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Abstract

In 1976, F. Loupekine created a method for constructing new snarks from already known ones. In the present work, we consider an infinite family of snarks constructed from the Petersen Graph using Loupekine's method, and show that this family verifies Fulkerson's Conjecture. In addition, we show that it is possible to extend this result to families constructed from snarks other than the Petersen Graph.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A graph G is *cubic* if every $v \in V(G)$ have degree three. An *edge-colouring* of G is a mapping $\pi : E(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set of *colours*, such that $\pi(e) \neq \pi(f)$ for any two adjacent edges $e, f \in E(G)$. If $|\mathcal{C}| = k$, then π is a *k-edge-colouring*. For each colour $c \in \mathcal{C}$, the set $\{e \in E(G) : \pi(e) = c\}$ is a *colour class*. A *matching* M of G is a set of pairwise nonadjacent edges of $E(G)$. Notice that a colour class is a matching. Let $v \in V(G)$. If there is an edge of M incident with v, then v is *M-saturated*. Otherwise, v is *M-unsaturated*. If every vertex of G is M-saturated, then M is a *perfect matching*. An edge $e \in E(G)$ is a *bridge* if $G \setminus e$ has more components than G.

This work deals with a conjecture independently formulated by C. Berge and D. R. Fulkerson [7]. It is usually called Fulkerson's Conjecture[1], since Fulkerson [3] first published it in 1971.

Conjecture 1 (Fulkerson's Conjecture)**.** *Every bridgeless cubic graph has six perfect matchings such that each edge belongs to exactly two of these matchings.*

In a cubic graph with a 3-edge-colouring, each colour class is a perfect matching. Thus, each edge of the graph belongs to one of three disjoint perfect matchings. A list of six perfect matchings such that each edge belongs to exactly two of them can be obtained by simply duplicating each colour class. Since Fulkerson's Conjecture clearly holds in this

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[1]Fulkerson's Conjecture is sometimes called the Berge-Fulkerson Conjecture [7].

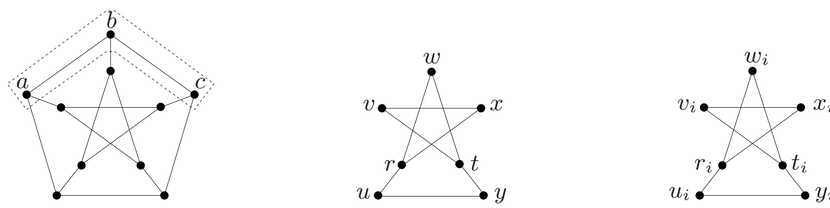
case, it remains to verify the conjecture for bridgeless cubic graphs not admitting a 3-edge-colouring.

Snarks are non-3-edge-colourable bridgeless cubic graphs. In 1880, P. G. Tait showed the equivalence of the Four Colour Problem with that of colouring the edges of cubic planar graphs [8]. The search for a counterexample to the Four Colour Conjecture brought great importance to the snarks. The first few snarks appeared as individual discoveries, sometimes with long intervals between them. The first snark was the Petersen Graph (Figure 1(a)), found in 1898. The first infinite families were described in 1975 by R. Isaacs [6]. F. Loupekine created a method for constructing new snarks from subgraphs of known snarks. This construction was described by Isaacs [5] in 1976.

Fulkerson's Conjecture has been verified for a few well known families of snarks, such as the flower snarks, the Goldberg snarks, and the generalized Blanuša snarks [4, 2]. In this work, we show that Fulkerson's Conjecture is also verified by an infinite family of snarks constructed from the Petersen Graph with Loupekine's method. Additionally, we show that it is possible to extend this result to families constructed from snarks other than the Petersen Graph.

2 LP_0-snarks

This section presents two infinite families of snarks, namely LP_0-snarks and LP_1-snarks[2], constructed with Loupekine's method [5]. Both families are constructed with *building blocks*, or simply *blocks*, derived exclusively from the Petersen Graph, denoted by P. A block B is formed by removing a path with three vertices from P. This process is illustrated in Figures 1(a) and 1(b). Since P has girth five, a block B has five distinct vertices with degree two. These vertices are named u, v, w, x, y as in Figure 1(b).



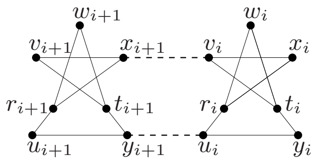
(a) Petersen Graph P. (b) Graph B. (c) Indexed block B_i.

Figure 1: Construction of a building block from the Petersen Graph.

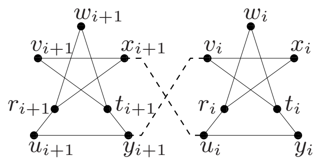
In order to construct an LP_1-snark, consider an odd number of indexed blocks, $B_1, \ldots, B_k$, isomorphic to the graph in Figure 1(b). Adjust notation so that vertex labels are indexed as shown in Figure 1(c). For each $i \in [1, k]$, link blocks B_i, B_{i+1} (indexes greater than k taken modulo k) by one of the pairs of *link edges* $\{u_i y_{i+1}, v_i x_{i+1}\}$ or $\{u_i x_{i+1}, v_i y_{i+1}\}$, as shown in Figure 2. Blocks B_i and B_{i+1} are *adjacent*.

To finalize the construction, the set of blocks is partitioned into parts with two or three blocks. For each part with two blocks, say B_i, B_j, add edge $w_i w_j$, creating a 2-*gadget*.

[2]The classes of Loupekine snarks referred here have their names taken from the work by L. Vaux [9].



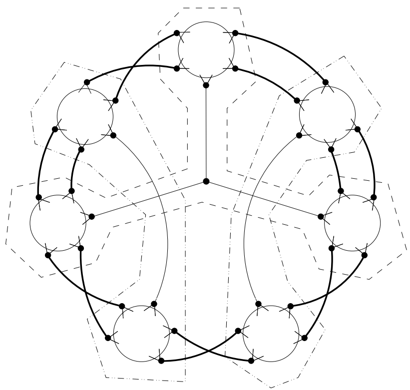
(a) A pair of *parallel* link edges.



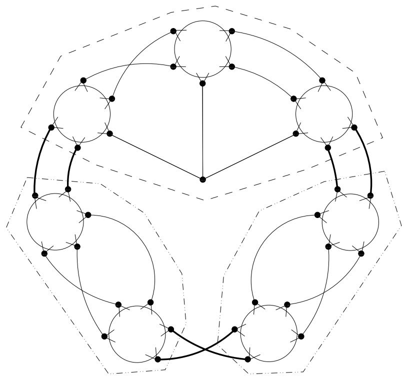
(b) A pair of *crossing* link edges.

Figure 2: The possible pairs of link edges (dashed lines) between two adjacent blocks.

For each part with three blocks, namely B_i, B_j, B_l, create a new vertex z_i and add edges z_iw_i, z_iw_j, z_iw_l, creating a 3-*gadget*. The resulting graph is cubic and is an LP_1-*snark*. Figure 3 shows sketches of LP_1-snarks and its gadgets.



(a) An LP_1-snark.



(b) An LP_1-snark which is an LP_0-snark.

Figure 3: Two LP_1-snarks. Only vertices u, v, w, x, y are represented in each block. Each graph has two 2-gadgets, marked with dashed-and-dotted boundaries, and one 3-gadget, marked with a dashed boundary. Thick lines are gadget-link edges.

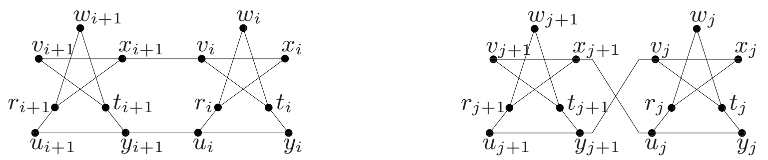
Different subfamilies of LP_1-snarks are obtained by restricting the way the partitioning of the set of blocks is done. If each gadget comprises blocks with consecutive indexes only, then the generated graphs are LP_0-*snarks*. Figure 3(b) represents an LP_0-snark.

If $k > 3$, then link edges between blocks of the same gadget are contained in the gadget. When $k = 3$, G has only one gadget. In this case, the link edges between B_3 and B_1 are not considered part of the gadget. A link edge not contained in any gadget is a *gadget-link edge*. A vertex of a gadget is a *border vertex* if it is incident with a gadget-link edge, which implies that this vertex has degree two in the gadget. Two gadgets are *adjacent* if they have adjacent border vertices. In Figure 3(b), for example, all gadgets are pairwise adjacent.

3 Main results

In this section, it is proved that every LP_0-snark verifies Fulkerson's Conjecture. Initially, we show some properties of LP_0-snarks that are useful in the proof of Theorem 4.

Proposition 2. *Let G be the graph comprised of blocks B_i, B_{i+1} and the pair of link edges $u_i y_{i+1}, v_i x_{i+1}$, as in* Figure 4(a)*. Let H be the graph comprised of blocks B_j, B_{j+1} and the pair of link edges $u_j x_{j+1}, v_j y_{j+1}$, as in* Figure 4(b)*. Then, G and H are isomorphic.*

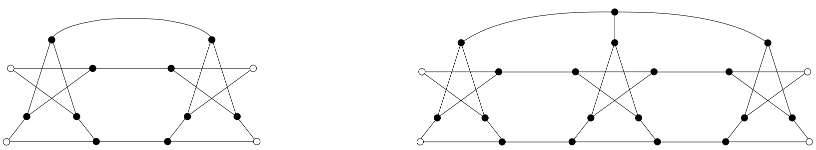


(a) (b)

Figure 4: Two isomorphic graphs, each comprised by two adjacent blocks and their link edges.

Proof. Let ϕ be a bijection from $V(G)$ to $V(H)$ defined as: $\phi(x_i) = y_j; \phi(y_i) = x_j; \phi(u_i) = v_j; \phi(v_i) = u_j; \phi(r_i) = t_j; \phi(t_i) = r_j; \phi(w_i) = w_j$; and for all $z_{i+1} \in V(B_{i+1})$, $\phi(z_{i+1}) = z_{j+1}$. The result follows by inspection of the adjacency relations in both graphs. □

Corollary 3. *Every gadget of an LP_0-snark is isomorphic to one of the graphs exhibited in* Figure 5*.*



(a) Graph isomorphic to every 2-gadget. (b) Graph isomorphic to every 3-gadget.

Figure 5: Graphs isomorphic to every gadget of an LP_0-snark.

Proof. The result follows by applying Proposition 2 in each gadget with crossing link edges. Note that this process may cross or uncross pairs of gadget-link edges. □

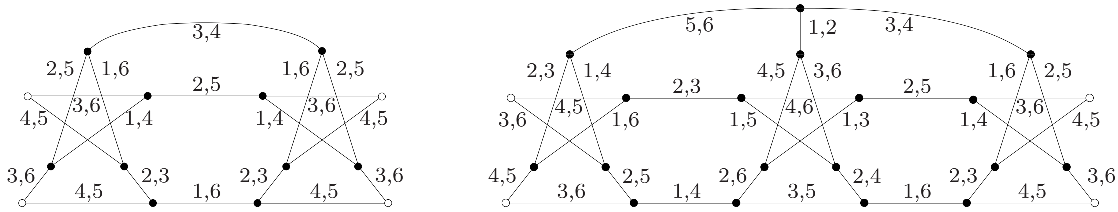
Theorem 4. *Every LP_0-snark has six perfect matchings such that each edge belongs to exactly two of them.*

Proof. Let G be an LP_0-snark comprised of l gadgets $G^1, \dots, G^l$. Recall that each gadget can be a 2-gadget or a 3-gadget, and that adjacent gadgets are connected by gadget-link edges (see Figure 3(b)). Adjust notation so that gadgets G^i, G^{i+1} are adjacent, for all $i \in [1, l]$, with indexes greater than l taken modulo l.

Let G^i be a gadget of G. By Corollary 3, G^i is isomorphic to one of the graphs in Figure 5. Figure 6 exhibits a labeling of these graphs, with two different labels assigned to each edge, and every label belonging to the set $\{1, \dots, 6\}$. If G^i is a 2-gadget, then we assign to every edge of G^i the labels of the corresponding edge of the graph in Figure 6(a). If G^i is a 3-gadget, we proceed analogously for the graph in Figure 6(b). For all $j \in \{1, \dots, 6\}$, let M_j^i be the set comprised of the edges of $E(G^i)$ with label j. We conclude, by inspection, that each M_j^i satisfies the following properties:

(a) $M_j^i, 3 \leq j \leq 6$, is a perfect matching;

(b) every border vertex of G^i (white vertices in Figures 5 and 6) is M_1^i-unsaturated and M_2^i-unsaturated.



(a) (b)

Figure 6: Labeling of LP_0 gadgets used to construct matchings for an arbitrary LP_0-snark.

For all $j \in \{1, \ldots, 6\}$, let $M_j := \bigcup_{i=1}^{l} M_j^i$. Since gadgets are pairwise vertex-disjoint, M_j is a matching. Moreover, by construction, every edge of each gadget is contained in exactly two of these matchings. In order to complete the proof, it is necessary to assign each gadget-link edge to two of these six matchings. By (a), M_3, M_4, M_5, M_6 are perfect matchings of G. By (b), all border vertices of G are M_1-unsaturated and M_2-unsaturated. The border vertices are the ends of the gadget-link edges. Since the gadget-link edges are pairwise nonadjacent, it is enough to assign each of them to M_1 and M_2. □

4 Final remarks

The previous section showed that every LP_0-snark verifies Fulkerson's Conjecture. It is possible to adapt the technique used there to show that the conjecture is verified by a wider class of snarks. For this purpose, let G be a snark with girth at least five[3]. Obtain $B \subseteq G$ by removing a path of three vertices from G, as sketched in Figure 7. Thus, B has five vertices of degree two, and is called a *(generic) block*.

(a) Snark G and a path with three vertices.

(b) Block B obtained from snark G.

Figure 7: Construction of a generic block.

[3]Snarks with girth less than five are considered trivial. Although Isaacs [5] avoids using trivial snarks, G can be a trivial snark as long as we choose a path of three vertices with no edges belonging to a cycle of size less than five.

L_0-snarks are constructed in the same way as LP_0-snarks, with the only difference that L_0-snarks are composed of generic blocks, while LP_0-snarks consist of blocks obtained exclusively from the Petersen Graph. Since Figure 3(b) represents only vertices u, v, w, x, y for each block, it can be regarded as an example of an L_0-snark. *Generic* 2-*gadgets* and *generic* 3-*gadgets* are obtained similarly as described for LP_0-snarks. Figure 8 sketches all possible generic 2-gadgets and 3-gadgets. Note that for a generic gadget it is necessary to consider more cases because we have no information about isomorphisms.

Now, we are ready to show how to generalise the main result of Section 3 to other subfamilies of L_0-snarks. Consider edge labels in Figure 6. These labels are used to obtain a labeling for each generic gadget of Figure 8. By analysing each block and its edge cut in this figure, and assigning labels 1 and 2 to the gadget-link edges, we get the six labeling cases sketched in Figure 9. Suppose that in each case it is possible to extend the labeling to the edges belonging to the blocks, satisfying the following properties:

(i) exactly two different labels are assigned to each edge;

(ii) for all $j \in \{1, \ldots, 6\}$, every vertex of the block is incident with an edge carrying label j.

Then, for an L_0-snark, these labelings can be combined to define labelings of the gadgets, and it is possible to proceed as in the proof of Theorem 4, by taking unions of the matchings induced by the gadget labelings. Thus, we end up with six perfect matchings which attest that the snark verifies Fulkerson's Conjecture.

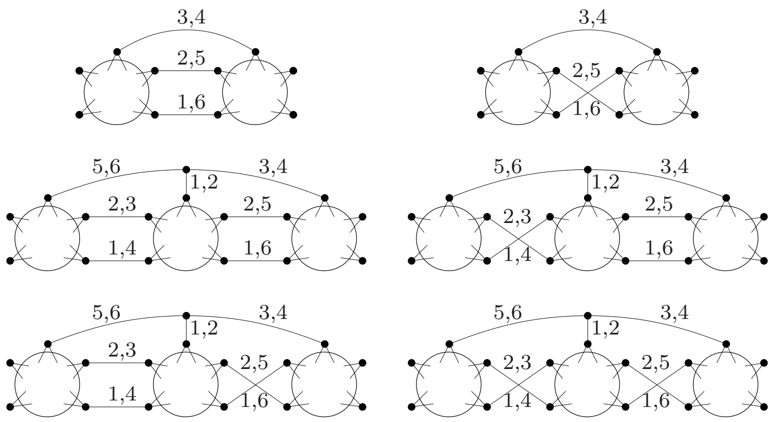


Figure 8: Composition and labeling of generic gadgets.

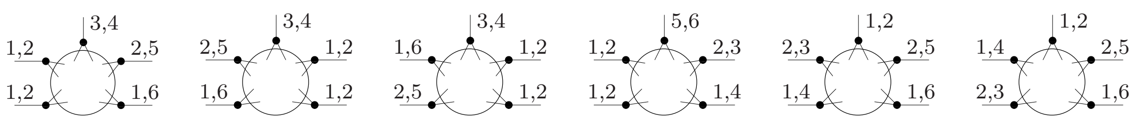


Figure 9: Generic blocks with labeled edge cuts.

As an example of a generic block, consider a block obtained from one of the Blanuša snarks [1], shown in Figure 10. Let this block be denoted by $B_{Blanuša}$. Figure 11 shows labelings satisfying (i) and (ii).

Corollary 5. *Every L_0-snark each block of which is either derived from the Petersen Graph or isomorphic to $B_{Blanuša}$ satisfies Fulkerson's Conjecture.* □

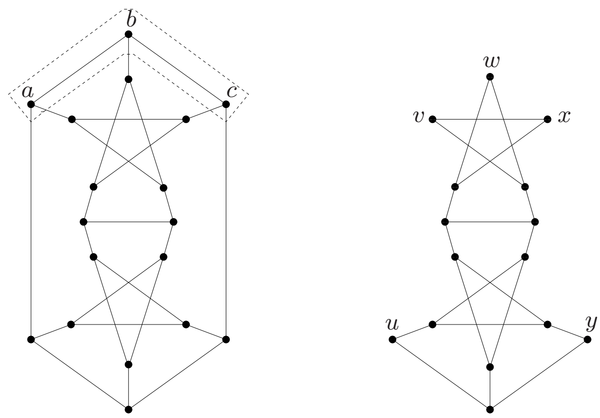


Figure 10: Building block obtained from a Blanuša snark.

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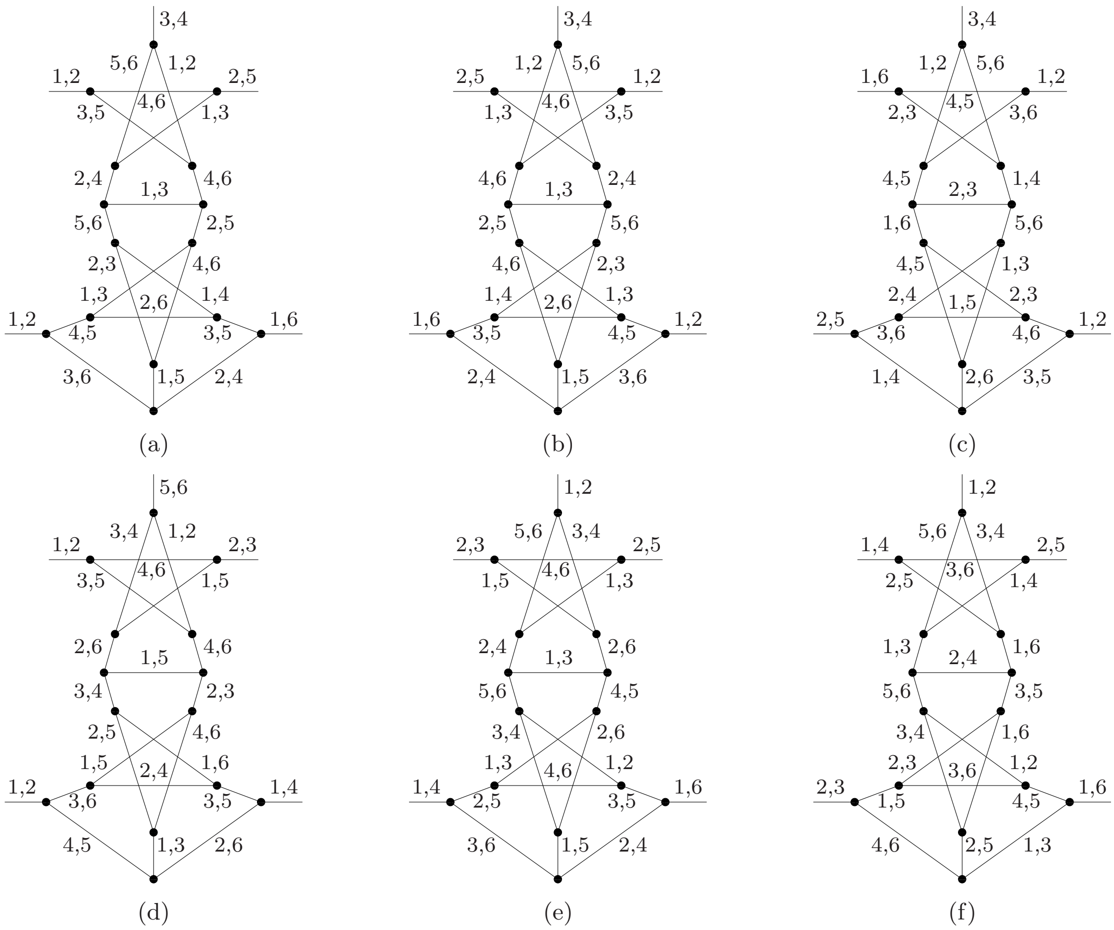

Figure 11: Labelings of a block obtained from the Blanuša snark and its edge cut.