{p\times q}$*-class* as the graphs obtained from $G_{p\times q}$ by arbitrarily triangulating all its faces up to the exterior face. Each graph in the $G^{\Delta}_{p\times q}$-class is called an *instance of* $G^{\Delta}_{p\times q}$. We label its vertices $v_{00}, v_{01}, ..., v_{(p-1)(q-1)}$, as illustrated in Figure 7(b). A graph resulting from triangulating the outer face of an instance of $G^{\Delta}_{p\times q}$ is a maximal planar graph. We define the $\mathcal{G}^{\Delta}_{p\times q}$-class as the set of graphs obtained from this operation.

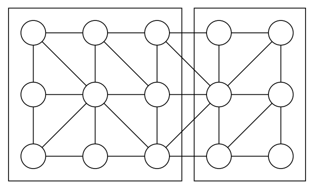
While searching for dominating sets for $G^{\Delta}_{p\times q}$, it is useful to divide its instances into disjoint subgraphs which are instances of $G^{\Delta}_{r\times s}$-class, for $2 \leq r \leq p$ and $2 \leq s \leq q$. We call one such subgraph a $G^{\Delta}_{r\times s}$*-block*. Figure 7(c) shows a division of a $G^{\Delta}_{3\times 5}$ instance. For this class, we define a *corner vertex* to be a vertex v such that $d_{G_{p\times q}}(v) = 2$.



(a) $G_{3\times 5}$.



(b) $G^{\Delta}_{3\times 5}$.



(c) A division of $G^{\Delta}_{3\times 5}$ into $G^{\Delta}_{3\times 3}$ and $G^{\Delta}_{3\times 2}$ blocks.

Figure 7: $G_{3\times 5}$ and $G^{\Delta}_{3\times 5}$.

Let G be a graph in the $\mathcal{G}^{\Delta}_{p\times q}$-class. We prove that $\gamma(G) \leq n/4$ for sufficiently large n by finding small dominating sets for the $G^{\Delta}_{p\times q}$-class. Through Lemmas 4 to 7, we obtain bounds for the domination number of the $G^{\Delta}_{p\times q}$-class for $2 \leq p \leq 5$.

Lemma 4 *Let G be an instance of $G^{\Delta}_{2\times q}$-class, $q \geq 2$. Then,*

$$\begin{cases} \gamma(G) \leq n/4,\ q \equiv 0 \pmod 2; \\ \gamma(G) = n/4 + 1/2,\ q \equiv 1 \pmod 2. \end{cases} \tag{1}$$

Proof. If q is even, we can partition $V(G)$ into $q/2$ sets of the form $\{v_{ij},\ v_{i(j+1)},\ v_{(i+1)j},\ v_{(i+1)(j+1)}\}$, $i, j \equiv 0 \pmod 2$. Each of these sets induces a subgraph isomorphic to $G^{\Delta}_{2\times 2}$, which requires only one vertex to be dominated. Since we can dominate G by dominating each of the induced subgraphs, G has a dominating set that uses $q/2 = n/4$ vertices.

Now, we show that $\gamma(G) > n/4$, $q \equiv 1 \pmod 2$, by exhibiting a subfamily of $G^{\Delta}_{2\times q}$ for which $\gamma(G) > n/4$. Let H be the graph obtained from $G_{2\times q}$ by adding a $v_{1j}v_{0(j+1)}$ edge if $j \equiv 0, 1 \pmod 4$ and a $v_{0j}v_{1(j+1)}$ edge otherwise. Then, H is an instance of the $G^{\Delta}_{2\times q}$-class. Figure 8 shows H for $q = 7$.

Let $S := \{v_{0j} : j \equiv 0 \pmod 4\} \cup \{v_{1j} : j \equiv 2 \pmod 4\}$. Then, $|S| = (q-1)/2 + 1$ and the neighborhoods of its vertices are mutually disjoint. Therefore, it is impossible to dominate H with less than $(q-1)/2 + 1 = n/4 + 1/2$ vertices. It can also be seen that $S \cup N(S) = V(H)$. In fact, that remains true even if we remove the triangulation from the interior faces, as shown in Figure 9. Therefore, S is a dominating set for $G_{2\times q}$, $q \equiv 1 \pmod 2$. Thus, any graph $G \in G^{\Delta}_{2\times q}$, $q \equiv 1 \pmod 2$, can be dominated with at most $(q-1)/2 + 1 = n/4 + 1/2$ vertices, as required. □

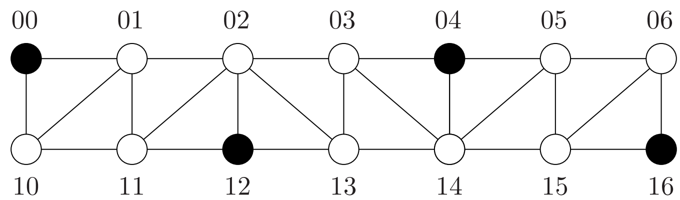


Figure 8: H with the vertices in S coloured black.

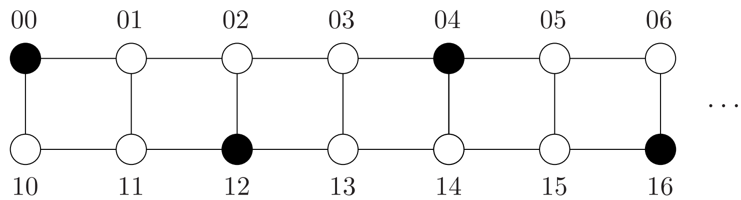


Figure 9: S is a dominating set for $G_{2\times q}$, $q \equiv 1 \pmod 2$.

Lemma 5 *Let G be an instance of $G^{\Delta}_{3\times q}$-class, $q \geq 3$. Then,*

$$\begin{cases} \gamma(G) \leq n/4, \ q \neq 5; \\ \gamma(G) > n/4, \ q = 5. \end{cases} \tag{2}$$

Proof. We divide the proof into three cases:

Case 1 $q \equiv 0 \pmod 3$, $q \geq 3$.

Figure 10 shows that any instance of $G^{\Delta}_{3\times 3}$ can be dominated with at most two vertices. Dominating a $G^{\Delta}_{3\times 3}$-block with a single vertex is only possible if the center vertex v has degree eight. The other possible cases, $d(v) = 7, 6, 5, 4$, admit a dominating set of size two. Thus, dominating G can be done by dividing the graph into $q/3$ $G^{\Delta}_{3\times3}$-blocks and dominating each of them individually. The resulting dominating set has at most $2q/3$ vertices. Since $n = 3q$, $\gamma(G) \leq 2q/3 = 2n/9 < n/4$.

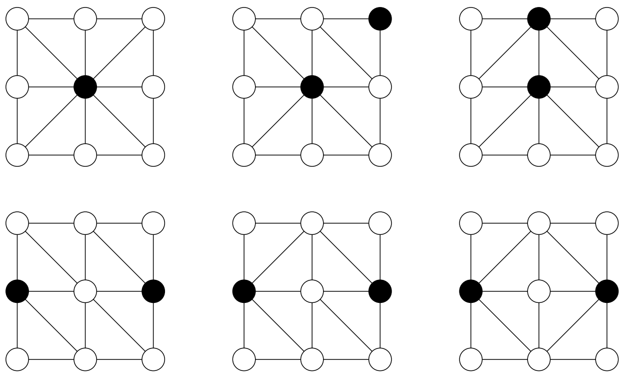


Figure 10: Dominating sets for instances of $G^{\Delta}_{3\times3}$.

Case 2 $q \equiv 1 \pmod 3$, $q \geq 4$.

In this case, we divide G into $(q-1)/3$ $G^{\Delta}_{3\times3}$-blocks and one subgraph isomorphic to P_3, which can be dominated with a single vertex. Dominating each $G^{\Delta}_{3\times3}$-block requires at most two vertices, so dominating each block and the P_3 subgraph individually requires at most $2(q-1)/3+1$ vertices. Since $n = 3q$, $\gamma(G) \leq 2(q-1)/3+1 = 2n/9+1/3 \leq n/4$ for $q \geq 4$.

Case 3 $q \equiv 2 \pmod 3$, $q \geq 5$.

We divide G into $(q-2)/3$ $G^{\Delta}_{3\times3}$-blocks and one $G^{\Delta}_{3\times2}$-block. Dominating each $G^{\Delta}_{3\times3}$-block and $G^{\Delta}_{3\times2}$-block requires two vertices. Dominating the blocks individually yields a dominating set of size at most $2(q-2)/3 + 2 = 2n/9 + 2/3 \leq n/4$ for $q \geq 8$.

It remains to show that $\gamma(G) > n/4$, $q = 5$. Figure 11 shows an instance of the $G^{\Delta}_{3\times5}$-class that needs at least four vertices in any dominating set. In order to see that, note that the neighborhoods of black vertices are disjoint. Figure 12 shows that $G^{\Delta}_{3\times5}$ can be dominated with four vertices, so $\gamma(G) \leq 4$. Therefore, we may conclude that $\gamma(G) = 4$. $\square$

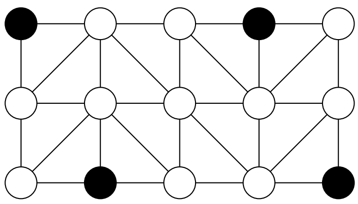


Figure 11: Black vertices have disjoint neighborhoods, so any dominating set for this graph has at least four vertices.

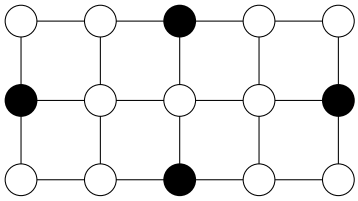
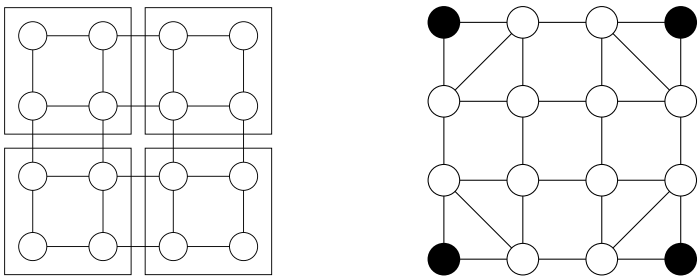


Figure 12: Black vertices form a dominating set.

Lemma 6 *Let G be an instance of $G^{\Delta}_{4\times q}$-class, $q \geq 4$. Then, $\gamma(G^{\Delta}_{4\times q}) \leq n/4$.*

Proof. We start by showing that dominating a $G^{\Delta}_{4\times 4}$-block requires four vertices. Figure 13(a) shows that it is possible to divide a $G^{\Delta}_{4\times 4}$-block into four $G^{\Delta}_{2\times 2}$-blocks. Dominating each block requires one vertex, so the resulting dominating set has four vertices. Therefore, $\gamma(G) \leq n/4$, $q = 4$. This bound is tight since Figure 13(b) shows a sketch of an instance of $G^{\Delta}_{4\times 4}$ which requires at least four vertices in any dominating set.



(a) A division of $G^{\Delta}_{4\times 4}$ into four $G^{\Delta}_{2\times 2}$.

(b) Black vertices have disjoint neighborhoods.

Figure 13: $G^{\Delta}_{4\times 4}$.

Now, we find dominating sets for $G^{\Delta}_{4\times q}$, $q > 4$. We divide the proof into four cases.

Case 1 $q \equiv 0 \pmod 4$, $q > 4$.

A dominating set with at most $n/4$ vertices can be obtained by dividing G into $q/4$ $G^{\Delta}_{4\times 4}$-blocks and dominating each of them individually. This yields a dominating set of size $q = n/4$.

Case 2 $q \equiv 1 \pmod 4$, $q \geq 5$.

Divide G into $(q-5)/4$ $G^{\Delta}_{4\times 4}$-blocks, one $G^{\Delta}_{4\times 3}$-block and one $G^{\Delta}_{4\times 2}$-block. Each $G^{\Delta}_{4\times 4}$-block adds four vertices to the dominating set. The $G^{\Delta}_{4\times 3}$-block and $G^{\Delta}_{4\times 2}$-block add three and two vertices, respectively. The size of the resulting dominating set will be at most

$4(q-5)/4+3+2=q=n/4$. Therefore, $\gamma(G) \leq n/4$.

Case 3 $q \equiv 2 \pmod 4$, $q \geq 6$.

Divide G into $(q-2)/4$ $G^{\Delta}_{4\times 4}$-blocks and one $G^{\Delta}_{4\times 2}$-block. Dominating each $G^{\Delta}_{4\times 4}$-block requires four vertices, while dominating the $G^{\Delta}_{4\times 2}$-block requires two vertices. The resulting dominating set has at most $4(q-2)/4+2=q=n/4$ vertices.

Case 4 $q \equiv 3 \pmod 4$, $q \geq 7$.

Divide G into $(q-3)/4$ $G^{\Delta}_{4\times 4}$-blocks and one $G^{\Delta}_{4\times 3}$-block. Since the $G^{\Delta}_{4\times 3}$-block adds three vertices to the dominating set, we obtain a dominating set of size at most $4(q-3)/4+3=q=n/4$, as required. □

Lemma 7 *Let G be an instance of $G^{\Delta}_{5\times q}$-class, $q \geq 5$. Then, $\gamma(G^{\Delta}_{5\times q}) \leq n/4$.*

Proof. We initially show that dominating a $G^{\Delta}_{5\times 5}$-block requires six vertices. Let H be an instance of $G^{\Delta}_{5\times 5}$. Figure 14(a) shows a sketch of a $G^{\Delta}_{5\times 5}$ instance that requires at least six vertices to be dominated, so $\gamma(H) \geq 6$. To show that $\gamma(H) \leq 6$, consider H triangulated so that at least one of its corner vertices has degree two. Figure 14(b) illustrates that, in this case, H has a dominating set of size six. Note that since every internal face is triangulated, one of the vertices $v_{33}, v_{34}, v_{43}, v_{44}$ is in the dominating set. Figures 14(c) and 14(d) depict dominating sets of size six for H when none of its corner vertices has degree two.

Now, we find dominating sets for $G^{\Delta}_{5\times q}$, $q > 5$. We divide the proof into five cases.

Case 1 $q \equiv 0 \pmod 5$, $q > 5$.

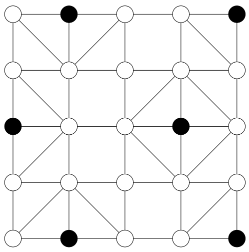
Divide G into $q/5$ $G^{\Delta}_{5\times 5}$-blocks and dominate each of them individually. This yields a dominating set with at most $6q/5 = 6n/25 < n/4$ vertices.

Case 2 $q \equiv 1 \pmod 5$, $q \geq 6$.

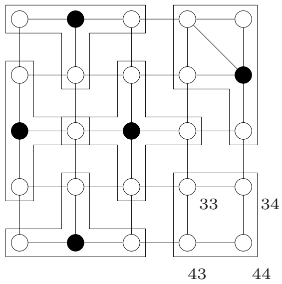
Divide G into $(q-1)/5$ $G^{\Delta}_{5\times 5}$-blocks and one subgraph isomorphic to P_5, which can be dominated with two vertices. The dominating set resulting from dominating each subgraph individually has at most $6(q-1)/5+2=(6n-30)/25+2 \leq n/4$ vertices for $q \geq 16$.

It remains to consider cases $q=6$ and $q=11$. Let $q=6$ and assume that at least one of the corner vertices has degree three. Then, Figure 15(a) shows that G has a dominating set of size $7 < n/4$. Note that since every internal face is triangulated, one of the vertices $v_{30}, v_{31}, v_{40}, v_{41}$ is in the dominating set. If all corner vertices have degree two, then G can be dominated using at most $n/4$ vertices, as shown on Figure 15(b).

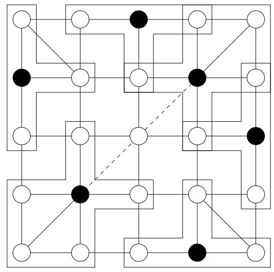
Let $q=11$. Then, we divide G into a $G^{\Delta}_{5\times 5}$-block and a $G^{\Delta}_{5\times 6}$-block and dominate each individually. The former can be dominated with six vertices, while the later requires at



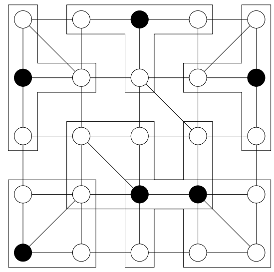
(a) Sketch of $G^{\Delta}_{5\times 5}$ showing that $\gamma(G^{\Delta}_{5\times 5}) \geq 6$. Black vertices have disjoint neighborhoods.



(b) A dominating set for the case in which one of the corner vertices has degree two.

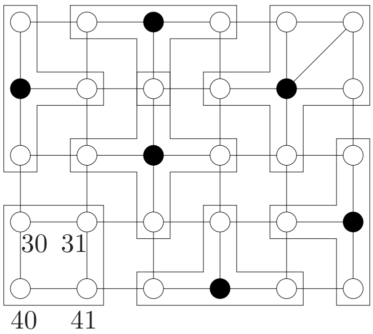


(c) The black vertices form a dominating set if any of the dotted edges is present.

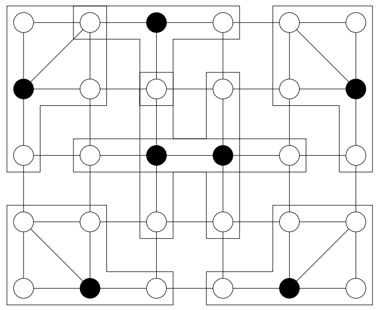


(d) If none of the dotted edges in 14(c) is present, then the graph can be dominated as shown.

Figure 14: $G^{\Delta}_{5\times 5}$.



(a)



(b)

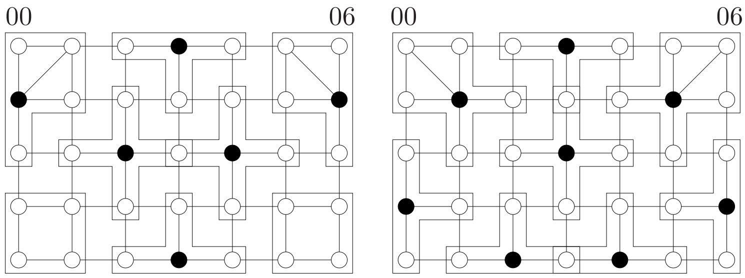
Figure 15: Dominating set for $G^{\Delta}_{5\times 6}$.

most seven vertices. This yields a dominating set with at most $13 < n/4$ vertices.

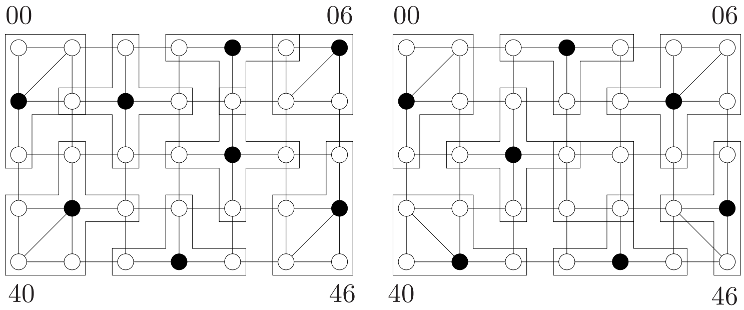
Case 3 $q \equiv 2 \pmod 5$, $q \geq 7$.

Divide G into $(q-2)/5$ $G^{\Delta}_{5\times 5}$-blocks and one $G^{\Delta}_{5\times 2}$-block, and dominate them individually. Each $G^{\Delta}_{5\times 5}$-block can be dominated with six vertices, while the $G^{\Delta}_{5\times 2}$-block can be dominated with three vertices. The resulting dominating set is of size at most $6(q-2)/5 + 3 = (6n-60)/25 + 3 \leq n/4$ for $q \geq 12$.

It remains to consider case $q = 7$. Consider vertices v_{00} and v_{06}. If both have the same degree, then G admits a dominating set of size $8 < n/4$, as shown in Figure 16(a). Thus, without loss of generality, we may assume $d(v_{00}) = 2$ and $d(v_{06}) = 3$. Consider now vertices v_{40} and v_{46}. The case when both have the same degree is equivalent to the case $d(v_{00}) = d(v_{06})$. Therefore, it remains to consider cases $d(v_{40}) = 2$ with $d(v_{46}) = 3$ and $d(v_{40}) = 3$ with $d(v_{46}) = 2$. Figure 16(b) shows that, in these cases, we can also find dominating sets of size $8 < n/4$. Therefore, $\gamma(G) \leq n/4$, $q = 7$.



(a) Dominating sets for $G^{\Delta}_{5\times 7}$ when $d(v_{00}) = d(v_{06})$.



(b) If v_{40} and v_{46} have the same degree, case (a) is applicable.

Figure 16: Dominating sets for $G^{\Delta}_{5\times 7}$.

Case 4 $q \equiv 3 \pmod 5$, $q \geq 8$.

Divide G into $(q-3)/5$ $G^{\Delta}_{5\times 5}$-blocks and one $G^{\Delta}_{5\times 3}$-block. Each $G^{\Delta}_{5\times 5}$-block can be dominated with six vertices, while the $G^{\Delta}_{5\times 3}$-block requires four vertices. The resulting dominating set has at most $6(q-3)/5+4 = (6n-90)/25+4 \leq n/4$ for $q \geq 8$. When $q = 3$, we have already shown that $\gamma(G) = 4 > n/4$.

Case 5 $q \equiv 4 \pmod 5$, $q \geq 9$.

Divide G into $(q-4)/5$ $G^{\Delta}_{5\times 5}$-blocks and one $G^{\Delta}_{4\times 5}$-block. Each $G^{\Delta}_{5\times 5}$-block can be dominated with six vertices, and the $G^{\Delta}_{4\times 5}$-block requires five vertices. The resulting dominating set has at most $6(q-4)/5+5 = (6n+5)/25 \leq n/4$, $q \geq 4$. □

Corollary 8 *Let G be an instance of $G^{\Delta}_{p\times q}$, $p, q \geq 2$. Then, $\gamma(G) \leq n/4$ except for the cases $G \in G^{\Delta}_{2\times q}$, $q \equiv 1 \pmod 2$, and $G \in G^{\Delta}_{3\times 5}$.*

Proof. Let G be an instance of $G^{\Delta}_{p\times q}$, $p, q \geq 2$. The cases when $2 \leq p \leq 5$ follow from Lemmas 4 to 7. Suppose that $p \geq 6$. In this case, we can find dominating sets for G by dividing it into $G^{\Delta}_{3\times q}$-blocks and $G^{\Delta}_{4\times q}$-blocks and dominating each of them individually. From Lemmas 5 and 6, each $G^{\Delta}_{3\times q}$-block and $G^{\Delta}_{4\times q}$-block can be dominated with at most

$n/4$ vertices except for the $G^{\Delta}_{3\times5}$-blocks. Therefore, it suffices to show that $G^{\Delta}_{p\times5}$, $p \geq 6$ can be dominated with at most $n/4$ vertices, which follows from Lemma 7. □

Let G be an instance of the $\mathcal{G}^{\Delta}_{p\times q}$-class. From Corollary 8, we know that G can be dominated with $n/4$ vertices when we are not in cases $p = 2$ with $q \equiv 1 \pmod 2$ or $p = 3$ with $q = 5$. As shown in Figure 17, there is an instance of $\mathcal{G}^{\Delta}_{2\times3}$ for which any dominating has at least $2 > n/4$ vertices, since there is no universal vertex.

Now, we verify Matheson and Tarjan's Conjecture for the $\mathcal{G}^{\Delta}_{2\times q}$-class, $q \equiv 1 \pmod 2$, when $q \geq 5$. This suffices to verify the Conjecture for $\mathcal{G}^{\Delta}_{p\times q}$-class when $n > 15$.

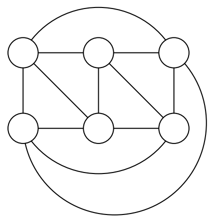


Figure 17: A triangulation of an instance of $G^{\Delta}_{2\times3}$ which requires two vertices in any dominating set.

Lemma 9 *Let G be an instance of $\mathcal{G}^{\Delta}_{2\times q}$, $q \equiv 1 \pmod 2$. Then, for $n \geq 10$, $\gamma(G) \leq n/4$.*

Proof. Let H be an instance of $G^{\Delta}_{2\times q}$, $q \geq 5$, $q \equiv 1 \pmod 2$, and let G be a graph resulting from the triangulation of the outer face of H.

Suppose that there is no degree five vertex in $V(H)$. Then, $E(H) = E(G_{2\times q}) \cup E'$, where $E' = \{v_{0j}v_{1(j+1)} : 0 \leq j < q-1\}$ or $E' = \{v_{1j}v_{0(j+1)} : 0 \leq j < q-1\}$. In the first case, vertices $\{v_{ij} : 0 \leq i \leq 1 \text{ and } 0 \leq j \leq 4\}$ can be dominated using v_{11} and v_{03}, while the remaining vertices can be dominated with $(q-5)/2$ vertices, by Lemma 4. The resulting dominating set has $(q-5)/2 + 2 = n/4 - 1/2$ vertices. The second case is similar, but uses vertices v_{01} and v_{13} instead of v_{11} and v_{03}. Thus, we may assume that there is at least one degree five vertex in H.

Suppose that there exists $v_{ij} \in V(H)$, $d_H(v_{ij}) = 5$, $j \equiv 1 \pmod 2$. Then, we can find a dominating set for H by dividing it into $(q-3)/2$ $G^{\Delta}_{2\times2}$-blocks and one $G^{\Delta}_{2\times3}$-block induced by $N_H[v_{ij}]$. Each subgraph can be dominated with one vertex, so the resulting dominating set is of size $(q-1)/2 = n/4 - 1/2 < n/4$.

Suppose now that there is no $j \equiv 1 \pmod 2$ such that $d_H(v_{ij}) = 5$. Let $0 \leq i' < q$, $0 \leq i'' < q$, $0 < l' < q$ and $0 < l'' < q$ so that sets $E' := \{v_{0j}v_{1(j+1)} : i' \leq j < i' + l'\}$ and $E'' := \{v_{1j}v_{0(j+1)} : i'' \leq j < i'' + l''\}$ be maximum. Figure 18 shows sets E' and E'' for an instance of $G^{\Delta}_{2\times7}$. Note that $E' \neq \emptyset$ and $E'' \neq \emptyset$, since there is a vertex v in H with $d_H(v) = 5$. Also, we note that i' is even since there is no v_{ij} with $d_H(v_{ij}) = 5$ and j odd. Similarly, we conclude that l' is even. Since E' and E'' are symmetric, we conclude that i'' and l'' are even.

Suppose that $l' \geq 4$, l' even. Then, we can divide H into $(q-5)/2$ $G^{\Delta}_{2\times2}$-blocks and one $G^{\Delta}_{2\times5}$-block so that the $G^{\Delta}_{2\times5}$-block is the subgraph of H induced by $\{v_{ij} : 0 \leq i \leq 1$ and

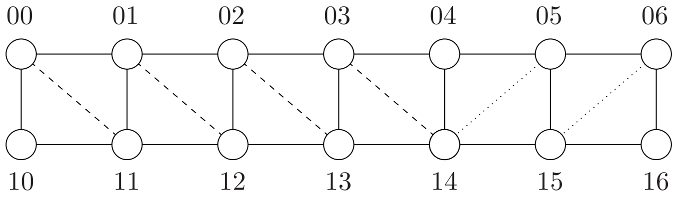


Figure 18: An instance of $G^{\Delta}_{2\times 7}$ with $i' = 0$, $l' = 4$, $i'' = 4$ and $l'' = 2$. Dashed edges are in E' and dotted edges are in E''.

$i' \leq j \leq i' + 4\}$. Each $G^{\Delta}_{2\times 2}$-block can be dominated with one vertex, while the $G^{\Delta}_{2\times 5}$-block can be dominated with two vertices, $v_{1(i'+1)}$ and $v_{0(i'+3)}$. The resulting dominating set is of size $(q-5)/2+2 = n/4-1/2$. Thus, in this case, $\gamma(G) \leq n/4$, and the result follows. If $l'' \geq 4$, a similar analysis applies, since sets E' and E'' are symmetric.

Therefore, we assume that $l' = l'' = 2$ and let H be the graph obtained from $G_{2\times q}$ by adding a $v_{0j}v_{1(j+1)}$ edge if $j \equiv 0, 1 \pmod 4$ and a $v_{1j}v_{0(j+1)}$ edge otherwise. An example of H is depicted in Figure 19. The case when H is obtained from $G_{2\times q}$ by adding a $v_{1j}v_{0(j+1)}$ edge if $j \equiv 0, 1 \pmod 4$, and a $v_{0j}v_{1(j+1)}$ edge otherwise, is symmetric.

Consider edge $e = v_{00}v_{10}$. Since G is a triangulated planar graph, e is in the boundary of two faces. Therefore, there exists $w \in V(G)$, $w \neq v_{11}$, such that w is adjacent to both v_{00} and v_{10}. Suppose that $w = v_{01}$. In this case we can find a dominating set for G that uses $n/4-1/2$ vertices. One such set is $D := \{v_{0j} : j \equiv 1 \pmod 4\} \cup \{v_{1j} : j \equiv 3 \pmod 4\}$. Note that $|D| = (q-1)/2 = n/4-1/2$ and each set of the form $\{v_{0j}, v_{1j}, v_{0(j+1)}, v_{1(j+1)}\}$, $j \equiv 1 \pmod 2$) is dominated by one vertex in D. Also v_{01} dominates v_{00} and v_{10}, since $w = v_{01}$ is adjacent to both. Therefore, all vertices of G are dominated by D, and the result follows.

Thus, we may assume that $w \in V(H_1)$, where H_1 is the subgraph of H induced by $V(H) \setminus \{v_{00}, v_{10}, v_{01}, v_{11}\}$.

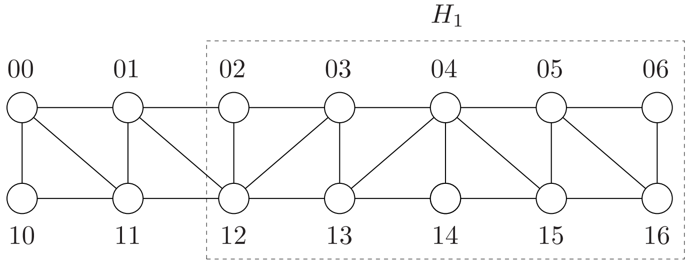
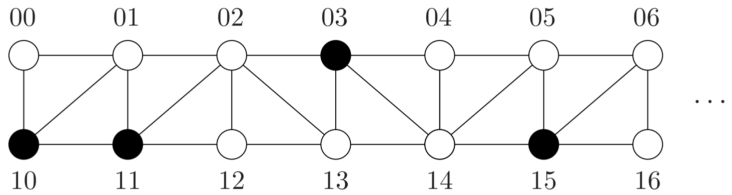


Figure 19: An instance of $G^{\Delta}_{2\times 7}$.

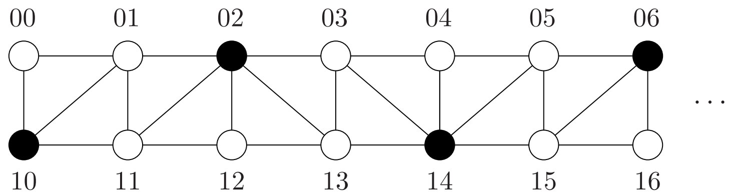
Claim 10 *Let* $G = G_{2\times q} \cup E$, $q \geq 3$, $q \equiv 1 \pmod 2$, *where* $E := \{v_{1j}v_{0(j+1)} : j \equiv 0, 1 \pmod 4\} \cup \{v_{0j}v_{1(j+1)} : j \equiv 2, 3 \pmod 4\}$. *Let* $v \in V(G)$, $v \notin \{v_{00}, v_{01}, v_{10}\}$. *Then, G has a dominating set D,* $|D| = (n-2)/4+1$, *that contains vertices* v_{10} *and* v.

Proof. We present four dominating sets D_1, D_2, D_3, D_4 such that $\bigcup_{1\leq i\leq 4} D_i = V(G) \setminus$

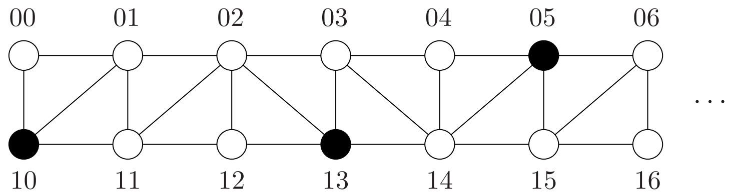
$\{v_{00}, v_{01}\}$ and $|D_1| = |D_2| = |D_3| = |D_4| = (n-2)/4+1$. The dominating sets are depicted in Figure 20.



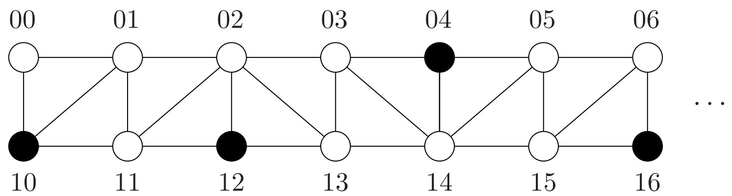
(a) $D_1 := \{v_{10}\} \cup \{v_{1j} : j \equiv 1 \pmod 4\} \cup \{v_{0j} : j \equiv 3 \pmod 4\}$.



(b) $D_2 := \{v_{1j} : j \equiv 0 \pmod 4\} \cup \{v_{0j} : j \equiv 2 \pmod 4\}$.



(c) $D_3 := \{v_{10}\} \cup \{v_{1j} : j \equiv 3 \pmod 4\} \cup (\{v_{0j} : j \equiv 1 \pmod 4\}/\{v_{01}\}) \cup \{v_{1(q-1)}\}$. Note that vertex $v_{1(q-1)}$ is not painted black, since the graph may continue beyond what is shown in this Figure.



(d) $D_4 := \{v_{10}\} \cup \{v_{1j} : j \equiv 2 \pmod 4\} \cup \{v_{0j} : j \equiv 0 \pmod 4\}/\{v_{00}\}$.

Figure 20: Dominating sets for $G^{\Delta}_{2\times q}$, $q \geq 3$, $q \equiv 1 \pmod 2$.

Let D_1 and D_2 be as defined in Figures 20(a) and 20(b). Then, for both D_1 and D_2, v_{10} dominates v_{00}. Moreover, for every $j \equiv 1 \pmod 2$, both D_1 and D_2 dominate each set $\{v_{0j}, v_{0(j+1)}, v_{1j}, v_{1(j+1)}\}$ using one vertex. Therefore, D_1 and D_2 are dominating sets for G of size $1 + (q-1)/2 = (n-2)/4 + 1$.

Let D_3 be as defined in Figure 20(c). Then, $v_{1(q-1)}$ dominates $v_{0(q-1)}$. Moreover, for every $j \equiv 0 \pmod 2$, D_3 dominates each set $\{v_{0j}, v_{0(j+1)}, v_{1j}, v_{1(j+1)}\}$ using one vertex. Therefore, D_3 is a dominating set of G of size $1 + (q-1)/2 = (n-2)/4 + 1$.

Finally, let D_4 be as defined in Figure 20(d). Then, v_{10} dominates v_{00} and v_{01}. The closed neighborhoods of the other vertices in D_4 are disjoint and include every vertex in $V(G) \setminus \{v_{10}, v_{00}, v_{01}\}$. Therefore, D_4 is a dominating set of G of size $1 + (q-1)/2 = (n-2)/4 + 1$.

In order to conclude the proof, note that $v_{10} \in D_i$, $1 \leq i \leq 4$. Also, each $v \in V(G) \setminus \{v_{00}, v_{01}\}$ belongs to at least one of D_1, D_2, D_3 and D_4. □

We assumed $w \in V(H_1)$, where $H_1 := H - \{v_{00}, v_{10}, v_{10}, v_{11}\}$ and w is adjacent to both v_{00} and v_{10}. Suppose that $w \notin \{v_{02}, v_{03}\}$. By Claim 10, there is a dominating set D for H_1 such that $|D| = (|V(H_1)| - 2)/4 + 1$ and $w, v_{12} \in D$. It can be seen in Figure 19 that v_{12} will dominate v_{01} and v_{11}. Also, w will dominate v_{00} and v_{10}. Thus, D is a dominating set for G of size $(|V(H_1)| - 2)/4 + 1 = (n - 4 - 2)/4 + 1 = n/4 - 1/2$. Therefore, we may assume that either $w = v_{02}$ or $w = v_{03}$.

Case 1 $w = v_{02}$.

As shown in Figure 21, if $w = v_{02}$ we can dominate vertices $\{v_{ij} : 0 \leq i \leq 1$ and $0 \leq j \leq 4\}$ with two vertices. The remaining vertices induce a subgraph of H isomorphic to an instance of the $G^{\Delta}_{2\times(q-5)}$-class. Thus, by Lemma 4, it can be dominated with $(q-5)/2$ vertices. The resulting dominating set is of size $(q-5)/2 + 2 = n/4 - 1/2$.

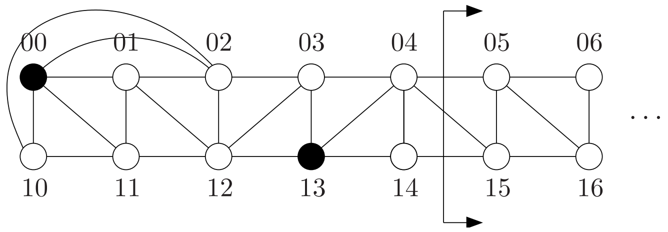


Figure 21: Black vertices dominate the first 10 vertices.

Case 2 $w = v_{03}$.

Since G is a triangulated planar graph, either $v_{00}v_{02}$ or $v_{01}v_{03}$ must be an edge of G. If $v_{00}v_{02} \in E(G)$, then the same dominating set of the previous case is a dominating set for this case, and we are done. Figure 22 illustrates graph G when $v_{00}v_{02} \in E(G)$.

It remains to show that G can be dominated with at most $n/4$ vertices if $v_{01}v_{03} \in E(G)$. Since G is triangulated, there exists $x \in V(G)$ such that x is adjacent to both v_{10} and v_{03} and $x \neq v_{00}$. Since G is a plane graph, $x \notin \{v_{01}, v_{02}\}$. Also, $x \notin \{v_{12}, v_{13}\}$, since it would result in the existence of multiple edges. This case is depicted in Figure 23.

Suppose that $x = v_{11}$. Then, Figure 24 shows that we can dominate the first 10 vertices with v_{03} and v_{13}. The remaining vertices induce a subgraph in the $G^{\Delta}_{2\times(q-5)}$-class that can

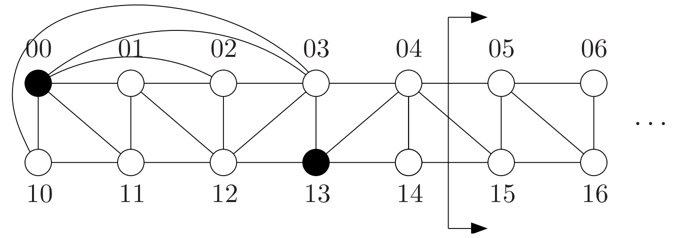


Figure 22: Black vertices dominate the first 10 vertices.

be dominated with $(q-5)/2$ vertices, by Lemma 4. Thus, we can find a dominating set of size $(q-5)/2+2 = n/4 - 1/2$ for G.

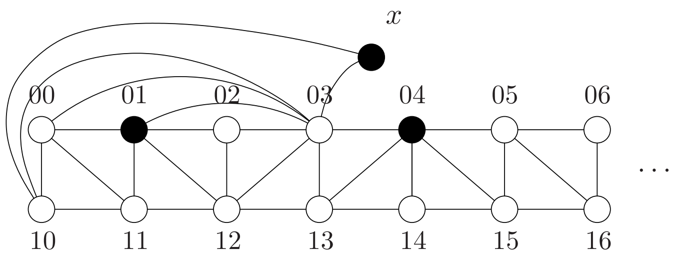


Figure 23: Black vertices dominate the first 8 vertices.

Therefore, we may assume that $x \in V(H_2)$, where H_2 is the subgraph of H induced by $V(H)\setminus\{v_{ij} : 0 \le i \le 1 \text{ and } 0 \le j \le 3\}$. If $q = 5$, then $\{x, v_{01}\}$ is a dominating set for G with $2 < n/4$ vertices. Suppose that $q \ge 7$ and $x \notin \{v_{14}, v_{15}\}$. By Claim 10, there is a dominating set D' for H_2 such that $x, v_{04} \in D'$ and $|D'| = (|V(H_2)| - 2)/4 + 1$. In this dominating set, x dominates v_{10} and v_{03}, and v_{04} dominantes v_{13}. Vertices $\{v_{00}, v_{01}, v_{02}, v_{11}, v_{12}\}$ can be dominated by adding vertex v_{01} to D', as shown in Figure 23. Thus, $D' \cup \{v_{01}\}$ is a dominating set for G that uses $(|V(H_2)|-2)/4+2 = (n-8-2)/4+2 = n/4-1/2$ vertices.

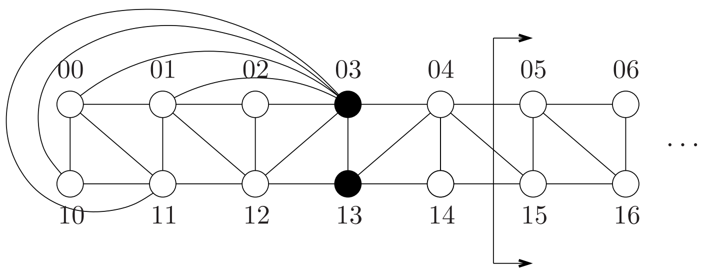


Figure 24: Black vertices dominate the first 10 vertices.

It remains to show that $\gamma(G) \le n/4$ when $x = v_{14}$ or $x = v_{15}$.

Case 2.1 $x = v_{14}$.

This case is illustrated in Figure 25. We can dominate the first 10 vertices using vertices v_{01} and v_{14}. The subgraph induced by the remaining vertices can be dominated using $(q-5)/2$ vertices, by Lemma 4. The resulting dominating set is of size $(q-5)/2+2 = n/4 - 1/2$.

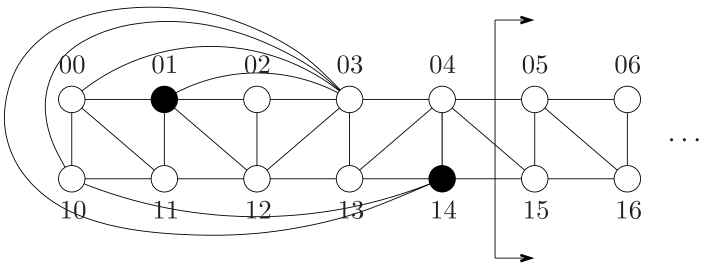


Figure 25: Black vertices dominate the first 10 vertices.

Case 2.2 $x = v_{15}$.

Then, either $v_{10}v_{14} \in E(G)$ or $v_{11}v_{15} \in E(G)$. If $v_{10}v_{14} \in E(G)$, the dominating set of Case 2.1 is a dominating set for this case, and we are done. If $v_{11}v_{15} \in E(G)$, then we can dominate vertices $\{v_{ij} : 0 \leq i \leq 1 \text{ and } 0 \leq j \leq 5\}$ with v_{03} and v_{05}, as shown in Figure 26. By Claim 10, the subgraph of H induced by $V(H) \setminus \{v_{ij} : 0 \leq i \leq 1 \text{ and } 0 \leq j \leq 5\}$ admits a dominating set with $(n-12-2)/4+1$ vertices if $q \geq 9$. This dominating set together with v_{03} and v_{05} form a dominating set for G with $(n-12-2)/4+3 = n/4-1/2$. If $q = 7$, $\{v_{03}, v_{15}, v_{06}\}$ is a dominating set for G with $3 < n/4$ vertices. □

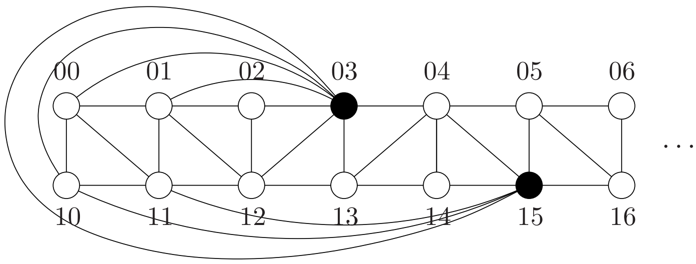


Figure 26: Black vertices dominate the first 12 vertices.

Finally, we present the final result for the $\mathcal{G}^{\Delta}_{p\times q}$-class, summarizing the previous results.

Theorem 11 *Let G be an instance of $\mathcal{G}^{\Delta}_{p\times q}$, $p, q \geq 2$. Then, $\gamma(G) \leq n/4$ for $n > 15$.*

Proof. It follows from Corollary 8 and Lemma 9. □

4 Final Remarks

The domination problem is NP-complete. For this reason, we restrict our attention to finding small dominating sets for some classes of graphs. This study was motivated by Matheson and Tarjan's conjecture [3], which claims that $\gamma(G) \leq n/4$ for n-vertex triangulated planar graphs with n greater than a constant n_0. Other studies on the domination number have investigated generalized Petersen graphs, products of graphs and cubic graphs, among many other classes, as well as dominating sets in conjunction with other properties. The classes of graphs studied in this work present an underlying grid-like structure, which simplifies the task of verifying the conjecture for them. Yet many other interesting restrictions may be made with regard to triangulated planar graphs, such as limiting the maximum/minimum degrees of the triangulation. Using this approach, King and Pelsmajer proved the conjecture for maximal planar graphs with maximum degree at most six and very large n, a result which could be further extended to graphs with greater maximum degree [2]. Reducing the constant n_0 obtained by King and Pelsmajer would also be a stimulating endeavor, and preliminary results show that it is possible to reduce this value by a constant factor.

References

[1] M. R. Garey and D. S. Johnson. *Computers and intractability.* W. H. Freeman and Co., San Francisco, Calif., 1979. a guide to the theory of NP- completeness, A Series of Books in the Mathematical Sciences.

[2] E. L.C. King and M. J. Pelsmajer. Dominating sets in plane triangulations. *Discrete Mathematics*, 310(17-18):2221–2230, 2010.

[3] L. R. Matheson and R. E. Tarjan. Dominating sets in planar graphs. *Europ. J. Combinatorics*, 17:565–568, 1996.