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Colorings and crossings

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Abstract

In 2007, Albertson conjectured that if a graph G has chromatic number k, then the crossing number of G is at least the crossing number of the complete graph with k vertices. To date, two papers were written on this subject trying to solve the conjecture for an arbitrary k-chromatic graph G, and after much effort the conjecture was proved true for $k \leq 16$. In this report, we present an overview of topics related to Albertson's Conjecture, such as: the crossing number problem, lower bounds on crossing number, lower bounds on the number of edges of color-critical graphs, and coloring of graphs with small crossing number and small clique number. While investigating Albertson's conjecture, J. Barát and G. Tóth proposed the following conjecture involving color-critical graphs: for every positive integer c, there exists a bound $k(c)$ such that for any k, where $k \geq k(c)$, any k-critical graph on $k + c$ vertices has a subdivision of K_k. In Section 9, we present counterexamples to this conjecture for every $c \geq 6$ and we prove that the conjecture is valid for $c = 5$.

1 Introduction

A graph G is *planar* if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing is called a *planar embedding* of the graph or a *plane graph*. Every planar graph is colorable with at most four colors by the Four Color Theorem [5, 49]. The efforts to solve the Four Color Problem played an important role in the development of Graph Theory.

The *crossing number* of a graph G, denoted *cr(G)*, is the smallest number of pairwise crossings of edges among all drawings of G in the plane. An *optimal drawing* of G is a drawing of G with exactly $cr(G)$ crossings. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum number of colors required to color the vertices of G, such that any pair of adjacent vertices receive distinct colors.

According to the Four Color Theorem, if a graph G has $cr(G) = 0$, then $\chi(G) \leq 4$. A further step on this problem would be to determine exact values of the maximum chromatic number for graphs with small crossing numbers. Oporowski and Zhao [43] proved that if $cr(G) \leq 2$, then $\chi(G) \leq 5$. They also showed that if $cr(G) = 3$ and G does not contains a copy of K_6, then $\chi(G) \leq 5$.

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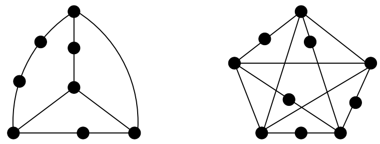
One way to study the relationship between crossings and colorings would be to determine what hypotheses on the crossings force the chromatic number to be relatively small. Albertson [3] studied how the relationship between pairs of crossings can interfere on the chromatic number of graphs; he determined conditions on the crossings that force the chromatic number to be at most 5. In 2007, Albertson conjectured that if a graph G has chromatic number k, then the crossing number of G is at least the crossing number of the complete graph with k vertices. Albertson's conjecture was proved true for $k \leq 16$, but it is still open for the remaining cases.

A *subdivided* K_k is a graph obtained by replacing edges $\{x, y\}$ of the complete graph K_k with $x - y$ paths. We say that G contains K_k as a subdivision if G has a subgraph H that is isomorphic to a subdivided K_k. The *subdivision number* $\sum(G)$ of a graph G is the largest integer k such that G contains a subdivision of K_k as a subgraph. Figure 1 shows subdivided graphs. Observe that if a k-chromatic graph G has K_k as a subdivision, then $cr(G) \geq cr(K_k)$.



(a) (b)

Figure 1: (a) A subdivided K_4. (b) A subdivided K_5.

Let $\omega(G)$ denote the *clique number* of G, the cardinality of a largest clique contained in G. A well-known lower bound on the chromatic number of a graph G states that $\chi(G) \geq \omega(G)$. It is also well-known that this bound is not sharp given that graphs with chromatic number k may not contain K_k as a subgraph, they can have clique number 2. A conjecture attributed to György Hajós states that graphs with chromatic number k contain a subdivision of K_k. Dirac [15] verified the conjecture for $k = 4$. Nevertheless, in 1979, Catlin [13] noticed that the lexicographic product of C_5 and K_3 is an 8-chromatic counterexample to the Hajós' conjecture. He generalized this construction to give counterexamples to Hajós' conjecture for all $k \geq 7$. Also, it was shown by Erdös and Fajtlowicz [20] that almost all graphs are counterexamples. However, the conjecture remains open for $k \in \{5, 6\}$. If Hajós' conjecture were true, Albertson's conjecture would follow, because the crossing number of any graph is at least the crossing number of any of its subdivisions. So, a main challenge of Albertson's conjecture lies on discovering the crossing number of k-chromatic graphs that do not contain K_k as a subdivision.

The *join* of two graphs G and H, denoted $G \vee H$, is the graph obtained by joining all the vertices from graph G to all the vertices of graph H. We say that a vertex v is of *full degree* if v has as neighboors all the other vertices of the graph. The join of a graph G with a vertex v is denoted by $G \vee v$.

The rest of this report is organized as follows. In Section 2 we present the crossing number problem and discuss it for some families of graphs. In Section 3 we present known lower bounds on the crossing number, in terms of the number of edges. In Section 4 we discuss some results on the relationship between crossings and colorings. In Section 5 we present basic results of color-critical graphs. In Section 6 we discuss some methods of construction of color-critical graphs. Lower bounds on the minimal number of edges of color-critical graphs are discussed in Section 7. In Section 8 we present a decomposition of color-critical graphs on small color-critical graphs. Section 9 is about color-critical graphs for which the Hajós conjecture holds. In Section 10 Albertson's conjecture is discussed. Finally, in Section 11 we present concluding remarks.

2 Crossings

We begin this section with some definitions. A *drawing* $\mathcal{D}$ of a graph G in the plane is a mapping of the vertices of G to distinct points of the plane, and the edges of G to open arcs in the plane containing no vertices, but having the corresponding vertex points as its ends, subject to three further conditions:

(i) no two arcs with a common end point meet;

(ii) no two arcs meet in more than one point;

(iii) no three arcs meet in a common point.

We speak about the images of vertices as vertices, and about the curves as edges. Figure 2 shows the prohibited configurations to a drawing and shows that these configurations can be avoided without relocating vertices in the drawing. The reason for the first two conditions is avoiding superfluous crossings; the third condition make the counting of crossings of edges unambiguous.

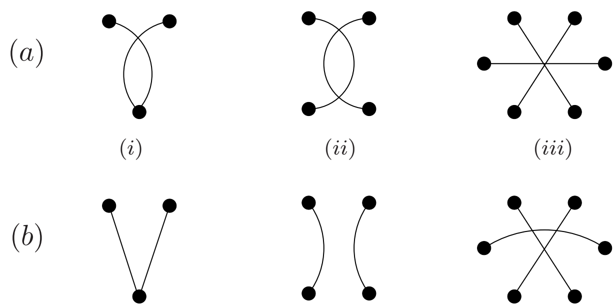


Figure 2: (*a*) Prohibited configurations in drawings. (*b*) How to avoid the prohibited configurations.

We say that two edges in a drawing *cross* in a point of the plane if this point belongs to the interiors of the arcs representing the edges. The point where two edges cross is called a *crossing*. The *crossing number* $cr(G)$ of a graph G is the minimum number of crossings over all drawings of G in the plane. We say that a drawing $\mathcal{D}$ of a graph G is *optimal* if the total number of crossings of $\mathcal{D}$ equals to $cr(G)$.

The study of crossing numbers has origins in the Second World War with Paul Turán. In [54] he tells how the problem ocurred to him while he was working in a brick factory during the war. In the factory, there were some kilns where the bricks were made and some storage yards where the bricks were stored. All kilns were connected by rail with all the storage yards. The work was to bring out bricks from the kilns and carry them on small wheeled trucks to an empty storage yard. The problem was that in the crossing of two rails the trucks jumped out, and the bricks fell down, causing a lot of extra work and loss of time. In this situation, Turán wondered how to design an efficient rail system from the kilns to the storage yards, i.e, how to lay the rails to reduce the number of crossings. This is the problem of finding the crossing number of the complete bipartite graph, and became known as *Turán's brick factory problem.*

In 1952, K. Zarankiewicz submitted a solution for Paul Turán's problem to *Fundamenta Mathematicae* [60]. For some years it was thought that the problem was solved, but flaws in the proof were discovered nearly twenty years later by the graph-theorists P. Kainen and G. Ringel (see [26]), and the problem remains open. Despite his unsuccessful attempt in solving the problem, Zarankiewicz provided the currently best upper bound for $cr(K_{m,n})$. In fact, the standard approach to crossing number problems is to find a drawing with a certain number of crossings, and then to show that no other drawing can have fewer. The following drawing of $K_{m,n}$ in the Cartesian plane gives the best-known upper bound to the crossing number of complete bipartite graphs.

Place $m_1 = \lceil \frac{m}{2} \rceil$ vertices equally spaced on the positive x-axis and $m_2 = \lfloor \frac{m}{2} \rfloor$ vertices equally spaced on the negative x-axis; do it similarly on the y-axis, place $n_1 = \lceil \frac{n}{2} \rceil$ vertices equally spaced on the positive y-axis and $n_2 = \lfloor \frac{n}{2} \rfloor$ vertices equally spaced on the negative y-axis. Then join all vertices on the x-axis to all those on the y-axis by straight-line segments. Figure 3 shows the drawing of $K_{5,4}$ as described in the above construction.

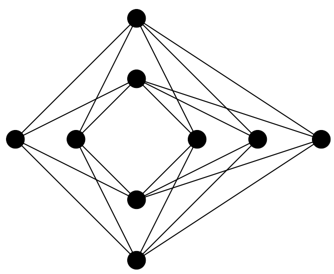


Figure 3: An optimal drawing of $K_{5,4}$.

The number of crossings produced by this construction can be computed from the following observation [8]: For each set of four vertices, with two in the same half of the x-axis and two in the same half of the y-axis, there is exactly one crossing. Therefore, the total number of crossings is

$$\binom{m_1}{2}\binom{n_1}{2} + \binom{m_1}{2}\binom{n_2}{2} + \binom{m_2}{2}\binom{n_1}{2} + \binom{m_2}{2}\binom{n_2}{2},$$

which equals

$$\left(\binom{m_1}{2}+\binom{m_2}{2}\right)\left(\binom{n_1}{2}+\binom{n_2}{2}\right). \tag{1}$$

To reduce expression (1) to a simpler one, we use some known mathematical results. Note that for every natural number k:

$$\left\lceil\frac{k}{2}\right\rceil=\left\lfloor\frac{k+1}{2}\right\rfloor,\quad \left\lceil\frac{k}{2}\right\rceil+\left\lfloor\frac{k}{2}\right\rfloor=k$$

and

$$\binom{\lceil\frac{k}{2}\rceil}{2}+\binom{\lfloor\frac{k}{2}\rfloor}{2}=\left\lfloor\frac{k}{2}\right\rfloor\left\lfloor\frac{k-1}{2}\right\rfloor. \tag{2}$$

Therefore, by applying (2) in expression (1), we obtain that the number of crossings of $K_{m,n}$ in this drawing is:

$$\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor. \tag{3}$$

Zarankiewicz conjectured that expression (3) is also a lower bound to $cr(K_{m,n})$. Therefore we have the following.

Conjecture 1. *(Zarankiewicz).* $cr(K_{m,n})=\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$.

In 1970, Kleitman [34] proved that Conjecture 1 holds for every $K_{m,n}$ with $min\{m,n\}\leq 6$. In 1993, Woodall [59] proved it for $m\leq 8, n\leq 10$. However, Conjecture 1 is still open for the remaining cases.

It is known that if Conjecture 1 holds for $K_{m,n}$ for any odd value of m (with n fixed), then it also holds for the next even value of m [58]. This is true because of the following general counting argument. Let m,n,r,s be integers, such that $m\geq r$ and $n\geq s$. Observe that each drawing of $K_{m,n}$ contains $\binom{m}{r}\binom{n}{s}$ drawings of $K_{r,s}$, each obtained by selecting r and s vertices, respectively, from the two partite sets; but each crossing involves two vertices from each partite set and is thus counted $\binom{m-2}{r-2}\binom{n-2}{s-2}$ times. Thus, for $m\geq r$ and $n\geq s$, we have

$$cr(K_{m,n})\geq\frac{\binom{m}{r}\binom{n}{s}*cr(K_{r,s})}{\binom{m-2}{r-2}\binom{n-2}{s-2}}.$$

Hence, if Conjecture 1 is true for $K_{r,s}$ with $r=2k-1$, $s=n$, and $m=2k$, we have

$$cr(K_{m,n})\geq\frac{m}{m-2}*cr(K_{r,s}). \tag{4}$$

Doing some calculation we find that lower bound (4) agrees with the upper bound for $cr(K_{m,n})$.

The *complete graph* on k vertices is the graph K_k having k vertices such that every pair is joined by an edge. This is another classic family of graphs for which the crossing number problem is considered. Similarly to the crossing number of complete bipartite graphs, the computation of $cr(K_k)$ has shown to be a challenging task. To date, the exact value of

$cr(K_k)$ is only known for $k \leq 12$. The following construction provides the best known upper bound to $cr(K_k)$; it was first known by Antony Hill [7], but an equivalent variation of it was shown by J.Blažek and N.Koman [9].

Label the vertices of K_k as $1, 2. \ldots, k$. Place the odd numbered vertices equally spaced around the inner of two concentric circles and the even ones around the outer circle. Then join all pairs of odd vertices inside the inner circle, join all pair of even vertices outside the outer circle, and join even vertices to odd vertices in the region between the circles. Figure 4 shows complete graphs drawn according to this construction.

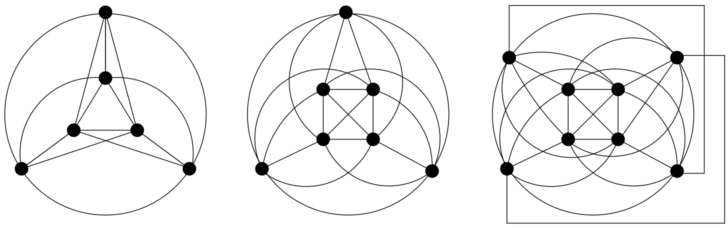


Figure 4: Optimal drawings of K_6, K_7, and K_8.

It can be shown that the above construction provides a drawing of K_k with exactly the following number of crossings: $\frac{1}{4}\lfloor\frac{k}{2}\rfloor\lfloor\frac{k-1}{2}\rfloor\lfloor\frac{k-2}{2}\rfloor\lfloor\frac{k-3}{2}\rfloor$. R. Guy [25, 27] conjectured that this expression is also a lower bound to $cr(K_k)$, thus we have the following conjecture:

Conjecture 2. *(Guy).* $cr(K_k) = \frac{1}{4}\lfloor\frac{k}{2}\rfloor\lfloor\frac{k-1}{2}\rfloor\lfloor\frac{k-2}{2}\rfloor\lfloor\frac{k-3}{2}\rfloor$.

In 1972, R. Guy [27] proved that Conjecture 2 holds for $k \leq 10$. In 2007, Pan and Richter [47] proved it to $k = 11, 12$. However, the conjecture remains open for the other cases.

It is known that if Conjecture 2 holds for K_{2p-1}, then it also holds for K_{2p}, where $p \geq 3$. This is true because of a general counting argument: Let k, s be integers, such that $k \geq s$. Observe that K_k contains $\binom{k}{s}$ subgraphs K_s. Each crossing of K_k is formed by exactly 4 vertices and so occurs in exactly $\binom{k-4}{s-4}$ such subgraphs. Thus

$$cr(K_k) \geq \frac{\binom{k}{s} * cr(K_s)}{\binom{k-4}{s-4}}. \tag{5}$$

In particular, if we put $s = 2p - 1$ and $k = 2p$, we see that if Conjecture 2 holds for K_s, then it also holds for K_k. This argument also implies that the removal of any vertex from an optimal drawing of K_{2p} leaves an optimal drawing of K_{2p-1}. If we put $r = 5$ in inequality (5), we get:

$$cr(K_k) \geq \frac{\binom{k}{5} * cr(K_5)}{\binom{k-4}{5-4}}.$$

which gives us

$$cr(K_k) \geq \frac{1}{120} n(n-1)(n-2)(n-3). \tag{6}$$

A second lower bound due to Guy [27] is obtained by using the following result of Kleitman [34]:

$$cr(K_{5,n}) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{7}$$

We can get a lower bound for $cr(K_k)$ in terms of $cr(K_{r,k-r})$. To obtain this, we note that $K_{r,k-r}$ can be taken as a subgraph of K_k in $\binom{k}{r}$ ways. In counting crossings in all such subgraphs, we count a given crossing in $4\binom{k-4}{r-2}$ ways, since there are 4 ways of choosing two of the crossing contributors to be in the set of r vertices, and $\binom{k-4}{r-2}$ ways of completing that set. Thus,

$$cr(K_k) \geq \frac{\binom{k}{r} * cr(K_{r,k-r})}{4\binom{k-4}{r-2}}. \tag{8}$$

In particular, with $r = 5$, we obtain

$$cr(K_k) \geq \frac{\binom{k}{5} * cr(K_{5,k-5})}{4\binom{k-4}{5-2}}.$$

Applying result (7) with $n = k - 5$ in inequality (8), we obtain

$$cr(K_k) \geq \frac{n(n-1)(n-2)(n-3)}{20(n-5)(n-6)} \left\lfloor \frac{n-5}{2} \right\rfloor \left\lfloor \frac{n-6}{2} \right\rfloor. \tag{9}$$

In [48], Richter and Thomassen analyse the relations between the crossing numbers of complete and complete bipartite graphs. The problem of computing the crossing number of a graph is NP-complete [24]. Székely [53] presents a survey on foundational issues related to the crossing number and a bibliography of papers on crossing number is maintained by I. Vrt'o [55].

3 Lower bounds on crossing number

A consequence of the Euler's formula[1] is that every planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges. This fact provides the following lower bound for the crossing number of a simple graph G with $n \geq 3$ vertices and m edges:

$$cr(G) \geq m - 3(n-2) \tag{10}$$

J. Pach et al. [45, 46] improved this idea and proved the following lower bounds on the crossing number:

$$cr(G) \geq \frac{7}{3}m - \frac{25}{3}(n-2) \tag{11}$$

[1] Let G be a connected plane graph. Euler's formula is: $|V(G)| - |E(G)| + f(G) = 2$, where $f(G)$ is the number of faces in the planar embedding of G.

$$cr(G) \geq 4m - \frac{103}{6}(n-2) \tag{12}$$

$$cr(G) \geq 5m - 25(n-2) \tag{13}$$

Pach et al. [45] also determined when these inequalities are best to be used: inequality (10) is best when $m \leq 4(n-2)$, inequality (11) is best when $4(n-2) \leq m \leq 5.3(n-2)$, inequality (12) is best when $5.3(n-2) \leq m \leq \frac{47}{6}(n-2)$, and inequality (13) is best for $m \geq \frac{47}{6}(n-2)$.

Using a probabilistic approach, J. Barát and G. Tóth [6] proved the following result.

Theorem 3. *[6] Suppose that $n \geq 10$, and $0 < p \leq 1$. Let*

$$cr(n,m,p) = \frac{4m}{p^2} - \frac{103n}{6p^3} + \frac{103}{3p^4} - \frac{5n^2(1-p)^{n-2}}{p^4}.$$

For any graph G with n vertices and m edges, the following holds:

$$cr(G) \geq cr(n,m,p).$$

Ajtai, Chvátal, Newborn, and Szemerédi [2], and independently Leighton [41], proved the following result conjectured by Erdös and Guy [28].

Lemma 4. *(The Crossing Lemma) Let G be a simple graph with $m \geq 4n$. Then*

$$cr(G) \geq \frac{1}{64}\frac{m^3}{n^2}. \tag{14}$$

The original constant factor was much larger. The constant $\frac{1}{64}$ comes from the well-known probabilistic proof of Chazelle, Sharir, and Welzl [1]. The basic ideia is to take a random induced subgraph and apply inequality (10) for that. Pach et al. [45] proved that the order of magnitude of this bound cannot be improved. The best known constant factor to this lemma is due to Pach et al. [45]. Using inequality (12) they showed for $m \geq \frac{103}{16}n$ that

$$cr(G) \geq \frac{1}{31.1}\frac{m^3}{n^2}. \tag{15}$$

4 Crossings and colorings

One of the first papers to consider coloring of graphs with a given crossing number is due to Oporowski and Zhao [43]. In their paper, the authors extended the Five Color Theorem to graphs with at most 2 crossings:

Theorem 5. *[43] If $cr(G) \leq 2$, then $\chi(G) \leq 5$.*

Since the graph K_6 has crossing number 3, it is not possible to extend Theorem 5 without additional hypotheses. Oporowski and Zhao [43] considered graphs with small crossing number and clique number:

Theorem 6. *[43] If $cr(G) \leq 3$ and $\omega(G) \leq 5$, then $\chi(G) \leq 5$.*

However, the authors highlighted that the assumption $cr(G) \leq 3$ could be relaxed. To illustrate it, they considered the following example: Let $K_3 \vee C_5$ denote the join of a K_3 and a 5-cycle. They proved that $\chi(K_3 \vee C_5) = 6$, $\omega(K_3 \vee C_5) = 5$, and $cr(K_3 \vee C_5) = 6$. Based on this graph, the authors posed the question: Can a graph G with $cr(G) \leq 5$ and $\omega(G) \leq 5$ be colorable with 5 colors? In 2011, Erman et al. [21] solved this question proving the following result.

Theorem 7. *[21] If $cr(G) \leq 4$ and $\omega(G) \leq 5$, then $\chi(G) \leq 5$.*

Additionaly, the authors also showed a graph G with $cr(G) \leq 5$ and $\omega(G) = 5$ which is not 5-colorable, thus answering in the negative the original question posed by Oporowski and Zhao. The counterexample can be constructed in the following way: take two copies of $K_6 - e$ with non-edges x_1y_1 and x_2y_2, respectively. Let $x_1u_1v_1$ and $x_2u_2v_2$ be two triangles in these copies, and identify the corresponding vertices in these triangles. Finally, add the edge y_1y_2. Figure 5 shows a drawing of this graph with 5 crossings.

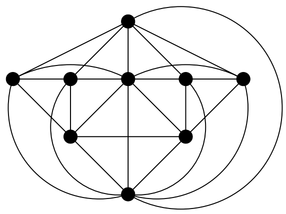


Figure 5: A graph G with $\chi(G) = 6$, $\omega(G) = 5$, and $cr(G) \leq 5$.

In the context of crossings and colorings, let us consider the following question:

Question 8. *What hypotheses on the crossings force the chromatic number of a graph to be relatively small?*

Albertson [3] worked on Question 8 examining the incidence relation between vertices and crossings. Let G be a simple graph, and let $e = uv$ and $f = xy$ be two edges that cross in a given drawing of G. We say that $\{u, v, x, y\}$ is the *cluster* of vertices associated with the crossing $\{ef\}$, and the crossing and the vertices of the cluster are said to be incident with each other. A *crossing cover* of a drawing of G is a set of vertices C such that every crossing has an edge incident with a vertex in C. Let $\mathcal{D}$ be an optimal drawing of G, and let $\mathcal{C}$ be a crossing cover of $\mathcal{D}$ with the smallest cardinality. The *crossing covering number* of G, denoted by $\rho(G)$, is equal to the cardinality of $\mathcal{C}$. The following result is an easy implication of the definition of crossing covers.

Theorem 9. *[3] $\chi(G) \leq 4 + cr(G)$.*

Proof. Let C be a minimum crossing cover in an optimal drawing of G, so $|C| = \rho(G)$. We have that $G - C$ is planar, so the vertices of $G - C$ can be colored using colors 1, 2, 3, 4. The vertices in C can each be assigned a unique color using colors $5, 6, \ldots, 4 + \rho(G)$. Therefore, we use $4 + \rho(G)$ colors to color G. Noting that $\rho(G) \leq cr(G)$ we immediately get $\chi(G) \leq 4 + cr(G)$. □

The upper bound in Theorem 9 is tight only for $cr(G) \leq 1$. In [4], Albertson et al. proved that $\chi(G) = O(cr(G)^{1/4})$. This result is best possible, since $\chi(K_k) = k$ and $cr(K_k) \leq \binom{|E(K_k)|}{2} = \binom{\binom{k}{2}}{2} \leq \frac{k^4}{8}$.

In a drawing of a graph G in the plane, two crossings are said to be *dependent* if their clusters have at least one vertex in common, and are said to be *independent* if no two are dependent. Similarly, two clusters are said to be *dependent* if they have at least one vertex in common, and are said to be *independent* if they have no vertex in common. A drawing is *independent* if its clusters are pairwise disjoint, and is *dependent* if it has at least two dependent clusters. Figure 6 illustrate these definitions.

Dependent clusters: $\{a, f, b, e\}$ and $\{b, g, c, f\}$
Independent clusters: $\{a, f, b, e\}$ and $\{c, h, d, g\}$

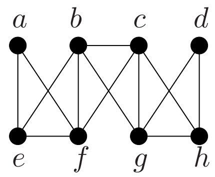


Figure 6: A dependent drawing and examples of dependent and independent clusters.

Albertson [3] studied the the effect of dependent crossings in the crossing number of a graph and proved the following result.

Theorem 10. *[3] If a graph G has an independent drawing in the plane in which there are at most three crossings, then $\chi(G) \leq 5$.*

Proof. By Theorem 5 we may assume that there are exactly three crossings[2]. For $1 \leq i \leq 3$, let $C_i = \{u_i, v_i, x_i, y_i\}$ denote the cluster of vertices associated with the i^{th} crossing. By the independence of the crossings these vertices are all distinct. We assume that the vertices are labeled so that u_iv_i crosses x_iy_i. We may assume that each C_i induces a K_4. If it does not, we can add edges to the cluster without creating any additional crossings by joining, for example, u_i to x_i, with an edge that is drawn within a small neighboorhood of the crossing pair of edges. This is illustrated with the dotted edge in Figure 7. Consider $G_0 = G[\cup_{j=1}^{3} C_j]$, the subgraph of G induced by the clusters C_1, C_2, and C_3. $|V(G_0)| = 12$. Since G_0 has three independent crossings, inequality (10) implies that $|E(G_0)| \leq 33$. Since we are assuming each cluster contains 6 edges, there are at most 15 edges joining vertices in one cluster with vertices in another.

Claim 11. *$\exists\ Z = \{z_1, z_2, z_3 : z_i \in C_i$ and Z is an independent set $\}$.*

Proof of claim. Since there are four choices for each z_i, there exist 64 candidates for Z. If the claim fails for a particular candidate set, there must be at least one edge joining two of the three vertices of Z. Each edge joining two vertices in different clusters is contained in 4 candidate sets. Thus, if the claim fails there must be at least 16 edges joining vertices in different clusters. Since there are at most 15 such edges, the claim holds. □

[2]The proof of Theorem 10 was written here exactly as it appears in Albertson's paper [3].

Continuing the proof of Theorem 10, given the independent set $Z = \{z_1, z_2, z_3\}$ for $1 \leq i \leq 3$ let e_i denote the crossing edge in C_i that is incident with z_i. Set $G_1 = G - \{\cup_{i=1}^{3} e_i\}$. Since G_1 is planar, we 4-color G_1 using colors $\{1, 2, 3, 4\}$. We then transfer this coloring to G and recolor each z_i with the color 5. □

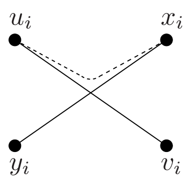


Figure 7: Clusters induce K_4's.

In the same paper [3], Albertson conjectured that all graphs having independent drawings are 5-colorable. In the proof of Theorem 10, the author used the fact that every independent drawing with exactly three crossings has a crossing cover C that is an independent set. Albertson [3] asked for the maximum k such that every independent drawing with k crossings has a crossing cover that is an independent set. P.Wenger [57] proved that $k \leq 4$, and showed a graph with five independent crossings that contains no crossing cover that is an independent set. Figure 8 shows P. Wenger's example.

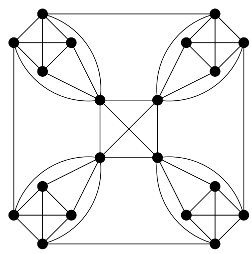


Figure 8: An independent drawing with five crossings such that no independent set of size 5 contains one vertex from each cluster.

Although an independent drawing with five or more crossings need not to have a crossing cover that is an independent set, nevertheless Král and Stacho [38] proved Albertson's conjecture that all graphs having independent drawings are 5-colorable:

Theorem 12. *[38] If a graph G has an independent drawing in the plane, then* $\chi(G) \leq 5$.

In 2007, at an American Mathematical Society special session, Albertson conjectured that if a graph G is k-chromatic, then $cr(G) \geq cr(K_k)$. Note that Albertson's conjecture is trivially true for $k \leq 4$, since every graph has crossing number at least zero. For $k = 5$ the conjecture is equivalent to the Four Colour Theorem. The case $k = 6$ of the conjecture is equivalent to the contrapositive of Theorem 5, so it was verified by Oporowski and Zhao. Albertson, Cranston, and Fox [4] verified the conjecture for $7 \leq k \leq 12$. Finally, Barát and Tóth [6] improved the previous results verifying Albertson's conjecture for $13 \leq k \leq 16$. Despite all these efforts, Albertson's conjecture is still open for the remaining cases.

So far, we have discussed the crossing number and relationships between crossings and colorings. In the next sections we present the class of color-critical graphs and discuss several results about this class of graphs. It seems that a deep understanding on the properties of color-critical graphs can be useful for approaching Albertson's conjecture, but only this will not be sufficient if, subject to it, we don't have better lower bounds on the crossing number of k-chromatic graphs with small number of edges. Since we even do not know the exact crossing number of complete graphs on more than 13 vertices, solving this conjecture is expected to be somewhat challenging.

5 Color-critical graphs

In 1951, Dirac [14] introduced the concept of *color-critical* graphs in order to simplify the study of coloring problems. Instead of investigating the class of all graphs with a given chromatic number k one studies those graphs of this class which are minimal in some respect. Let G be an arbitrary k-chromatic graph. We say that G is *color-k-critical* if $\chi(H) < \chi(G)$ for every proper subgraph H of G. A graph G is *vertex-critical* if G is k-chromatic and for any vertex $v \in V(G)$, $\chi(G - v) = k - 1$. Similarly, G is *edge-critical* if G is k-chromatic and for any edge $e \in E(G)$, $\chi(G - e) = k - 1$. It is easy to see that every edge-critical graph is also vertex-critical, but the converse is not true. In 1970, Dirac conjectured that, for every integer $k \geq 4$, there is a vertex-k-critical graph with no critical edges [19]. In fact, for almost all values of $k \geq 5$, this conjecture is known to be true [12, 31, 40], but it is open for the other cases. For simplicity, we abbreviate the term 'edge-critical' to 'critical'. A *k-critical* graph is one that is k-chromatic and critical. The next lemmas and theorems are classic results about k-critical graphs.

Lemma 13. *[14] Every k-chromatic graph contains a k-critical subgraph.*

Lemma 14. *[14] If G is k-critical, then $\delta(G) \geq k - 1$.*

Theorem 15. *[16] If G is k-critical, then G is $(k-1)$-edge-connected.*

Theorem 16. *If G is k-critical, then G has no cutset of vertices inducing a clique.*

A maximal subgraph H of a graph G such that any two edges of H are contained in a circuit of G is called a *block* of G. A vertex $v \in V(G)$ shall be called a *low vertex* when $d(v) = k - 1$. The subgraph of G induced by the set of low vertices is called the *low vertex subgraph* of G, denoted by $L(G)$. Gallai [22] proved the following fundamental result about the structure of k-critical graphs.

Theorem 17. *[22] If G is k-critical, then $L(G)$ is a graph whose blocks are complete graphs or odd cycles.*

6 Construction of color-critical graphs

In this section, some known methods of construction of color-critical graphs are presented. For a detailed review of the methods of constructing critical graphs, see the survey written by M. Stiebtiz and M. Sachs [51].

Let S be a vertex cut of a graph G, and let the components of $G - S$ have vertex sets $V_1, V_2, \ldots, V_t$. We define the *S-components* of G as the subgraphs $G_i := G[V_i \cup S]$. By Theorem 16, if a k-critical graph has a 2-vertex-cut $\{u, v\}$, then u and v cannot be adjacent. We say that a $\{u, v\}$-component G_i of G is of *Type 1* if every $(k-1)$-coloring of G_i assigns the same color to u and v, and of *Type 2* if every $(k-1)$-coloring of G_i assigns distinct colors to u and v.

Let G be a k-critical graph that has a 2-vertex cut $S = \{u, v\}$, so G has at least two S-components. Since G is k-critical, we have that each S-component G_i of G is $(k-1)$-colorable. Let G_i be an arbitrary S-component of G. Observe that in the various $(k-1)$-colorings of G_i, it cannot occur that in one coloring the vertices u and v are assigned the same color and in another coloring u and v are assigned distinct colors. If that were possible, we could color the S-components G_i in such a way that the colors in the vertices u, v were the same in each of the S-components, which would lead to a $(k-1)$-coloring of G, a contradiction. Therefore, all S-components of G fall into two classes, depending on the colors at u and v in any $(k-1)$-coloring. So let G_1 and G_2 be two S-components of G such that, G_1 is Type 1 and G_2 is Type 2. Obviously, $G_1 \cup G_2$ is k-colorable, and since G is k-critical, we have that $G = G_1 \cup G_2$. Therefore, we conclude that every k-critical graph G with a 2-vertex cut $S = \{u, v\}$ has exactly two S-components G_1 and G_2, such that, G_1 is Type 1 and G_2 is Type 2.

Based on this property of the S-components of a k-critical graph with a 2-vertex cut, Dirac [16] showed that given a k-critical graph G with a 2-vertex cut, we can obtain from it two smaller k-critical graphs H_1 and H_2. This is a *decomposition procedure* and is explained as follows: Let G be a k-critical graph with a 2-vertex cut $S = \{u, v\}$. As discussed above, G has two S-components G_1, G_2 labelled so G_1 is Type 1 and G_2 is Type 2. We can obtain new k-critical graphs H_1 and H_2 from G in the following way:

1. $H_1 = G_1 + uv$. Since G_1 is Type 1, it is clear that H_1 is k-colorable. To prove that H_1 is critical we remove any edge e from it and show that $H_1 - e$ is $(k-1)$-colorable. If $e = uv$, then $H_1 - e = G_1$, and so it is $(k-1)$-colorable. So, let e be any other edge of H_1. Observe that $G - e$ has a $(k-1)$-coloring in which the vertices u and v receive distinct colors, because G_2 is a subgraph of $G - e$. The restriction of such a coloring to the subgraph G_1 is a $(k-1)$-coloring of $H_1 - e$. Thus, H_1 is k-critical.

2. H_2 is obtained from G_2 by identifying the two vertices u and v. Since G_2 is Type 2, it is clear that H_2 is k-colorable. Now, let us prove that H_2 is critical. Let e be any edge of H_2. Observe that $G - e$ has a $(k-1)$-coloring in which the vertices u and v both receive the same color, because G_1 is a subgraph of $G - e$. The restriction of such a coloring to the subgraph G_2 is a $(k-1)$-coloring of $H_2 - e$. Therefore, H_2 is k-critical.

Complementarily to this decomposition procedure there exist a *composition procedure* [44, Chapter 11]. This procedure is described as follows.

Composition procedure. Let H_1 and H_2 be two disjoint k-critical graphs, $k \geq 3$. We remove an edge $e_1 = ab$ from H_1 to obtain the graph $G_1 = H_1 - e_1$. This is a connected graph since H_1 has no separating vertices. In H_2 we select a vertex v and divide the edges at v into two disjoint non empty classes A and B. This is possible since there are at least $k-1 \geq 2$ edges at v. We separate v into two new vertices v_1 and v_2 letting the edges in A go to v_1 and the edges in B to v_2. The resulting graph is denoted by G_2; it is also connected. Finally, we obtain the composed graph $G = G_1 \cup G_2$ by identifying a and v_1 as well as b and v_2.

Theorem 18. *[44] The graph G obtained by the composition procedure is k-critical provided the graph G_2 can be colored in $k-1$ colors.*

In 1961, Hajós [29] presented a simple procedure to construct all the graphs of chromatic number k, $k \geq 3$. A graph is *Hajós-k-constructible* if it can be obtained from complete graphs K_k by repeated applications of the following two operations:

1. **Conjunction.** Let G_1 and G_2 be already obtained disjoint graphs with edges x_1y_1 and x_2y_2, respectively. Remove x_1y_1 and x_2y_2, identify x_1 and x_2, and join y_1 and y_2 by an edge.

2. **Coalition.** Identify independent vertices, replacing multiple by single edges if any result from the identification.

It is not difficult to see that the chromatic number of any graph obtained from k-chromatic graphs by these operations is at least k. In fact, if a graph G' obtained after a coalition operation into two non-adjacent vertices $x, y \in V(G)$ has a coloring with fewer than k colors, then this defines such a coloring also for G, a contradition. Similarly, in any coloring of a graph constructed by a conjunction operation, the vertices y_1 and y_2 do not both have the same color as x, so this coloring induces a coloring of either G_1 or G_2 and hence uses at least k colors. Hajós [29] proved that the converse also holds:

Theorem 19. *(Hajós' Theorem [29]) Let G be a graph and $k \in \mathbb{N}$. Then $\chi(G) \geq k$ if and only if G has a Hajós-k-constructible subgraph.*

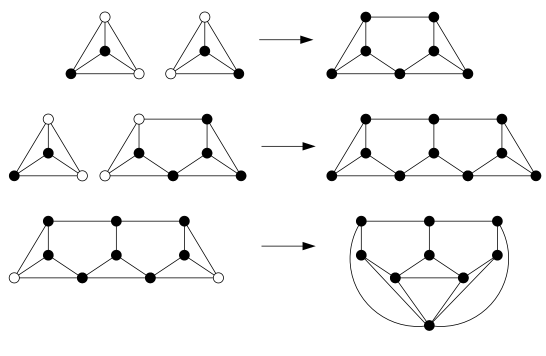


Figure 9: Hajós-4-construction of a graph G.

Hajós' Theorem offers a method of proving that a given graph G has $\chi(G) \geq k$. Figure 9 presents a Hajós-4-construction of a graph G proving that G has chromatic number at least

4. Hajós asked whether any k-critical G can be obtained in such a way that all intermediary graphs are also k-critical (see [32], Problem 11.7). The answer to Hajós' question is trivially true for $k \leq 3$. However, an example given by P.A.Catlin [13] shows, as proved in [30], that the answer is negative for $k = 8$, that is, any Hajós-8-construction of Catlin's example produces a noncritical intermediary graph. T. Jensen and G. Royle [33] proved that the answer to Hajós' question is negative for all $k \geq 4$.

Theorem 20. *[33] For every $k \geq 4$ there exists a k-critical graph that allows no Hajós-k-construction such that all intermediary graphs are k-critical.*

In fact, in the proof of Theorem 20 the authors showed that, for each $k \geq 4$, there exist a k-critical graph G such that the last step of any possible Hajós-k-consruction of G consists in an application of a coalition operation to a graph that is not critical. Therefore, the following question remains unsolved for all $k \geq 4$.

Open Problem 21. *[33] Does every k-critical graph allow a Hajós-k-construction in which every intermediary graph ocurring in the conjunction operation is k-critical?*

7 Lower bounds on the number of edges of color-critical graphs

Let G be a k-critical graph with n vertices and m edges. Since G is k-critical, we have that $\delta(G) \geq k-1$, and therefore, $2m \geq (k-1)n$. The value $2m-(k-1)n$ is the *excess* of G, denoted $\epsilon_k(G)$, and is formally defined as

$$\epsilon_k(G) = \sum_{x \in V(G)} (deg(x) - (k-1)) = 2m - (k-1)n.$$

Since every vertex of G has degree at least $k-1$, we have that $\epsilon_k(G) \geq 0$. By Brook's theorem[3] [11], $\epsilon_k(G) = 0$ if and only if G is a complete graph or an odd cycle. In 1957, Dirac [17] proposed the problem of determine the minimum number of edges of a k-critical graph with n vertices, $n \geq k \geq 4$, and he proved the following result.

Theorem 22. *[17] Let G be a k-critical graph with n vertices and m edges. For $k \geq 4$ and $n \geq k+2$,*

$$2m \geq (k-1)n + (k-3).$$

Short proofs of Theorem 22 were given by Kronk and Mitchem [39] and Weinstein [56]. For $k \geq 4$, Dirac [18] gave a complete characterization of the k-critical graphs that have excess $k-3$.

For $k \geq 3$, let $\mathcal{D}_k$ denote the family of all graphs G whose vertex set consists of three non-empty pairwise disjoint sets A, B_1, B_2 with $|B_1| + |B_2| = |A_1| + 1 = k-1$ and two additional vertices a, b such that A and $B_1 \cup B_2$ are cliques in G not joined by any edge, $N_G(a) = A \cup B_1$ and $N_G(b) = A \cup B_2$. Figure 10 shows a scheme of this family. It is easy

[3]Brook's theorem: If G is a connected graph other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$.

to see that the graphs in family $\mathcal{D}_k$ have $2k-1$ vertices. Observe that these graphs have chromatic number k, and they contain a subdivision of K_k. Therefore, they satisfy the Hajós' conjecture.

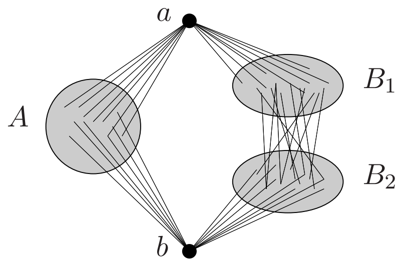


Figure 10: The family $\mathcal{D}_k$.

In 1963, Gallai [23] proved that every k-critical graph with at most $2k-2$ vertices is the join of two smaller graphs, which implies that the complement of such graphs is disconnected. Based on this observation, Gallai proved that non-complete k-critical graphs on at most $2k-1$ vertices have much larger excess than in Dirac's result.

Theorem 23. *[23] Let $k \geq 4$ be an integer and let G be a k-critical graph with n vertices and m edges. If $p := n-k$ is such that $2 \leq p \leq k-1$, then $2m \geq (k-1)n + p(k-p) - 2$. Equality holds if and only if G is the join of K_{k-p-1} and a graph in $\mathcal{D}_{p+1}$.*

Since every graph G in $\mathcal{D}_{p+1}$ contains a subdivision of K_{p+1}, the join of K_{k-p-1} and G contains a subdivision of K_k. This observation yields the following corollary.

Corollary 24. *[6] Let k, p be integers, $k \geq 4$ and $2 \leq p \leq k-1$. If G is a k-critical graph with n vertices and m edges, where $n = k+p$, and G does not contain a subdivision of K_k, then $2m \geq (k-1)n + p(k-p) - 1$.*

Kostochka and Stiebitz [37] presented a slightly different family of k-critical graphs with $2k-1$ vertices. For $k \geq 3$, let $\mathcal{E}_k$ denote the family of all graphs G whose vertex set consists of four non-empty pairwise disjoint sets A_1, A_2, B_1, B_2, and one additional vertex c. We have $|B_1| + |B_2| = |A_1| + |A_2| = k-1$ and $|A_2| + |B_2| \leq k-1$. Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Each of the sets A and B induce a clique in G. We also have $N_G(c) = A_1 \cup B_1$. A vertex a in A is adjacent to a vertex b in B if and only if $a \in A_2$ and $b \in B_2$. Figure 11 presents a scheme of this family. Obviously, every such graph G has $2k-1$ vertices and independence number 2. Consequently, $\chi(G) \geq k$. Moreover, the deletion of any edge results in a graph that is $(k-1)$-colorable. Therefore, G is k-critical with $2k-1$ vertices. Additionally, observe that $\mathcal{D}_k \subseteq \mathcal{E}_k$. Kostochka and Stiebitz [37] improved Dirac's bound proving the following result.

Theorem 25. *[37] Let k be a positive integer, $k \geq 4$, and let G be a k-critical graph. If G is neither a K_k nor a member of $\mathcal{E}_k$, then $2m \geq (k-1)n + 2(k-3)$.*

Barát and Tóth [6] proved that every k-critical graph G of the family $\mathcal{E}_k$ contains a subdivision of K_k. Based on this result they derived the the following corollary from Theorem 25.

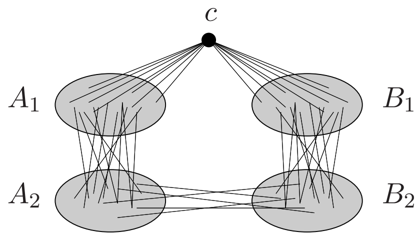


Figure 11: The family $\mathcal{E}_k$.

Corollary 26. *[6] Let k be a positive integer, $k \geq 4$, and let G be a k-critical graph. If G does not contain a subdivision of K_k, then $2m \geq (k-1)n + 2(k-3)$.*

In the following, $f_k(n)$ denotes the minimum number of edges of a k-critical n-vertex graph. In his fundamental papers on color-critical graphs [22, 23], Gallai found exact values of $f_k(n)$ for $k+2 \leq n \leq 2k-1$.

Theorem 27. *[23] If $k \geq 4$ and $k+2 \leq n \leq 2k-1$, then*

$$f_k(n) = \frac{1}{2}((k-1)n + (n-k)(2k-n)) - 1.$$

First, observe that Theorem 27 is an extension of Theorem 23 considering only the k-critical graphs with at most $2k-1$ vertices. Second, observe that Theorem 27 does not include graphs with $2k$ vertices. However, these graphs are considered in the following result due to Kostochka and Stiebitz [37].

Theorem 28. *[37] If $k \geq 4$, then $f_k(2k) = k^2 - 3$.*

Proof. Theorem 25 implies $f_k(2k) \geq k^2 - 3$. To prove that $f_k(2k) \leq k^2 - 3$, we present a family of k-critical graphs on $2k$ vertices and $k^2 - 3$ edges. For $k \geq 4$, let $\mathcal{F}_k$ denote the family of all graphs G whose vertex set consists of four non-empty pairwise disjoint sets A, B_1, B_2, B_3 where $|B_1| + |B_2| + |B_3| = |A| + 1 = k - 1$ and three aditional vertices c_1, c_2, c_3 such that A and $B_1 \cup B_2 \cup B_3$ are cliques in G not joined by any edge and, for $i \in \{1, 2, 3\}$, $N_G(c_i) = A \cup B_i$.

Obviously, every such graph G has $2k$ vertices. To see that G has $k^2 - 3$ edges, note that

$$\begin{aligned} m(G) &= m(B_1 \cup B_2 \cup B_3) + m(A) + |N_G(c_1)| + |N_G(c_2)| + |N_G(c_3)| \\ &= m(K_{k-1}) + m(K_{k-2}) + 3(k-2) + (k-1) \\ &= \frac{(k-1)(k-2)}{2} + \frac{(k-2)(k-3)}{2} + 4k - 7 = k^2 - 3. \end{aligned}$$

To see that G is k-colorable, let us color G with k colors. First, color A with $k-2$ colors. Second, color c_1, c_2, c_3 with a same $(k-1)^{th}$ color, we can do it since c_1, c_2, c_3 is an independent set. Since $B_1 \cup B_2 \cup B_3$ is a $(k-1)$-clique, we use the $(k-2)$ colors of A plus a k^{th} additional color to distinguish from the color used on the vertices c_1, c_2, c_3. Therefore, G is k-colorable. Finally, it is straightfoward to see that G is edge-critical. □

Figure 12 shows a scheme of the family $\mathcal{F}_k$. Note that the vertices c_1, c_2, and c_3 in the family $\mathcal{F}_k$ form a 3-cut set.

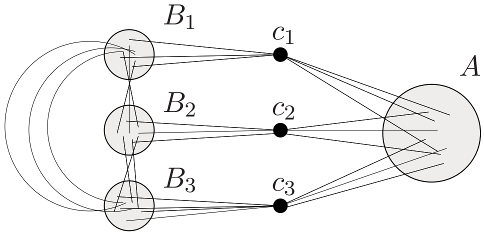


Figure 12: The family $\mathcal{F}_k$.

In 1967, Ore showed in his book *The four color problem* [44, Chapter 11], that, for each $n \geq k \geq 4$, $n \neq k+1$ there exist k-critical graphs with n vertices. His proof was constructive. For each $k \geq 4$, he showed that once we know how to construct the k-critical graphs on at most $2k$ vertices, we could construct all the other k-critical graphs on $n \geq 2k+1$ vertices. Ore showed that this could be made by taking an appropriate k-critical graph between those on at most $2k$ vertices and applying over it the Hajós' construction with a complete graph K_k by successive times, until reach on the desired number of vertices. Ore also showed how to construct the k-critical graphs on at most $2k$ vertices.

In fact, Ore observed that Hajós' construction implies the following

$$f_k(n+k-1) \leq f_k(n) + \frac{(k-2)(k+1)}{2}. \tag{16}$$

Inequality (16) is directly derived from Ore's proof. He also conjectured that the k-critical graphs constructed by this way would have the minimum number of edges, giving rise to his conjecture on the minimum number of edges of k-critical graphs:

Conjecture 29. *(Ore's Conjecture [44]) If $k \geq 4$, then*

$$f_k(n+k-1) = f_k(n) + \frac{(k-2)(k+1)}{2}.$$

According to Theorems 27 and 28, the exact values of $f_k(n)$ for $k+2 \leq n \leq 2k$ are known. This implies that if Ore's Conjecture is true, then it provides an exact formula for the minimum number of edges of all k-critical graphs. In fact, Ore's construction of such graphs provides an upper bound on $f_k(n)$.

In 2012, Kostochka and Yancey [36] proved the following result.

Theorem 30. *[36] If $k \geq 4$ and G is k-critical, then*

$$f_k(n) \geq \left\lceil \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)} \right\rceil.$$

This bound is exact for $k = 4$ and every $n \geq 6$. For every $k \geq 5$, the bound is exact for every $n \equiv 1(mod\ k-1)$, $n \neq 1$. The result also confirms Ore's conjecture for these cases. Using Theorem 30, Kostochka et al. [10, 35] gave short proofs of known theorems on 3-coloring of planar graphs.

8 Characterization of some color-critical graphs

It is known that the complete graphs K_1 and K_2 are the only 1-critical and 2-critical graphs. It is also known that a graph G has $\chi(G) \geq 3$ if and only if G contains an odd circuit. Therefore, the only 3-critical graphs are the odd circuits C_{2n+1}, $n \geq 1$. For graphs with higher chromatic number we have no characterization.

The following result was proved by M. Stehlík in his thesis [52], and it is useful in an attempt to characterize vertex-k-critical graphs on $k + c$ vertices, $c \geq 2$, $k \geq c + 1$. Since every edge-critical graph G is also vertex-critical, the following result is also valid if we consider only the family of the edge-critical graphs. Theorem 31 implies that a k-critical graph G on $k + c$ vertices can be constructed by the addition of vertices of full degree to k-critical graphs of smaller cardinality and size than G.

Theorem 31. *[52, Chapter 4] Suppose G is a vertex-k-critical graph on $2k - t$ vertices, where $1 \leq t \leq k$. If $t = k$ then $G = K_k$, otherwise $G = K_{k-\chi(H)} \vee H$, where $H = G_1 \vee \ldots \vee G_q$ and $G_1, \ldots G_q$ are non-trivial simple vertex-critical graphs such that*

(i) $1 \leq q \leq (k - t)/2$,

(ii) $\chi(H) \leq k - t + q$,

(iii) $\chi(H) \geq 3q$ *if* $k - t$ *is even,*

(iv) $\chi(H) \geq 3q + 1$ *if* $k - t$ *is odd,*

(v) $|V(H)| = k - t + \chi(H)$.

Theorem 31 implies the following two results, both due to Dirac.

Theorem 32. *There exist no vertex-k-critical graphs on $k + 1$ vertices.*

Theorem 33. *If G is a vertex-k-critical graph on $k + 2$ vertices, then $G = C_5 \vee K_{k-3}$.*

Gallai [23] showed that there exist precisely two edge-4-critical graphs on seven vertices. They are shown in Figure 13.

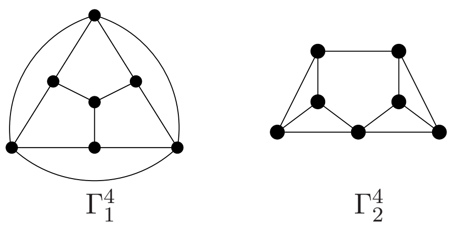


Figure 13: The two edge-4-critical graphs on 7 vertices.

Lemma 34. *(Gallai [23]) If G is an edge-4-critical graph on 7-vertices, then $G = \Gamma_1^4$ or Γ_2^4.*

Let $G_{k,n}$ denote a simple vertex-k-critical graph of order n. We can use Theorem 31 and Lemma 34 to prove the following result.

Theorem 35. *(Gallai [23]) If G is a vertex-k-critical graph on $k+3$ vertices, then $G = K_{k-4} \vee G_{4,7}$. If G is edge-k-critical on $k+3$ vertices, then $G = K_{k-4} \vee \Gamma_1^4$ or $G = K_{k-4} \vee \Gamma_2^4$.*

Table 1 shows the possible decompositions of vertex-k-critical graphs on $2k - t$ vertices, for $0 \leq k - t \leq 5$.

$k-t$	G
0	K_k
1	$-$
2	$K_{k-3} \vee C_5$
3	$K_{k-4} \vee G_{4,7}$
4	$K_{k-3} \vee C_7$
	$K_{k-5} \vee G_{5,9}$
	$K_{k-6} \vee C_5 \vee C_5$
5	$K_{k-4} \vee G_{4,9}$
	$K_{k-5} \vee G_{5,10}$
	$K_{k-6} \vee G_{6,11}$
	$K_{k-7} \vee C_5 \vee G_{4,7}$

Table 1: Decomposition of vertex-k-critical graphs of order $2k - t$.

9 Color-k-critical graphs that contain a subdivision of the complete graph K_k

Gallai [23] found the lower bound on the number of vertices of full degree in color-critical graphs. This result is presented in the following theorem.

Theorem 36. *(Gallai [23]) Let $k \geq 3$ and $n < \frac{5}{3}k$. If G is a k-critical, n-vertex graph, then it contains at least $\lceil \frac{3}{2}(\frac{5}{3}k - n) \rceil$ vertices of full degree.*

In Section 8 we showed that, for $k \geq 3$, there exist characterizations for k-critical graphs with $k+2$ and $k+3$ vertices. Using Theorem 36 and Royle's complete search on small graphs [50], J.Barát and G.Tóth [6] provided an inductive proof of the following result.

Theorem 37. *[6] For $k \geq 6$, there are precisely twenty-two k-critical graphs on $k+4$ vertices. Each of them can be constructed by adding vertices of full degree to a graph in the following list:*

- *the 3-critical graph on seven vertices,*
- *the four 4-critical graphs on eight vertices,*
- *the sixteen 5-critical graphs on nine vertices, or*

- *the* 6*-critical graph on ten vertices.*

The graphs in this list can be found in the Royle's Table [50]. Following the same approach used by J.Barát and G.Tóth [6], we proved the following result.

Theorem 38. *For $k \geq 7$, there are precisely 395 k-critical graphs on $k+5$ vertices. Each of them can be constructed by adding vertices of full degree to a graph in the following list:*

- *the twenty-one 4-critical graphs on 9 vertices,*
- *the one hundred and forty-one 5-critical graphs on 10 vertices,*
- *the two hundred and thirty-one 6-critical graphs on 11 vertices, or*
- *the two 7-critical graphs on 12 vertices.*

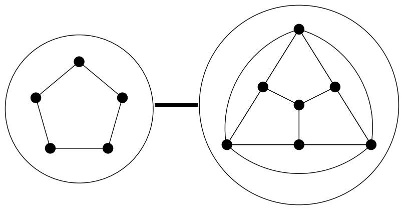
Proof. By induction on k. For the base case ($k = 7$), Royle's complete search on small graphs (see Royle's table [50]) shows that there exist exactly 395 7-critical graphs on 12 vertices. All these graphs are built by adding vertices of full degree to any graph of the list above. For the induction step we suppose $k \geq 8$ and we use Theorem 36 to see that any k-critical graph G on $k+5$ vertices contains at least $\lceil k - 7.5 \rceil$ vertices of full degree. Since $k \geq 8$, G always has a vertex v of full degree. Removing v, we use induction to complete the proof. □

It is known that any k-critical graph on at most $k+4$ vertices has a subdivision of K_k [6]. In Section 7 were presented other k-critical graphs that also contain a subdivision of K_k. In fact, we checked the 395 7-critical graphs on 12 vertices of Theorem 38 and we found that these graphs also have a subdivision of K_7. From Royle's table [50] we can see that there exist 21 4-critical graphs on 9 vertices, and it is known that all 4-chromatic graphs have a subdivision of K_4 [15]. There exist 162 5-critical graphs on 10 vertices, 21 of them are constructed by adding a vertex of full degree to the 21 4-critical graphs on 9 vertices, so they have a subdivision of K_5. We checked the other 141 5-critical graphs on 10 vertices, partly by hand and partly using Mader's result [42], and found that these graphs also have a subdivision of K_5. There are 393 6-critical graphs on 11 vertices, and all these graphs have a subdivision of K_6; 393 7-critical graphs on 12 vertices are constructed by adding a vertex of full degree to the 393 6-critical graphs on 11 vertices. The other two 7-critical graphs on 12 vertices were checked by hand and it was found that they have a subdivision of K_7. In fact, these two graphs are constructed by joining the cycle C_5 to one of the two 4-critical graphs on 7 vertices. Figure 14 illustrates these graphs. From all these facts we have the following result.

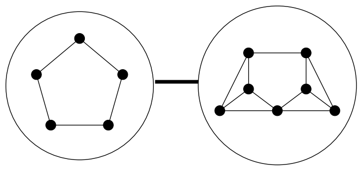
Theorem 39. *Any k-critical graph on at most $k+5$ vertices has a subdivision of K_k.*

J.Barát and G.Tóth [6] conjectured the following.

Conjecture 40. *For every positive integer c, there exists a bound $k(c)$ such that for any k, where $k \geq k(c)$, any k-critical graph on $k+c$ vertices has a subdivision of K_k.*



(a) 7-critical graph on 12 vertices.



(b) 7-critical graphs on 12 vertices.

Figure 14: The two 7-critical graphs on 12 vertices that are formed by the join of C_5 and one of the two 4-critical graphs on 7 vertices. Thick edges joining two circles indicate the join of the graphs.

This conjecture is true for $c \in \{0, 2, 3, 4\}$ (see [4, 6]), and Theorem 39 shows that it is true for $c = 5$. However, we note that the conjecture is false for $c \geq 6$.

For $c \geq 6$, let F_c be the graph whose vertex set consists of five non-empty pairwise disjoint sets A_1, A_2, C_1, C_2, and C_3 where $|C_1| = |C_2| = |C_3| = 3$ and $|A_1| = |A_2| = c - 4$, such that these sets are cliques in F_c. The sets A_1, A_2, C_1, C_2, and C_3 are joined in the following way: $A_1 \vee C_1$, $A_1 \vee C_2$, $C_1 \vee A_2$, $A_2 \vee C_3$, and $C_2 \vee C_3$. Figure 15 shows a scheme of the graphs F_c.

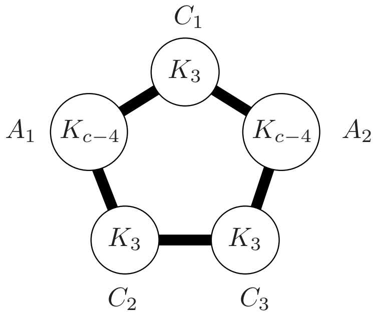


Figure 15: Scheme of a graph F_c. Heavy edges indicate that every vertex in one circle is adjacent to every vertex in the other.

We claim that, for $c \geq 6$, F_c has $\chi(F_c) = c+1$, and that F_c is a $(c+1)$-critical graph that does not contain a subdivision of $K_{(c+1)}$. If this is true, it is easy to see that any member of this family is a minimal counterexample to Conjecture 40, for each $c \geq 6$.

Now we prove our claims. First, note that every graph F_c has $2c + 1$ vertices.

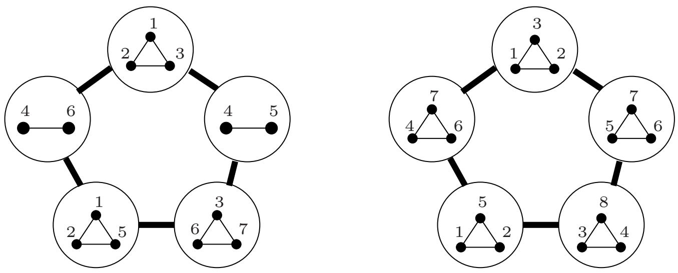
Proposition 41. *For $c \geq 6$, $\chi(F_c) = c + 1$.*

Proof. To see that F_c requires at least $c + 1$ colors note that

$$\chi(F_c) \geq \frac{n(F_c)}{\alpha(F_c)} = \frac{2c+1}{2} = c + \frac{1}{2} \implies \chi(F_c) \geq c + 1.$$

Moreover, F_c is $(c+1)$-colorable. For $6 \le c \le 7$, Figure 16 shows the required $(c+1)$-colorings to graphs F_6 and F_7. For $c \ge 8$, the graph F_c can be assigned a $(c+1)$-coloring such that the vertices in the sets A_1, A_2, C_1, C_2, and C_3 receive the following colors: $C_1 = \{1,2,3\}$, $C_2 = \{1,4,5\}$, $C_3 = \{2,6,7\}$, $A_1 = \{6,7,8,9,\dots,c+1\}$, and $A_2 = \{4,5,8,9,\dots,c+1\}$.

It is easy to check that, for each $c \ge 8$, this is a $(c+1)$-coloring for F_c. Therefore, F_c is $(c+1)$-chromatic. $\square$



(a) F_6: 7-critical graph on 13 vertices that does not contain a subdivision of K_7.

(b) F_7: 8-critical graph on 15 vertices that does not contain a subdivision of K_8.

Figure 16: Graphs F_c. Heavy edges indicate that every vertex in one circle is adjacent to every vertex in the other.

Proposition 42. *For $c \ge 6$, F_c is edge-color-critical.*

Proof. For $c \ge 6$, we show that, after the removal of an arbitrary edge e, the graph $F_c - e$ is c-colorable. There are 6 different cases to consider.

Case 1. $e \in E(C_1)$. For $c \ge 6$, the vertex subsets of $F_c - e$ can be assigned the following colors. $C_1 = \{1,2\}$, $C_2 = \{1,6,7\}$, $C_3 = \{2,4,5\}$, $A_1 = \{4,5,8,\dots,c+1\}$, and $A_2 = \{6,7,8,\dots,c+1\}$. Observe that the color 3 does not appear in this coloring. Therefore, $F_c - e$ is colorable with c colors.

Case 2. e is an edge with one endpoint in the set C_1 and the other endpoint in the set A_1. For $c \ge 6$, the vertex subsets of $F_c - e$ can be assigned the following colors: $C_1 = \{1,2,3\}$, $C_2 = \{2,6,7\}$, $C_3 = \{1,3,5\}$, $A_1 = \{1,5,8,\dots,c+1\}$, and $A_2 = \{6,7,8,\dots,c+1\}$. Observe that the color 4 does not appear in this coloring. Therefore, $F_c - e$ is colorable with c colors.

Case 3. $e \in E(A_1)$. For $c \ge 6$, the vertex subsets of $F_c - e$ can be assigned the following colors: $C_1 = \{1,2,3\}$, $C_2 = \{1,6,7\}$, $C_3 = \{2,3,5\}$, $A_1 = \{5,8,\dots,c+1\}$, and $A_2 = \{6,7,8,\dots,c+1\}$. Observe that the color 4 does not appear in this coloring. Therefore, $F_c - e$ is colorable with c colors.

Case 4. e is an edge with one endpoint in the set A_1 and the other endpoint in the set C_2. For $c \ge 6$, the vertex subsets of $F_c - e$ can be colored with the following colors: $C_1 = \{1,2,3\}$, $C_2 = \{1,6,7\}$, $C_3 = \{2,3,5\}$, $A_1 = \{5,6,8,\dots,c+1\}$, and $A_2 = \{6,7,8,\dots,c+1\}$. Observe that the color 4 does not appear in this coloring. Therefore, $F_c - e$ is colorable with c colors.

Case 5. $e \in E(C_2)$. For $c \geq 6$, the vertex subsets of $F_c - e$ can be assigned the following colors: $C_1 = \{1, 2, 3\}$, $C_2 = \{1, 7\}$, $C_3 = \{2, 3, 5\}$, $A_1 = \{5, 6, 8, \ldots, c+1\}$, and $A_2 = \{6, 7, 8, \ldots, c+1\}$. Observe that the color 4 does not appear in this coloring. Therefore, $F_c - e$ is colorable with c colors.

Case 6. e is an edge with one endpoint in the set C_2 and the other endpoint in the set C_3. For $c \geq 6$, the vertex subsets of $F_c - e$ can be assigned the following colors: $C_1 = \{1, 2, 3\}$, $C_2 = \{1, 2, 7\}$, $C_3 = \{1, 3, 5\}$, $A_1 = \{5, 6, 8, \ldots, c+1\}$, and $A_2 = \{6, 7, 8, \ldots, c+1\}$. Observe that the color 4 does not appear in this coloring. Therefore, $F_c - e$ is colorable with c colors.

□

Proposition 43. *For $c \geq 6$, F_c does not contain a subdivision of $K_{(c+1)}$.*

Proof. Suppose $c \geq 8$ and F_c has a subdivision of K_{c+1}; let W be the set of branching vertices of a subdivision of K_{c+1}. Within the subdivision, any two vertices of W are joined by c pairwise internally-disjoint paths.

Only if $c = 8$ does $C_1 \cup C_2 \cup C_3$ have enough vertices to contain W. In this case, $W = V(C_1 \cup C_2 \cup C_3)$ and there must be nine disjoint paths through A_1 representing the edges from C_1 to C_2. Since A_1 has only four vertices, this is impossible.

Thus, in all cases, at least one vertex of W is in $A_1 \cup A_2$. Since $V(C_1 \cup C_2)$ is a cut-set in F_c of size $6 < c$, there cannot be vertices of W in both $V(A_1)$ and $V(A_2)$. Therefore, we may assume $W \cap V(A_1) \neq \emptyset$ and $W \cap V(A_2) = \emptyset$.

Since $V(C_1 \cup C_2)$ also separates A_1 from C_3, we deduce that $W \cap V(C_3) = \emptyset$. Therefore, $W \subseteq V(C_1 \cup A_1 \cup C_2)$. Since $|W| = c+1$ and $|V(C_1 \cup A_1 \cup C_2)| = c+2$, exactly one vertex v in $C_1 \cup A_1 \cup C_2$ is not in W.

It follows that there are at least 6 internally-disjoint C_1C_2-paths in K_{c+1} representing the edges from $W \cap V(C_1)$ to $W \cap V(C_2)$. Since at most one vertex of A_1 is not in W, at most one of the 6 C_1C_2-paths can go through A_1. In the other direction, at most three can go through C_3. This is impossible, so there is no subdivision of K_{c+1} in F_c.

There is a subdivision of K_c, consisting of the $c-4$ vertices in A_1, three vertices in C_1, and one vertex in C_2. This uses only three internally disjoint C_1C_2-paths through $C_3 \cup A_2$. □

Since any graph F_c is a $(c+1)$-critical graph that does not contain a subdivision of $K_{(c+1)}$, any graph built from F_c by adding vertices of full degree will have the same property. Therefore, the graphs F_c are counterexamples to Conjecture 40.

- Despite of the fact that the graph F_c is $(c+1)$-critical but does not contain a subdivision of $K_{(c+1)}$, we conjecture that Albertson's conjecture holds for F_c. In fact, this is true for $6 \leq c \leq 15$, because Albertson's conjecture is known to be true for k-critical graphs with $k \leq 16$. **But, how can we prove it for the remaining cases?** We do not have adequate estimates on $cr(F_c)$.

10 Albertson's conjecture

In 2007, Albertson posed the following conjecture.

Conjecture 44. *(Albertson's Conjecture). If $\chi(G) = k$, then $cr(G) \geq cr(K_k)$.*

As discussed in the end of Section 4, Albertson's conjecture is trivially true for $k \leq 4$, since every graph has crossing number at least zero. For $k = 5$ the conjecture is equivalent to the Four Colour Theorem. The case $k = 6$ of the conjecture is equivalent to the contrapositive of Theorem 5, so it was verified by Oporowski and Zhao. Albertson, Cranston, and Fox [4] verified the conjecture for $7 \leq k \leq 12$. Finally, Barát and Tóth [6] improved the previous results verifying Albertson's conjecture for $13 \leq k \leq 16$. Albertson's conjecture is still open for the remaining cases.

According to Lemma 13, every k-chromatic graph G has a k-critical subgraph. Observe that if H is a subgraph of a graph G, then $cr(H) \leq cr(G)$. Therefore, to prove Albertson's conjecture for a given k, it suffices to prove it only for k-critical graphs. Also observe that if a k-chromatic graph G has a subdivision of K_k, then $cr(G) \geq cr(K_k)$. According to Theorem 39, any k-critical graph on at most $k+5$ vertices has a subdivision of K_k. So, for $k \geq 3$, Albertson's conjecture holds for any k-critical graph G with at most $k+5$ vertices.

To attack Albertson's conjecture, Albertson et al. [4], J. Barát and G. Tóth [6] used the following strategy. They calculated a lower bound for the number of edges of a k-critical n-vertex graph G. Next they substituted this into one of the lower bounds on the crossing number of a graph G presented in Section 3. Finally, they compared the result and the value for $cr(K_k)$ conjectured by Guy (Conjecture 2). The problem with this method is that, for large k, the method is not sufficient. Using the lower bounds on the number of edges of k-critical graphs presented in Section 7 jointly with the lower bounds on the crossing number of a graph G presented in Section 3, the authors were able to verify Albertson's conjecture for $k \leq 16$. However, for $k \geq 17$, this strategy showed to be insufficient to aproach the problem.

We present below the proof of Albertson's conjecture for the cases $7 \leq k \leq 13$.

Proposition 45. *If $\chi(G) = 7$, then $cr(G) \geq 9 = cr(K_7)$.*

Proof. Let G be a 7-critical graph that does not contain a subdivision of K_7. Let n be the number of vertices of G and m be the number of edges of G. According to Corollary 26, $m \geq 3n + 4$. Applying this result in the lower bound (10) we obtain

$$cr(G) \;\geq\; m - 3n + 6 \;\geq\; 3n + 4 - 3n + 6 \;=\; 10 \;>\; 9 \;=\; cr(K_7).$$

□

Proposition 46. *If $\chi(G) = 8$, then $cr(G) \geq 18 = cr(K_8)$.*

Proof. Let G be an 8-critical graph that does not contain a subdivision of K_8. Let n be the number of vertices of G and m be the number of edges of G. According to Theorem 39, we may suppose $n \geq 14$, and according to Corollary 26, $m \geq \frac{7}{2}n + 5$. Applying this result in the lower bound (10) we obtain

$$cr(G) \;\geq\; m - 3n + 6 \;\geq\; \frac{7}{2}n + 5 - 3n + 6 \;=\; \frac{n}{2} + 11 \;\geq\; \frac{14}{2} + 11 \;=\; 18 \;=\; cr(K_8).$$

□

Proposition 47. *If $\chi(G) = 9$, then $cr(G) \geq 36 = cr(K_9)$.*

Proof. Let G be a 9-critical graph that does not contain a subdivision of K_9. Let n be the number of vertices of G and m be the number of edges of G. According to Theorem 39, we may suppose $n \geq 15$, and according to Corollary 26, $m \geq 4n + 6$. Hence, inequality (11) gives:

$$cr(G) \;\geq\; \frac{7}{3}m - \frac{25}{3}(n-2) \;\geq\; n + \frac{92}{3} \;\geq\; 15 + \frac{92}{3} \;> 46 \;>\; 36 \;=\; cr(K_9).$$

□

Proposition 48. *If $\chi(G) = 10$, then $cr(G) \geq 60 = cr(K_{10})$.*

Proof. Let G be a 10-critical graph that does not contain a subdivision of K_{10}. Let n be the number of vertices of G and m be the number of edges of G. According to Theorem 39, we may suppose $n \geq 16$, and according to Corollary 26, $m \geq \frac{9}{2}n + 7$. Hence, inequality (12) gives:

$$cr(G) \;\geq\; 4m - \frac{103}{6}(n-2) \;\geq\; \frac{5}{6}n + \frac{187}{3} \;>\; 75 \;>\; 60 \;=\; cr(K_{10}).$$

□

Proposition 49. *If $\chi(G) = 11$, then $cr(G) \geq 100 = cr(K_{11})$.*

Proof. Let G be a 11-critical graph that does not contain a subdivision of K_{11}. Let n be the number of vertices of G and m be the number of edges of G. According to Theorem 39, we may suppose $n \geq 17$, and according to Corollary 26, $m \geq 5n + 8$. Hence, inequality (12) gives:

$$cr(G) \;\geq\; 4m - \frac{103}{6}(n-2) \;\geq\; \frac{17}{6}n + \frac{199}{3} \;\geq\; \frac{17}{6} \cdot 17 + \frac{199}{3} \;>\; 114 \;>\; 100 \;=\; cr(K_{11}).$$

□

Proposition 50. *If $\chi(G) = 12$, then $cr(G) \geq 150 = cr(K_{12})$.*

Proof. Let G be a 12-critical graph that does not contain a subdivision of K_{12}. Let n be the number of vertices of G and m be the number of edges of G. According to Theorem 39, we may suppose $n \geq 18$, and according to Corollary 26, $m \geq \frac{11}{2}n + 9$. Hence, inequality (12) gives:

$$cr(G) \;\geq\; 4m - \frac{103}{6}(n-2) \;\geq\; \frac{29}{6}n + \frac{211}{3} \;\geq\; \frac{29}{6} \cdot 18 + \frac{211}{3} \;>\; 157 \;>\; 150 \;=\; cr(K_{12}).$$

□

Proposition 51. *If $\chi(G) = 13$, then $cr(G) \geq 225 \geq cr(K_{13})$.*

Proof. Let G be a 13-critical graph that does not contain a subdivision of K_{13}. Let n be the number of vertices of G and m be the number of edges of G. According to Theorem 39, we may suppose $n \geq 19$, and according to Corollary 26, $m \geq 6n + 10$.

Case 1. For $n \geq 22$, the last lower bound combined with inequality (12) gives the desired result:

$$cr(G) \;\geq\; 4m - \frac{103}{6}(n-2) \;\geq\; 4(6n+10) - \frac{103}{6}(n-2) \;\geq\; \frac{41}{6}n + \frac{223}{3} \;>\; 224.$$

Case 2. For $n = 21$, we use Gallai's bound (Corollary 24) to get that $m \geq 146$. Applying this result to inequality (12) we obtain:

$$cr(G) \;\geq\; 4m - \frac{103}{6}(n-2) \;\geq\; 4 \cdot 146 - \frac{103}{6} \cdot 19 \;\geq\; 258 \;>\; 225 \;\geq\; cr(K_{13}).$$

Case 3. For $n = 20$, we use Gallai's bound (Corollary 24) to get that $m \geq 141$. Applying this result to inequality (12) we obtain:

$$cr(G) \;\geq\; 4m - \frac{103}{6}(n-2) \;\geq\; 4 \cdot 141 - \frac{103}{6} \cdot 18 \;\geq\; 255 \;>\; 225 \;\geq\; cr(K_{13}).$$

Case 4. For $n = 19$, we use Gallai's bound (Corollary 24) to get that $m \geq 135$. Applying this result to inequality (12) we obtain:

$$cr(G) \;\geq\; 4m - \frac{103}{6}(n-2) \;\geq\; 4 \cdot 135 - \frac{103}{6} \cdot 17 \;\geq\; 249 \;>\; 225 \;\geq\; cr(K_{13}).$$

□

Using the Crossing Lemma (bound 15), Albertson et al. [4] were able to give an upper bound on the number of vertices of a counterexample to Albertson's conjecture. They proved the following result:

Theorem 52. *[4] If G is a k-critical graph with $n \geq 4k$ vertices, then $cr(G) \geq cr(K_k)$.*

Proof. It is known that $cr(G) \geq cr(K_k)$ holds for $k \leq 16$. Thus, we may assume $k \geq 17$. Let m be the number of edges of G. Since G is k-critical, then $m \geq n(k-1)/2$. In particular, since $k \geq 17$, we have that $m \geq 8n \geq \frac{103}{16}n$. Therefore, we can apply the Crossing Lemma (bound 15), which gives

$$\begin{aligned} cr(G) &\geq \frac{1}{31.1}\frac{m^3}{n^2} \geq \frac{1}{8 \cdot 31.1}(k-1)^3 n \geq \frac{1}{64}(k-1)^3 k \\ &\geq \frac{1}{4}\left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{k-1}{2} \right\rfloor \left\lfloor \frac{k-2}{2} \right\rfloor \left\lfloor \frac{k-3}{2} \right\rfloor \geq cr(K_k). \end{aligned}$$

□

J. Barát and G. Toth [6] improved this result and proved the following theorem.

Theorem 53. *[6] If G is a k-critical graph with $n \geq 3.57k$ vertices, then $cr(G) \geq cr(K_k)$.*

Therefore, according to Theorem 39 and to Theorem 53, for any $k \geq 17$, any counterexample to Albertson's conjecture must have more than $k + 5$ vertices and less than $3.57k$ vertices.

11 Concluding Remarks

During the investigation of Albertson's conjecture, the following question came up:

Question 54. *For each $k \geq 5$ and $s \geq 1$, if $cr(G) \geq cr(K_k)$, then $cr(G \vee K_s) \geq cr(K_{k+s})$?*

Very soon we discovered that the join of the complete bipartite graph $K_{3,3}$ with the complete graph K_2 provides a negative answer to this question. In fact, we have that $cr(K_{3,3} \vee v) = 3 = cr(K_6)$, and we also have that $cr(K_{3,3} \vee K_2) \leq 8 < 9 = cr(K_7) = cr(K_{(5+2)})$. Figure 17 shows a drawing of $K_{3,3} \vee K_2$ with 8 crossings. Figures 18 and 19 show other counterexamples to Question 54.

Given these examples we see that the answer to question 54 is negative. Following this question, it would be interesting to know how much the join of a full degree vertex in a graph can change the crossing number of the new graph. We propose another question considering the addition of a full degree vertex in a graph G.

Question 55. *For each $k \geq 1$, what is the biggest integer $f(k)$ so that*

$$cr(G) \geq k \implies cr(G+v) \geq f(k)?$$

It is easy to see that $f(1) = 3$, since $cr(K_5 + v) = cr(K_6) = 3$ and $cr(K_{3,3} + v) = 3$. But it would be interesting to find $f(k)$ for $k \geq 2$.

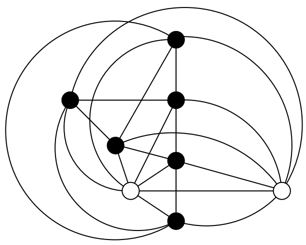


Figure 17: Drawing of $K_{3,3} \vee K_2$ showing that $cr(K_{3,3} \vee K_2) \leq 8$.

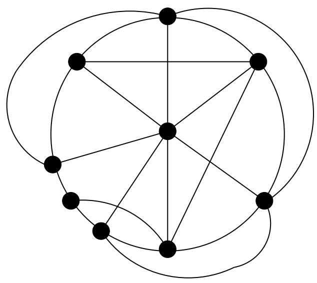
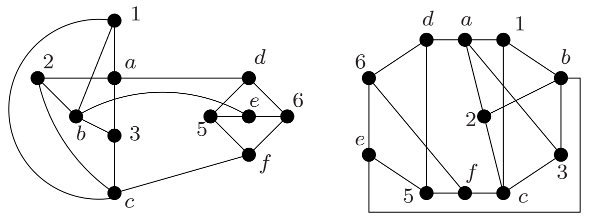


Figure 18: Drawing of a graph G with conjectured crossing number $cr(G) = 3$. Observe that $cr(G+v) \leq 7 < 9 = cr(K_7)$.



(a) (b)

Figure 19: (a) Drawing of a graph G with conjectured crossing number 3. (b) Drawing of G with 4 crossings. Note that adding a new vertex v in the outer face and joining it to all the other vertices will result in a new graph $G + v$ with only 8 crossings. Thus, $cr(G + v) \leq 8 < 9 = cr(K_7)$.

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