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Fulkerson's Conjecture and Loupekine Snarks

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Abstract

In 1976, F. Loupekine created a method for constructing new snarks from already known ones. In the present work, we consider an infinite family of Loupekine's snarks constructed from the Petersen Graph, and verify Fulkerson's Conjecture for this family.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *degree* of a vertex $v \in V(G)$ is denoted by $d_G(v)$, or simply $d(v)$. A graph is *k-regular* if all of its vertices have degree equal to k. A 3-regular graph is also called a *cubic* graph. The *maximum degree* of G, denoted by $\Delta(G)$, is the number $\max\{d(v) \colon v \in V(G)\}$. Let $X \subseteq V(G)$. The *edge cut* $\partial(X)$ is the set comprised of all edges with exactly one end in X. The edge cut $\partial(\{v\})$ is also denoted by $\partial(v)$. An edge $e \in E(G)$ is a *cut edge* or *bridge* if, by removing e from G, the resulting graph has more connected components than G. A *cyclic edge cut* is an edge cut whose removal creates connected components each containing a cycle. A graph is *cyclically k-edge-connected* if it does not contain a cyclic edge cut with less than k edges.

An *edge-colouring* of G is a map $\pi \colon E(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set of *colours*, such that for any two adjacent edges $e, f \in E(G)$, $\pi(e) \neq \pi(f)$. If $|\mathcal{C}| = k$, then π is a *k-edge-colouring* and G is *k-edge-colourable*. For each $i \in \mathcal{C}$, set $E_i := \{e \in E(G) \colon \pi(e) = i\}$ is a *colour class* of π. Thus, an edge-colouring is a partition $\{E_i \colon i \in \mathcal{C}\}$ of $E(G)$. A *matching* of G is a set $M \subseteq E(G)$ such that no two edges in M are adjacent. If $v \in V(G)$ is incident with an edge $e \in M$, then v is *saturated* by M or *M-saturated*. A matching M is *perfect* if every vertex in $V(G)$ is M-saturated. Since each colour class is a matching, a k-edge-coloring is a partition of $E(G)$ into k matchings. If G is k-regular, each matching in the partition is perfect.

The following conjecture was independently formulated by C. Berge and D. R. Fulkerson [13]. It was first published by Fulkerson [4], and it is called *Fulkerson's Conjecture*[1].

Conjecture 1 (Fulkerson's Conjecture)**.** *Every bridgeless cubic graph has six perfect matchings such that each edge belongs to exactly two of them.*

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[1]It is sometimes called Berge-Fulkerson Conjecture [10].

Despite its very simple statement, this conjecture has challenged researchers since its publication in 1971. Not many partial results have been published since then. In a cubic graph with a 3-edge-colouring, such a collection of six perfect matchings is constructed by duplicating each colour class. Thus, the problem is reduced to verify the conjecture for non-3-edge-colourable bridgeless cubic graphs. A few well-known infinite families of these graphs have been shown to satisfy Fulkerson's Conjecture [2, 3, 7].

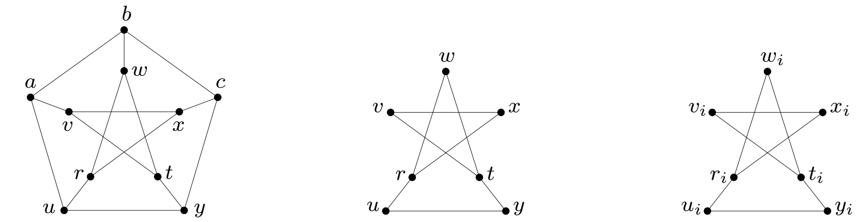
The first discovery of a non-3-edge-colourable bridgeless cubic graph is due to Julius Petersen [12] in 1898. Petersen found the smallest graph with these properties, the *Petersen Graph*, depicted in Figure 1(a). After many years, only few discoveries happened, showing that non-3-edge-colourable bridgeless cubic graphs are very hard to find. Because of this difficulty, M. Gardner [5] named these graphs *snarks*, inspired by the elusive creature in Lewis Caroll's poem *The Hunting of the Snark*. In his definition of snarks, Gardner excluded some trivial cases, such as graphs with cycles of size at most three, which can be easily derived from smaller snarks. In order to also avoid other trivialities, snarks are currently defined as cyclically 4-edge-connected non-3-edge-colourable cubic graphs with girth at least five. Currently, many infinite families of snarks are known [6, 8, 9, 11]. There are several good texts with more details on snarks, their motivation, history, and constructions [1, 11, 15, 16].

Fulkerson's Conjecture was verified for the families of flower snarks, Goldberg snarks, generalised Blanuša snarks and for the Szekeres snark [3, 7]. In 1976, Isaacs described a method proposed by F. Loupekine for constructing new snarks from already known ones [8]. This paper proposes a technique tied up with this construction to verify the conjecture for a family of Loupekine's snarks constructed from the Petersen Graph.

2 Loupekine's snarks

F. Loupekine proposed a construction of two infinite families of snarks, using subgraphs of other known snarks. Loupekine's construction is presented here as described by Isaacs [8]. This section also provides some additional notation used in this paper.

Let G be a snark. Any subgraph of G obtained by removing a path of three vertices is denoted by $B(G)$. Figure 1 illustrates this operation for the Petersen Graph P. Since the girth of G is at least five, B(G) has five different degree-two vertices, namely u, v, w, x, y, called *border vertices* and labeled relatively to vertices of the path removed as shown in Figure 1(b).



(a) Petersen Graph P. (b) Graph $B(P)$. (c) A block B_i.

Figure 1: Construction of a block from the Petersen Graph.

Let $G_1, G_2, \ldots, G_k$ be snarks. The subgraphs $B(G_i)$, with $1 \leq i \leq k$, are called *blocks* and are used in the construction of a new snark. Denote each block $B(G_i)$ by B_i, and attach index i to its vertex names, as in Figure 1(c). For each i, with $1 \leq i \leq k$, add a pair of edges linking each border vertex of $\{u_i, v_i\}$ to a different border vertex of $\{x_{i+1}, y_{i+1}\}$. Here, and throughout this text, indexes greater than k are taken modulo k.The resulting graph, denoted by G_B, is the *block subgraph* of the new snark. Figure 2(a) shows an example.

Notice that G_B has exactly k vertices with degree two: $w_1, \ldots, w_k$. Let G_C be a graph with exactly k vertices of degree one, namely $z_1, \ldots, z_k$, and all the other vertices of degree three. The edge incident with vertex z_i is denoted by e_i^z. The graph G_C is the *central subgraph* of the new snark. An example of a central subgraph is depicted in Figure 2(b). For each i, with $1 \leq i \leq k$, identify vertices w_i and z_i. The resulting graph is cubic and denoted by G_k^L.

If k is odd, then G_k^L is a snark, as shown by Isaacs [8]. If k is even, with $k \geq 6$, it is necessary to add a constraint to G_C. The central subgraph cannot admit a 3-edge-colouring π which satisfies the following property: $\pi(e_i^z) = \pi(e_{i+1}^z)$ either for all odd $i \leq k$, or for all even $i \leq k$. A central subgraph satisfying this constraint is, for example, one that has three vertices z_i, z_{i+1}, z_{i+2} adjacent to a common vertex. In any 3-edge-coloring of this graph, e_{i+1}^z can never exhibit the same color of neither e_i^z or e_{i+2}^z. These two families of snarks, with odd $k \geq 3$ or even $k \geq 6$, are the families proposed by Loupekine. These families are called *Loupekine snarks* or *L-snarks*[2]. Figure 2 shows an example of Loupekine's construction with k odd. Notice that G_B and G_C form a decomposition of G into edge-disjoint subgraphs.

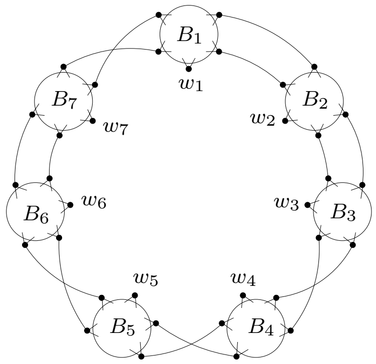
Let G be an L-snark such that each connected component of G_C is isomorphic to one of the graphs K_2 and S_3, with K_2 being a complete graph with two vertices, and S_3 a star with three vertices of degree one. The graph G is an *L_1-snark*. The graph of Figure 2(c) is an L_1-snark. Suppose that G is an L-snark such that each of its blocks is isomorphic to $B(P)$ (see Figure 1(b)). Then, G is an *LP-snark*. Moreover, an L_1-snark which is also an LP-snark is called an *LP_1-snark*.

Let B_i be a block of an LP-snark. Figure 3 shows B_i together with the edpge cut $\partial(V(B_i)) = \{e_i^u, e_i^v, e_i^w, e_i^x, e_i^y\}$, also denoted by ∂_{B_i}. The edges of ∂_{B_i} and their ends $u_i, u_i', v_i, v_i', w_i, w_i', x_i, x_i', y_i, y_i'$ are named as shown in the figure. The *extended block* B_i^+ is the graph with vertex set $V(B_i^+) = V(B_i) \cup \{u_i', v_i', w_i', x_i', y_i'\}$ and edge set $E(B_i^+) = E(B_i) \cup \partial_{B_i}$. For simplicity, the index i is omitted whenever it is clear in the context.

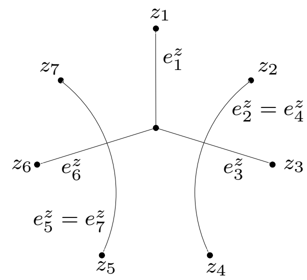
3 Main Results

In this section, it is proved that every LP_1-snark verifies Fulkerson's Conjecture. Before proceeding to the main results, some definitions and concepts are necessary. Let G be a graph with $\Delta(G) \leq 3$ and let L be an index set such that $|L| = 6$. A *Fulkerson Collection* is a family $\mathcal{F} = \{M_l : l \in L\}$ of matchings of G such that each edge of $E(G)$ belongs to exactly two members of $\mathcal{F}$. By this definition, each vertex of G with degree three is saturated by the six matchings of $\mathcal{F}$. As a consequence, if G is cubic, then every matching of $\mathcal{F}$ is perfect. Therefore, the following is true.

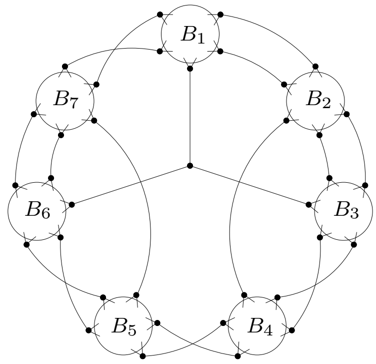
[2]The names of families of Loupekine snarks used here were taken from L. Vaux's work [14].



(a) A graph G_B.



(b) A graph G_C.



(c) An L-snark G_7^L.

Figure 2: An L-snark G_7^L and its subgraphs G_B and G_C. The graph G_7^L is also an L_1-snark.

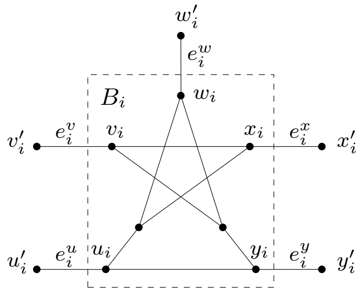


Figure 3: The edges of ∂_{B_i} and its end vertices in the graph B_i^+.

Claim 2. *A cubic graph satisfies Fulkerson's Conjecture if and only if it admits a Fulkerson Collection.*

Let $L^{(2)}$ be the set of 2-element subsets of L. The function $\lambda\colon E(G) \to L^{(2)}$, defined as $\lambda(e) = \{l \in L\colon e \in M_l, M_l \in \mathcal{F}\}$, is *induced* by $\mathcal{F}$. It is easy to see that λ satisfies the following property.

F1 For all adjacent $e, f \in \mathrm{Dom}(\lambda)$, $\lambda(e) \cap \lambda(f) = \emptyset$.

A *Fulkerson Function* of G is a function $\lambda\colon E' \to L^{(2)}$, with $E' \subseteq E(G)$ and $|L| = 6$, satisfying F1. The image of λ is a set of unordered pairs of L. An unordered pair $\{p, q\}$ is also denoted by pq and p, q. A Fulkerson Function of G is *complete* if its domain is $E(G)$. Complete Fulkerson Functions and Fulkerson Collections are equivalent and are used indistinctively in this paper.

In order to prove that LP_1-snarks satisfy Fulkerson's Conjecture, a complete Fulkerson Function is constructed for an arbitrary LP_1-snark. Initially, some useful properties of (not necessarily complete) Fulkerson Functions of LP-snarks are shown.

Let $\pi\colon E' \to L^{(2)}$, with $E' \subseteq E(G)$ and L an index set with cardinality six. Let $\mathcal{L} := \{L_1, L_2, L_3\}$ be a partition of L with each part of cardinality two. Consider edges e^u, e^v, e^w, e^x, e^y of an extended block B^+ and the following properties.

P1 $\pi(e^w) \in \mathcal{L}$.

P2 π satisfies exactly one of:
- **(a)** $\pi(e^x) = \pi(e^y) \in \mathcal{L}$
- **(b)** $\pi(e^x), \pi(e^y) \notin \mathcal{L}$;
  $\pi(e^x)$ and $\pi(e^y)$ are disjoint; and
  $\pi(e^x) \cup \pi(e^y) = L_s \cup L_t,\ \ L_s, L_t \in \mathcal{L}$.

P3 π satisfies exactly one of:
- **(a)** $\pi(e^u) = \pi(e^v) \in \mathcal{L}$;
- **(b)** $\pi(e^u), \pi(e^v) \notin \mathcal{L}$;
  $\pi(e^u)$ and $\pi(e^v)$ are disjoint; and
  $\pi(e^u) \cup \pi(e^v) = L_s \cup L_t,\ \ L_s, L_t \in \mathcal{L}$.

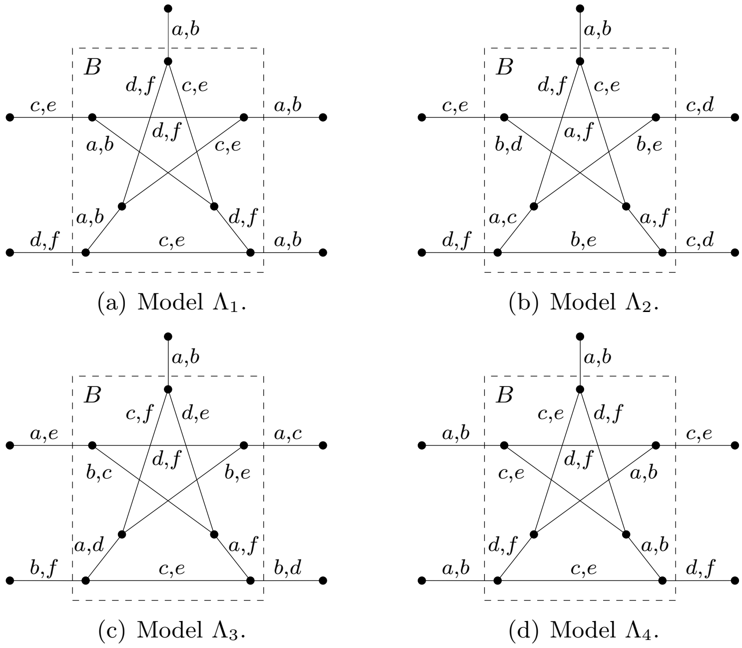
Let $\lambda\colon E' \to L^{(2)}$, with $E' \subseteq E(G)$, be a Fulkerson Function. Function λ is $(B^+, \mathrm{P1})$-*strong* if it satisfies property P1 concerning the edges of B^+. Similarly for properties P2 and P3.

Let B^+ be an extended block and let $L = \{a, b, c, d, e, f\}$. Figure 4 exhibits four different Fulkerson Functions $\Lambda_j\colon E(B^+) \to L^{(2)}$, with $1 \leq j \leq 4$, called *models*. These models are used in the construction of a complete Fulkerson Function for an LP_1-snark. The next property relates properties P1, P2, and P3 to the functions presented in Figure 4. It is easily verified by inspection.

Property 3. *Let $\mathcal{L} = \{ab, cd, ef\}$. Each model $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ is $(B^+, \mathrm{P1})$-strong, $(B^+, \mathrm{P2})$-strong, and $(B^+, \mathrm{P3})$-strong. Moreover, the following statements are true:*
- (i) Λ_1 *satisfies* P2(a) *and* $\Lambda_1(e^w) = \Lambda_1(e^x)$*;*
- (ii) Λ_2 *satisfies* P2(a) *and* $\Lambda_2(e^w) \cap \Lambda_2(e^x) = \emptyset$*;*
- (iii) Λ_3 *satisfies* P2(b) *and* $\Lambda_3(e^w) \subset (\Lambda_3(e^x) \cup \Lambda_3(e^y)$*;*
- (iv) Λ_4 *satisfies* P2(b) *and* $\Lambda_4(e^w) \not\subset (\Lambda_4(e^x) \cup \Lambda_4(e^y))$. □

The following lemma is used in the proof of Theorem 6 for constructing a Fulkerson Function for an LP_1-snark.

Lemma 4. *Let B^+ be an extended block of an LP-snark. Let $L := \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{L} := \{12, 34, 56\}$. If $\lambda\colon \{e^w, e^x, e^y\} \to L^{(2)}$ is a $(B^+, \mathrm{P1})$-strong and $(B^+, \mathrm{P2})$-strong Fulkerson Function, then there exists a Fulkerson Function $\lambda^+\colon E(B^+) \to L^{(2)}$ which is $(B^+, \mathrm{P3})$-strong and such that λ is a restriction of λ^+.*



(a) Model Λ_1. (b) Model Λ_2.

(c) Model Λ_3. (d) Model Λ_4.

Figure 4: Fulkerson Functions $\Lambda_j\colon E(B^+) \to L^{(2)}$, $1 \leq j \leq 4$, with $L = \{a, b, c, d, e, f\}$.

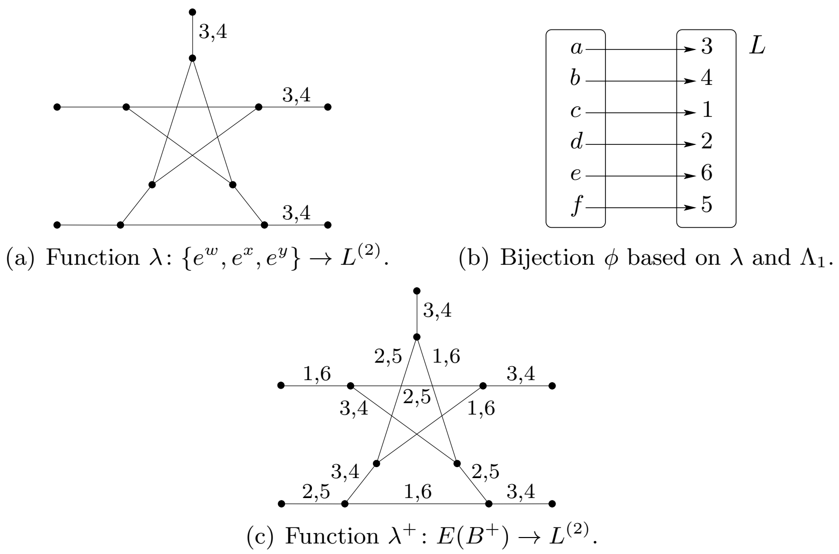
Proof. Let B^+, L, and $\mathcal{L}$ be defined as in the hypothesis. Let $\lambda\colon \{e^w, e^x, e^y\} \to L^{(2)}$ be a Fulkerson Function which is $(B^+, \text{P1})$-strong and $(B^+, \text{P2})$-strong. We construct a Fulkerson Function $\lambda^+\colon E(B^+) \to L^{(2)}$ using one of the models in Figure 4. Then, we show that λ^+ is $(B^+, \text{P3})$-strong and that λ is a restriction of λ^+.

Since λ is $(B^+, \text{P1})$-strong, $\lambda(e^w) \in \mathcal{L}$. Moreover, since λ is $(B^+, \text{P2})$-strong, λ satisfies either P2(a) or P2(b). Suppose λ satisfies P2(a). Therefore, we have to consider two cases: either $\lambda(e^w) = \lambda(e^x)$ or $\lambda(e^w) \cap \lambda(e^x) = \emptyset$. If λ satisfies P2(b), it is also necessary to consider two cases: either $\lambda(e^w) \subset (\lambda(e^x) \cup \lambda(e^y))$ or $\lambda(e^w) \not\subset (\lambda(e^x) \cup \lambda(e^y))$. Notice that, by Property 3, each of the functions $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ falls into exactly one of these four cases. Using the appropriate function as a model, it is possible to define a function $\lambda^+\colon E(B^+) \to L^{(2)}$ such that λ is a restriction of λ^+. It can be done by finding a suitable bijection from $\{a, b, c, d, e, f\}$ to L. Moreover, since every $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ is $(B^+, \text{P3})$-strong, λ^+ naturally is $(B^+, \text{P3})$-strong. Figure 5 exhibits an example of a function λ, a bijection, and a function λ^+ for the extended block B^+. □

Definition 5. *Let G be an LP-snark with k blocks. The sequence $\{G_j\}, 0 \leq j \leq k$, of subgraphs of G is defined as*

$$G_j = \begin{cases} G_C \cup G[\{x_1, x_1', y_1, y_1'\}], & j = 0 \\ G_{j-1} \cup B_j^+, & 1 \leq j < k \end{cases}.$$

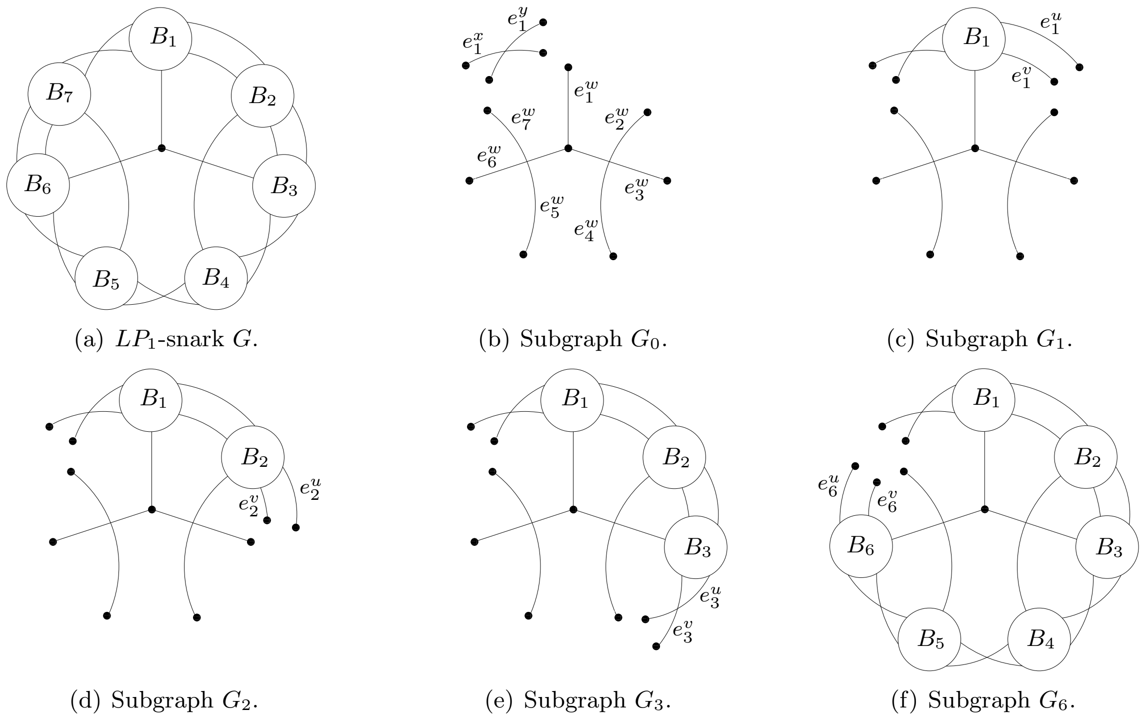
Figure 6 shows examples of subgraphs in the sequence $\{G_j\}$ of an LP_1-snark G with seven blocks.



(a) Function λ: $\{e^w, e^x, e^y\} \to L^{(2)}$. (b) Bijection ϕ based on λ and Λ_1.

(c) Function λ^+: $E(B^+) \to L^{(2)}$.

Figure 5: Example of a construction based on Model Λ_1 of a Fulkerson Function λ^+ for an extended block.



(a) LP_1-snark G. (b) Subgraph G_0. (c) Subgraph G_1.

(d) Subgraph G_2. (e) Subgraph G_3. (f) Subgraph G_6.

Figure 6: LP_1-snark G with 7 blocks and its subgraphs G_0, G_1, G_2, G_3, G_6 as established in Definition 5.

Theorem 6. *Every LP_1-snark has a complete Fulkerson Function.*

Proof. Let G be an LP_1-snark with k blocks. Let $L := \{1,2,3,4,5,6\}$. We construct a complete Fulkerson Function $\lambda\colon E(G) \to L^{(2)}$. For this purpose, let $\mathcal{L} := \{12, 34, 56\}$ be a partition of L.

First, we construct a complete Fulkerson Function λ_C for central subgraph G_C such that $\lambda_C(e^w_k) = 12$. Graph G_C is 3-edge-colourable since its connected components are isomorphic to K_2 and S_3. Then, there exists a 3-edge-colouring $\lambda_C\colon E(G_C) \to \mathcal{L}$ such that $\lambda_C(e^w_k) = 12$. It is easy to see that λ_C is a complete Fulkerson Function for G_C.

As a second step, we prove by induction on j that there exists a complete Fulkerson Function λ_j for the subgraph G_j, $j \in [0, k-1]$, such that

(i) $\lambda_j(e^w_i) = \lambda_C(e^w_i)$, $i \in [1,k]$;
(ii) $\lambda_j(e^x_1) = \lambda_j(e^y_1) = 12$;
(iii) λ_j is $(B^+_{j+1}, \text{P2})$-strong.

For $j = 0$, let function λ_0 be defined as: $\lambda_0(e) := \lambda_C(e)$, if $e \in \text{Dom}(\lambda_C)$; $\lambda_0(e) := 12$, if $e \in \{e^x_1, e^y_1\}$. It is clear that λ_0 satisfies conditions (i) and (ii). Since $\lambda_0(e^x_1) = \lambda_0(e^y_1) = 12 \in \mathcal{L}$, then λ_0 is $(B^+_1, \text{P2})$-strong. Thus, λ_0 satisfies condition (iii). Moreover, e^x_1 (e^y_1) is not adjacent to any other edge of $E(G_0)$. We conclude that λ_0 is a complete Fulkerson Function for G_0 and the result follows for $j = 0$.

For $j > 0$, suppose that there exists λ_{j-1} satisfying conditions (i), (ii), and (iii). By (i) and by the construction of λ_C, function λ_{j-1} is $(B^+_j, \text{P1})$-strong. By (iii), λ_{j-1} is $(B^+_j, \text{P2})$-strong. Then, we can apply Lemma 4 by letting λ be the restriction of λ_{j-1} to $\{e^w_j, e^x_j, e^y_j\}$. Suppose that $\lambda^+\colon E(B^+_j) \to L^{(2)}$ is the Fulkerson Function obtained from Lemma 4. Then, λ_j is defined as:

$$\lambda_j(e) := \begin{cases} \lambda_{j-1}(e), & e \in \text{Dom}(\lambda_{j-1}) \\ \lambda^+(e), & e \in \text{Dom}(\lambda^+) \setminus \{e^w_j, e^x_j, e^y_j\} \end{cases}. \tag{1}$$

Note that $\text{Dom}(\lambda_{j-1}) \cap \text{Dom}(\lambda^+) = \{e^w_j, e^x_j, e^y_j\}$, and that $\lambda_{j-1}(e) = \lambda^+(e)$, for all $e \in \{e^w_j, e^x_j, e^y_j\}$. Therefore, since λ_{j-1} and λ^+ are Fulkerson Functions, we conclude that λ_j also is.

Since λ_{j-1} satisfies conditions (i) and (ii), and $\{e^w_i : i \in [1,k]\} \cup \{e^x_1, e^y_1\} \subseteq \text{Dom}(\lambda_{j-1})$, then λ_j satisfies conditions (i) and (ii). Recall that $\{e^u_j, e^v_j\} = \{e^x_{j+1}, e^y_{j+1}\}$. By Lemma 4, λ^+ is $(B^+_j, P3)$-strong. Moreover, P2 is essentially the same statement as P3 applied to e^x_{j+1}, e^y_{j+1}. Therefore, λ_j satisfies (iii). This completes the induction.

Now, consider subgraph G_{k-1} and its complete Fulkerson Function $\lambda_{k-1}\colon E(G_{k-1}) \to L^{(2)}$ previously constructed, satisfying conditions (i), (ii), and (iii). Figure 7 shows a representation of G with G_{k-1}. Note that $E(G_{k-1}) = E(G) \setminus E(B_k)$.

Let $\mathcal{F}_{k-1} := \{M_l : l \in L\}$, where each M_l is the set $\{e \in E(G_{k-1}) : l \in \lambda_{k-1}(e)\}$. Remark that, from F1, M_l is a matching. Let $v \in V(G_{k-1})$. Then v has degree either one or three in G_{k-1} (see Figure 7). If v has degree three, then v is saturated by every matching of $\mathcal{F}_{k-1}$. If v has degree one, then it is saturated by precisely two matchings of $\mathcal{F}_{k-1}$, and it is incident with exactly one edge of ∂_{B_k}, with $\partial_{B_k} = \{e^u_k, e^v_k, e^w_k, e^x_k, e^y_k\}$.

Let $V_3(G_{k-1})$ be the set of vertices with degree three in G_{k-1}. Observe from Figure 7 that $|V_3(G_{k-1})| = |V(G)| - |V(B_k)|$. From the fact that G is cubic, we have that $|V(G)|$ is

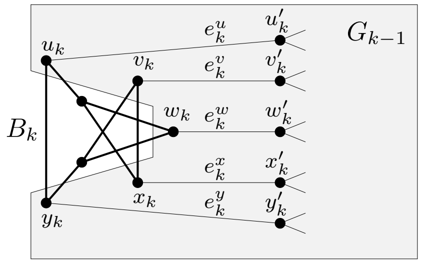


Figure 7: A decomposition of G into two edge disjoint subgraphs: G_{k-1}, in the shaded part, and B_k, in bold lines.

even. Also, by the construction of a block, $|V(B_k)|$ is odd. Therefore,

$$|V_3(G_{k-1})| \equiv 1 \pmod 2. \tag{2}$$

Let $l \in L$. Recall that M_l saturates every vertex in $V_3(G_{k-1})$. Also, every edge in $M_l \cap \partial_{B_k}$ has only one end in $V_3(G_{k-1})$. Thus, $|V_3(G_{k-1})| = 2|M_l| - |M_l \cap \partial_{B_k}|$. By (2), we have that $|M_l \cap \partial_{B_k}|$ is odd. Therefore,

$$|M_l \cap \partial_{B_k}| \geq 1. \tag{3}$$

Remark that $\lambda_{k-1}(e_1^x) = \lambda_{k-1}(e_1^y) = \lambda_{k-1}(e_k^w) = 12$. Therefore, $|M_1 \cap \partial_{B_k}| \geq 3$ and $|M_2 \cap \partial_{B_k}| \geq 3$. Considering this and (3), and from the fact that $|\partial_{B_k}| = 5$,

$$|M_l \cap \partial_{B_k}| = 3, \qquad l = 1, 2 \tag{4}$$
$$|M_l \cap \partial_{B_k}| = 1, \qquad l = 3, 4, 5, 6. \tag{5}$$

By construction, λ_{k-1} is $(B_k^+, \text{P2})$-strong. By P2 and by (5), we have that $\{\lambda_{k-1}(e_k^x), \lambda_{k-1}(e_k^y)\}$ is equal to either $\{35, 46\}$ or $\{36, 45\}$, thus satisfying P2(b). By (i) and by construction of λ_C, $\lambda_{k-1}(e_k^w) = 12$. Then, $\lambda_{k-1}(e_k^w) \not\subset (\lambda_{k-1}(e_k^x) \cup \lambda_{k-1}(e_k^y))$. Therefore, by Property 3(iv), we can use model Λ_4 (Figure 4(d)) to define a Fulkerson Function $\lambda_k^+ \colon E(B_k^+) \to L$. Figure 8 exhibits two examples. The other possibilities, by exchanging $\lambda_{k-1}(e_k^x)$ and $\lambda_{k-1}(e_k^y)$, are very similar. Consider the function λ defined as

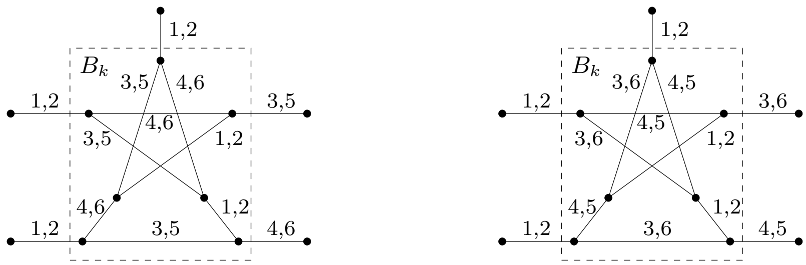
$$\lambda(e) := \begin{cases} \lambda_{k-1}(e), & e \in \text{Dom}(\lambda_{k-1}) \\ \lambda_k^+(e), & e \in \text{Dom}(\lambda_k^+) \setminus \{e_k^u, e_k^v, e_k^w, e_k^x, e_k^y\} \end{cases}. \tag{6}$$

Remark that $\lambda_{k-1}(e) = \lambda_k^+(e)$ for every $e \in \text{Dom}(\lambda_{k-1}) \cap \text{Dom}(\lambda_k^+) = \{e_k^u, e_k^v, e_k^w, e_k^x, e_k^y\}$, and that $\text{Dom}(\lambda) = \text{Dom}(\lambda_{k-1}) \cup \text{Dom}(\lambda_k^+) = E(G_{k-1}) \cup E(B_k^+) = E(G)$. Thus, λ is a complete Fulkerson Function of G. □

Corollary 7. *Every LP-snark G such that G_C is 3-edge-colorable admits a Fulkerson Collection.* □

Note that the family of Goldberg snarks [6] can be obtained by Loupekine's construction[3]. A simple definition of this family is given by Hao et. al. [7]. Every Goldberg snark is

[3]It is worth noting that Goldberg's construction [6], which encompasses the so-called Goldberg snarks, is more general than Loupekine's construction.



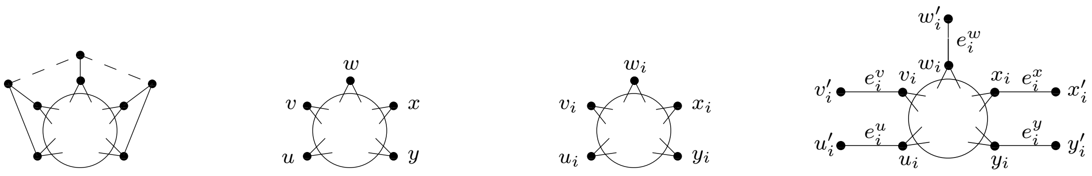
(a) $\lambda_k^+(e_k^x) = 35$ and $\lambda_k^+(e_k^y) = 46$ (b) $\lambda_k^+(e_k^x) = 36$ and $\lambda_k^+(e_k^y) = 45$

Figure 8: Fulkerson Function $\lambda_k\colon E(B_k^+) \to L$ based on model MD_4.

also an LP-snark. Moreover, the central subgraph of a Goldberg snark is 3-edge-colourable. Thus, Corollary 7 shows that Goldberg snarks verify Fulkerson's Conjecture, as an alternative to the proofs given by Hao et. al. [7] and Fouquet et. al. [3].

4 Application to additional subfamilies of Loupekine snarks

The technique used in the previous section can be adapted to show that Fulkerson's Conjecture is verified by a larger subfamily of Loupekine snarks. For this purpose, let G be any snark. The construction of a block $B(G)$ is sketched in Figure 9, which also shows the indexed block B_i. Since Figure 2(c) represents only vertices u, v, w, x, y of each block, it can be regarded as the sketch of an L_1-snark, assuming that its blocks are obtained from arbitrary snarks.

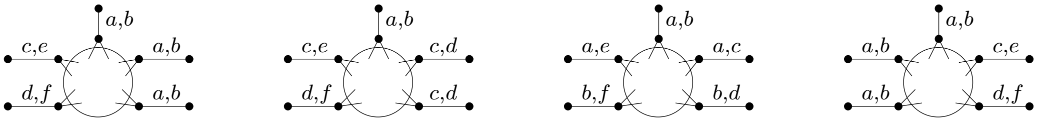


(a) A snark G. (b) A block $B(G)$. (c) Indexed block B_i. (d) Extended block B^+.

Figure 9: Construction of a generic block and an extended generic block.

Let B^+ be an extended block obtained from a generic block B, in a way similar to the extended block derived from $B(P)$. Figure 9(d) shows a sketch of the extended generic block B^+. Let $L := \{a, b, c, d, e, f\}$. Consider the labels of edges e^u, e^v, e^w, e^x, e^y in the models of Figure 4. We use these labels in the extended block of Figure 9(d) to define *generic models* $\Lambda_j\colon \partial_B \to L^{(2)}$, $1 \leq j \leq 4$, depicted in Figure 10. It is possible to generalise Theorem 6 to subfamilies of L_1-snarks other than LP_1-snarks. Let $\mathcal{B}$ be a set of non-isomorphic generic blocks. An *$L\mathcal{B}$-snark* is a Loupekine snark each block of which is isomorphic to a block of $\mathcal{B}$.

Suppose that, for each block $B \in \mathcal{B}$, there exist four Fulkerson Functions from $E(B^+)$ to $L^{(2)}$, with $L = \{a, b, c, d, e, f\}$, such that each generic model is a restriction of one of these functions. Then, we have four models for each block of $\mathcal{B}$. These models can be

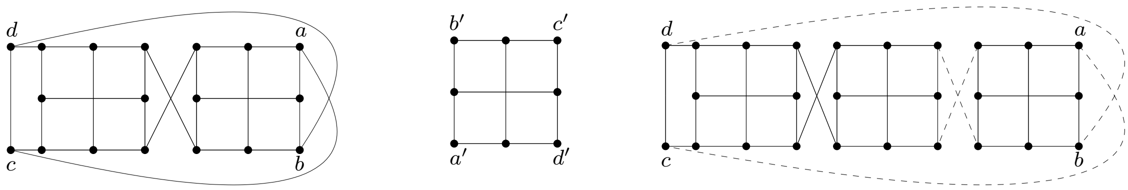


(a) Model Λ_1. (b) Model Λ_2. (c) Model Λ_3. (d) Model Λ_4.

Figure 10: Fulkerson Functions $\Lambda_j\colon \partial_B \to L^{(2)}$, $1 \le j \le 4$, with $L = \{a, b, c, d, e, f\}$.

used in the constructions in Lemma 4 and Theorem 6, in order to prove that $L\mathcal{B}_1$-snarks admit complete Fulkerson Functions. Moreover, analogously to Corollary 7, we can prove that every $L\mathcal{B}$-snark with a 3-edge-colourable central subgraph has a complete Fulkerson Function.

As an example, consider the first of the two infinite families of generalised Blanuša snarks, described by John. J. Watkins [16]. Let $G^1, G^2, \ldots$ denote the graphs of this family. Figure 11(a) shows G^1, with four of its vertices labeled a, b, c, d. A member $G^i, i > 1$, is recursively obtained in the following way. Remove edges ac and bd from G^{i-1}. Take a copy of the link graph G', shown in Figure 11(b) with its degree-two vertices labeled a', b', c', and d'. Add edges aa', bb', cc', and dd'. The resulting graph is G^i. As an example, Figure 11(c) shows member G^2 with the new edges drawn as dashed lines. Additionaly, in G^i, rename vertices c' and d' from the link graph to a and b, respectively, and maintain the labels of vertices c and d from G^{i-1}. Then, G_i has four vertices labeled a, b, c, d, used to construct G^{i+1}.



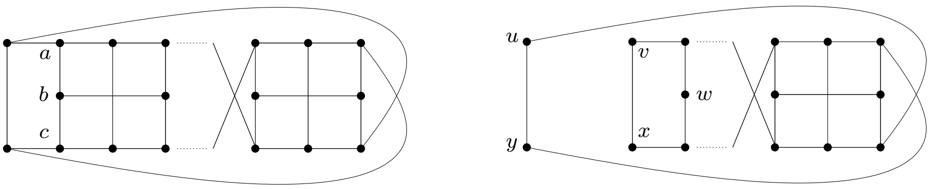
(a) Member G^1 of the family. (b) Link graph G'. (c) Member G^2 of the family.

Figure 11: First family of generalized Blanuša snarks.

Let G^i be a member of the first family of generalised Blanuša snarks. Let abc be the path of three vertices of G^i shown in Figure 12(a). The block $B^i \subseteq G^i$, depicted in Figure 12(b), is obtained by removing a, b, and c. Let $\mathcal{B}_{Blanuša} := \{B^1, B^2, B^3, \ldots\}$ be a set of non-isomorphic blocks, with each B^i obtained from G^i by removing vertices a, b, and c, as shown in Figure 12. Observe that each block B^i is uniquely defined.

Lemma 8. *Every extended block B^{i+}, with $B^i \in \mathcal{B}_{Blanuša}$ and $i \ge 1$, admits Fulkerson Functions $\Lambda^i_j\colon E(B^{i+}) \to L^{(2)}$, with $L = \{a, b, c, d, e, f\}$ and $1 \le j \le 4$, such that model Λ_j is a restriction of Λ^i_j.*

Proof. We prove the statement by induction on i. Take B^1 of $\mathcal{B}_{Blanuša}$. The extended block B^{1+} is depicted in Figure 13. Let $L := \{a, b, c, d, e, f\}$. Let Λ^1_1, Λ^1_2, Λ^1_3, Λ^1_4 be the Fulkerson



(a) A generalised Blanuša snark. (b) Block obtained from the snark in (a).

Figure 12: Construction of a block from a generalised Blanuša snark.

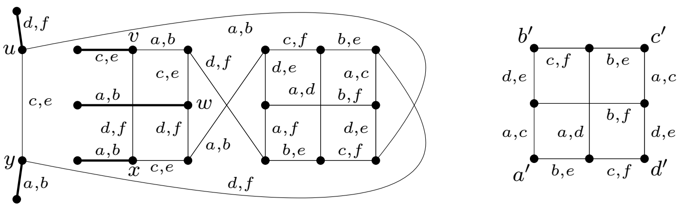
Functions from $E(B^{1+})$ to $L^{(2)}$ exhibited in Figure 13. By inspection of Figure 10 and Figure 13, it is easy to see that Λ_j is a restriction of Λ_j^1, $1 \leq j \leq 4$.

Suppose that $B^{(i-1)+}$, with $i > 1$, admits Fulkerson Functions $\Lambda_j^{(i-1)}, 1 \leq j \leq 4$, as stated in the lemma. The extended block B^{i+} can be constructed from $B^{(i-1)+}$ by attaching the link graph G' to $B^{(i-1)+}$, in a way similar to the construction of G^i. For each $j = 1, 2, 3,$ and 4, proceed as follows. Take a copy of $B^{(i-1)+}$ with each edge e labeled with $\Lambda_j^{(i-1)}(e)$. Let Λ'_j be the Fulkerson Function of G' exhibited in Figure 13. Take a copy of G' with each edge e labeled with $\Lambda'_j(e)$. Then, attach the copy of G' to the copy of $B^{(i-1)+}$ to obtain B^{i+}. New edges are added with this operation. For each new edge e, proceed as follows. Let $u \in V(B^{(i-1)+})$ and $v \in V(G')$ be the ends of e. Assign to e the labels of the removed edge of $B^{(i-1)+}$ which was incident with u. Note that, by construction, these labels are exactly the labels missing in the edges of G' incident with v. Thus, the resulting labeling of B^{i+} induces a Fulkerson Function $\Lambda_j^i \colon E(B^{i+}) \to L^{(2)}$. Notice that the labels of the edges of ∂_{B^i} are equal to their labels in $B^{(i-1)+}$. Since Λ_j is a restriction of Λ_j^{i-1}, Λ_j is also a restriction of Λ_j^i. □

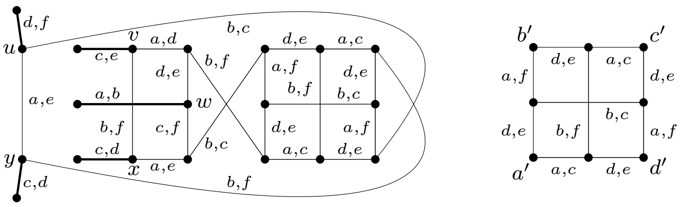
By applying the method described in this section, the following result is obtained.

Theorem 9. *Every Loupekine snark G each block of which is isomorphic to a block of $\mathcal{B}_{Blanuša} \cup \{B(P)\}$, and such that the central subgraph G_C is 3-edge-colourable, admits a complete Fulkerson Function.* □

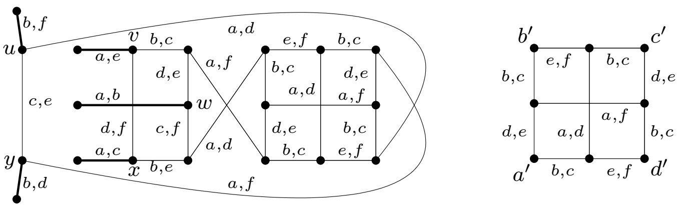
In this work, we verified Fulkerson's Conjecture [4] for the infinite family of LP_1-snarks, constructed from the Petersen Graph using Loupekine's method. Moreover, we showed how the technique used in this proof can be adapted to verify the conjecture for other families of L-snarks constructed from snarks other then the Petersen Graph. This technique can also be applied to some Loupekine snarks without a 3-edge-colourable central subgraph. It can be applied without great effort to snarks produced by connecting two independently generated block subgraphs, as suggested by Issacs [8]. These results contribute as one more evidence that Fulkerson's Conjecture is true.



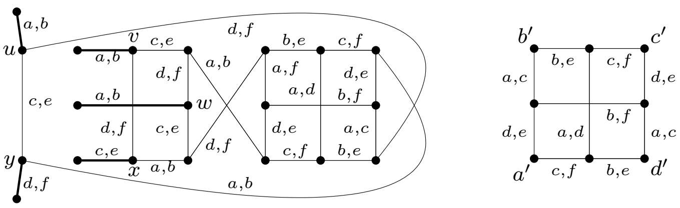
(a) Fulkerson Functions Λ_1^1 for B^{1+} and Λ_1' for G'.



(b) Fulkerson Functions Λ_2^1 for B^{1+} and Λ_2' for G'.



(c) Fulkerson Functions Λ_3^1 for B^{1+} and Λ_3' for G'.



(d) Fulkerson Functions Λ_4^1 for B^{1+} and Λ_4' for G'.

Figure 13: Models Λ_1^i, Λ_2^i, Λ_3^i, and Λ_4^i for the generalised Blanuša snarks. In each case, models are provided for B^{1+} and G'.

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