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The total chromatic number of some families of snarks*

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Abstract

The *total chromatic number* $\chi_T(G)$ is the least number of colours needed to colour the vertices and edges of a graph G, such that no incident or adjacent elements (vertices or edges) receive the same colour. It is known that the problem of determining the total chromatic number is NP-hard and it remains NP-hard even for cubic bipartite graphs. *Snarks* are simple connected bridgeless cubic graphs which are not 3-edge colourable. In this paper, we show that the total chromatic number is 4 for three infinite families of snarks, namely, the Flower Snarks, the Goldberg Snarks and the Twisted Goldberg Snarks. This result reinforces the conjecture that all snarks are type 1. Moreover, we give recursive procedures to construct 4-total colourings in each case.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. An *element* of G is a vertex or an edge of G. A subset of $V(G) \cup E(G)$ is *independent* if its elements are pairwise nonadjacent and nonincident. If $S \subseteq E(G)$, then $V(S)$ is the set of the ends of the edges of S. The *graph induced by* S, $G[S]$, is the graph whose vertex set is $V(S)$ and edge set is S. As usual, we denote by $d(v)$ the *degree* of $v \in V(G)$ and by $\Delta(G)$ the maximum degree of G.

Let $S = E(G)$ and let $\mathcal{C}$ be a set of colours. An *edge colouring* of G is a mapping $\phi : S \to \mathcal{C}$ such that, for each adjacent elements $x, y \in S$, we have $\phi(x) \neq \phi(y)$. It is well known that the mininum number of colours needed to colour the edges of a simple graph G is either $\Delta(G)$, or $\Delta(G) + 1$ [1]. We say that G is *class 1* in the first case and *class 2* in the second case.

Let $S = V(G) \cup E(G)$. A *total colouring* of G is a mapping $\phi : S \to \mathcal{C}$ such that, for each adjacent or incident elements $x, y \in S$, we have $\phi(x) \neq \phi(y)$. If $|\mathcal{C}| = k$, then mapping ϕ is called a *k-total colouring*. Let π be an assignment of colours to a set $S \subseteq V(G) \cup E(G)$. Let $x \in S$; we say that c *occurs* in x if either $\pi(x) = c$, or there exists $y \in S$ adjacent to, or incident with, x such that $\pi(y) = c$. If there exist $x, y \in S$ which are adjacent or incident and such that $\pi(x) = \pi(y)$, we say that π has a *conflict*.

The *total chromatic number* of G, $\chi_T(G)$, is the least integer k for which G admits a k-total colouring. Clearly, $\chi_T(G) \geq \Delta(G) + 1$. Sánchez-Arroyo [2] has shown that

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deciding whether $\chi_T(G) = \Delta(G) + 1$ is NP-complete. McDiarmid and Sánchez-Arroyo [3] have shown that even the problem of determining the total chromatic number of k-regular bipartite graphs is NP-hard, for each fixed $k \geq 3$. The *Total Colouring Conjecture (TCC)*, posed independently by Behzad [4] and Vizing [1], states that every simple graph G has $\chi_T(G) \leq \Delta(G) + 2$. If $\chi_T(G) = \Delta(G) + 1$, then G is a *type 1* graph; if $\chi_T(G) = \Delta(G) + 2$, then G is a *type 2* graph.

Although the names type 1 and type 2 were inspired by their counterparts for edge-colourings, the two concepts are independent, as illustrated by the following classes of graphs.

- *Class 1/Type 1*: $K_{m,n}$, $m = n$;
  C_n, n even and $n \equiv 0 \pmod 3$;
- *Class 1/Type 2*: $K_{m,n}$, $m \neq n$;
  C_n, n even, $n \equiv 1, 2 \pmod 3$;
  K_n, n even;
- *Class 2/Type 1*: C_n, n odd and $n \equiv 0 \pmod 3$;
  K_n, n odd;
- *Class 2/Type 2*: C_n, n odd and $n \equiv 1, 2 \pmod 3$.

Considering the importance of cubic graphs for Graph Theory, we restricted our attention to them. Initially, we remark that the TCC was verified for cubic graphs [5, 6]. Moreover, we know classes of cubic graphs which are class 1/type 1 and which are class 1/type 2: *near ladders*, which are bipartite graphs with k vertices in each part of the bipartition are type 1, when k is even, and type 2, when k is odd [7]. Therefore, we turned our attention to cubic graphs that are class 2.

Snarks are simple connected bridgeless cubic graphs whose edges cannot be coloured with three colours. The study of these graphs began in 1880, when Tait proved that the four-colour theorem is equivalent to the statement that every cubic map is 3-edge-colourable. This equivalence justifies the historic importance of snarks and the search for planar cubic graphs whose edges cannot be coloured with three colours. The Petersen graph was the first discovered snark and it remained the only known snark until 1946, when the Blanuša Snarks were found [8]. The next snark was discovered by Blanche Descartes (pseudonymous of Tutte et al.) [9]. In 1975, Isaacs found two infinite sets of snarks [10], including the *Flower Snarks*. In 1981, Goldberg found an additional class of snarks [11]. The name snarks was given by Martin Gardner [12] in 1976, inspired on the "The Hunting of the Snark", by Lewis Carroll.

In 2003, Cavicchioli et al. [13] showed, using computers, that every snark of order less than 30 is type 1, without presenting a colouring for them. In that work the authors posed the problem of finding (if any) a snark which is type 2 and has the smallest number of vertices.

In this work, we consider that problem and prove that all graphs in three infinite families of snarks, the Flower Snarks, the Goldberg Snarks, and the Twisted Goldberg Snarks, are type 1. We also give recursive procedures to construct 4-total colourings in each case.

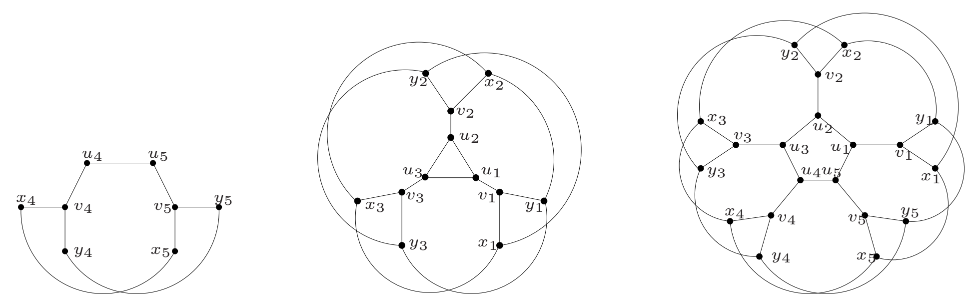
2 Main results

In this section we determine the total chromatic number of Flower Snarks, Goldberg Snarks and Twisted Goldberg Snarks. Graphs in these families share a common property: they can be built from a suitable glueing of some special graphs which we call *basic blocks*.

2.1 Flower Snarks

Let F_3, F_5, $\ldots$, F_{2i+1}, $i \geq 1$, be the members of the family of Flower Snarks, where F_i has $4i$ vertices. For this family we define the *basic block* B_i as the graph with vertex set $V(B_i) = \{u_i, v_i, x_i, y_i\}$ and edge set $E(B_i) = \{u_iv_i, x_iv_i, y_iv_i\}$. We define the set of *link edges* as $E_{ij} = \{u_iu_j, x_ix_j, y_iy_j\}$, and the *link graph* L_i, i odd and $i \geq 5$, as the union of B_{i-1}, B_i, and the graph induced by $E_{(i-1)i}$. Figure 1(a) shows L_5.

The first Flower Snark, F_3, is defined as the union of B_1, B_2, B_3, and the graph induced by $E_{23} \cup E_{31} \cup \{u_1u_2, x_1y_2, y_1x_2\}$; depicted in Figure 1(b). For each i odd and $i \geq 5$, F_i is obtained from graphs F_{i-2} and L_i as follows: $V(F_i) = V(F_{i-2}) \cup V(L_i)$, and $E(F_i) = (E(F_{i-2}) \backslash E^{out}_{i-2}) \cup E(L_i) \cup E^{in}_i$, where $E^{out}_{i-2} = E_{(i-2)1}$, and $E^{in}_i = E_{(i-2)(i-1)} \cup E_{i1}$. Figure 1(c) shows F_5, constructed from graphs F_3 and L_5. The next result, Theorem 1, states that Flower Snarks are type 1 graphs.



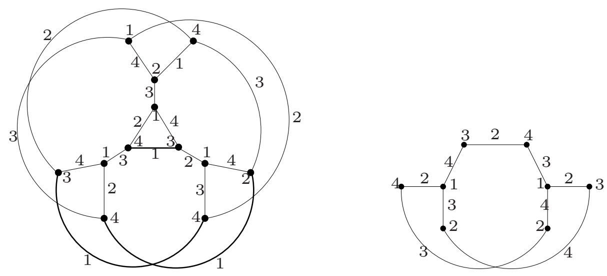
(a) Link graph L_5 (b) Graph F_3. (c) Graph F_5

Figure 1: Graph F_5 constructed from graphs F_3 and L_5.

Theorem 1. *Each Flower Snark F_i, i odd and $i \geq 3$, is a type 1 graph.*

Proof. We prove that each F_i admits a 4-total colouring such that all edges of E^{out}_i have the same colour 1. The proof is by induction and based on the recursive procedure described above. Figure 2(a) shows π_3, a 4-total colouring of graph F_3. Note that the edges of E^{out}_3, which are in bold, have the same colour 1. Figure 2(b) shows π, a fixed 4-total colouring of L_i, where $\pi(u_{i-1}) = 3$ and $\pi(u_i) = 4$.

Graph F_i is recursively constructed from F_{i-2} and L_i. By induction hypothesis, F_{i-2} admits a 4-total colouring π_{i-2} such that the E^{out}_{i-2} edges have the same colour 1. We obtain π_i as follows. Assign colour 1 to the edges of E^{in}_i (recall that E^{in}_i is an independent set of



(a) (b)

Figure 2: The 4-total colouring π_3 and π for graphs F_3 and L_i, respectively.

edges). Assign to each element of $(V(F_i) \cup E(F_i)) \setminus E_i^{in}$ the colour of its equivalent element in F_{i-2} or L_i.

Now, we show that π_i is a 4-total colouring for F_i. First, note that the total colouring of subgraph $F_i[V(F_i) \cap V(F_{i-2})]$ is a 4-total colouring because π_{i-2} is a 4-total colouring for F_{i-2}. The same applies to the total colouring of $F_i[V(F_i) \cap V(L_i)]$, obtained from π. Moreover, $V(E_i^{in}) = V(E_i^{out}) \cup (V(L_i) \setminus \{v_{i-1}, v_i\})$, therefore colour 1 does not occur in any vertex of $V(E_i^{in})$.

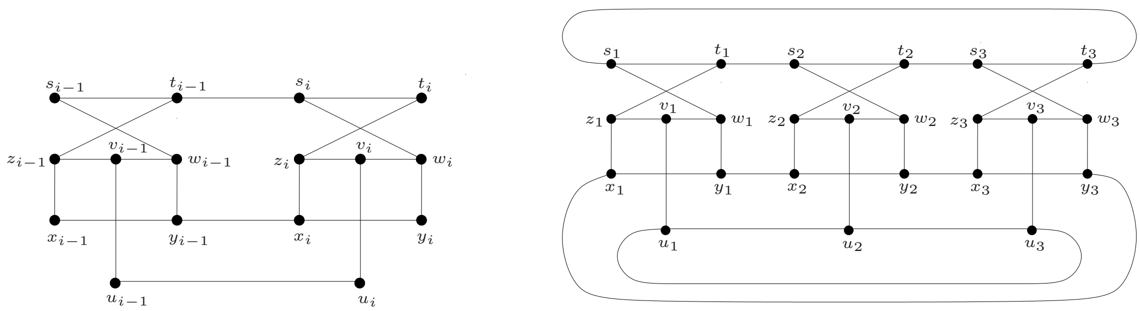
We complete the proof showing that the ends of edges in E_i^{in} have distinct colours. Since colouring π is fixed, we know the colours of vertices $V(E_i^{in}) \cap V(L_i)$. We also know the colours of u_1, x_1, y_1, because they are in F_3. It remains to determine the colours of vertices u_{i-2}, x_{i-2} and y_{i-2}. First, note that these vertices belong to F_{i-2}. If $i = 5$, these vertices are u_3, x_3 and y_3, that belong to F_3; and for $i \geq 7$, these vertices belong to L_{i-2}, with fixed colouring π. Therefore, we conclude that: $\pi_i(u_{i-2}) = 4$ and $\pi_i(u_{i-1}) = 3$; $\pi_i(x_{i-2}) \in \{2, 3\}$ and $\pi_i(x_{i-1}) = 4$; $\pi_i(y_{i-2}) \in \{3, 4\}$ and $\pi_i(y_{i-1}) = 2$; $\pi_i(u_i) = 4$ and $\pi_i(u_1) = 3$; $\pi_i(x_i) = 2$ and $\pi_i(x_1) = 4$; $\pi_i(y_i) = 3$ and $\pi_i(y_1) = 2$; this ends the proof. □

2.2 Goldberg and Twisted Goldberg Snarks

For the second family of snarks considered, Goldberg Snarks, the *basic block* B_i is the graph with vertex set $V(B_i) = \{u_i, v_i, x_i, y_i, z_i, w_i, s_i. t_i\}$ and edge set $E(B_i) = \{u_iv_i, x_iy_i, x_iz_i, y_iw_i, z_iv_i, z_it_i, v_iw_i, w_is_i, s_it_i\}$. The set of *link edges* E_{ij} is $\{t_is_j, y_ix_j, u_iu_j\}$. Figure 3(a) shows the *link graph* L_i, which is obtained from the basic blocks B_{i-1} and B_i, connected by $E_{(i-1)i}$.

The first Goldberg Snark, G_3, is defined as the union of B_1, B_2, B_3, and the graph induced by $E_{12} \cup E_{23} \cup E_{31}$; it is depicted in Figure 3(b). For each i odd and $i \geq 5$, graph G_i is obtained from G_{i-2} and L_i as follows: $V(G_i) = V(G_{i-2}) \cup V(L_i)$; and $E(G_i) = (E(G_{i-2}) \setminus E_i^{out}) \cup E(L_i) \cup E_i^{in}$, where $E_i^{out} = E_{(i-2)1}$ and $E_i^{in} = E_{(i-2)(i-1)} \cup E_{i1}$.

Theorem 2. *Each Goldberg Snark G_i, i odd and $i \geq 3$, is a type 1 graph.*

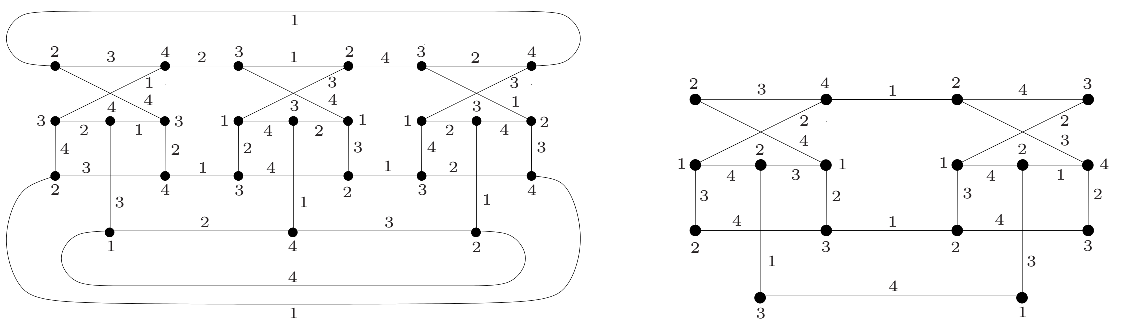


(a) Link graph L_i. Note that blocks B_{i-1} and B_i appear consecutively in the figure

(b) Graph G_3. Blocks B_1, B_2, B_3 appear consecutively in the figure.

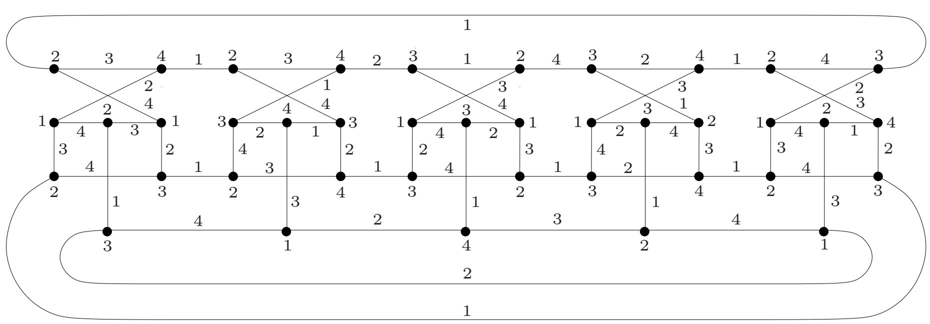
Figure 3: Link graph L_i and graph G_3.

Proof. The proof is similar to the previous one. We construct 4-total colourings π_3 and π_5 for G_3 and G_5, as depicted in figures 4(a) and 4(c), respectively. We also construct a 4-total colouring π for L_i, as shown in Figure 4(b).



(a) A 4-total colouring π_3 for G_3.

(b) A 4-total colouring π for L_i. Blocks B_{i-1}, B_i appear consecutively in the figure.



(c) A 4-total colouring π_5 for G_5. Blocks B_1, B_2, B_3, B_4, B_5 appear consecutively in the figure.

Figure 4: 4-total colourings for G_3, L_i and G_5.

For each i odd and $i \geq 7$, a 4-total colouring π_i for G_i is obtained from the colourings

π_{i-2} and π, using the recursive definition of Goldberg Snarks. To obtain π_i, we assign to each element of $V(G_i) \cup E(G_i) \setminus E_i^{in}$ the colour of its equivalent element in G_{i-2} or L_i. The edges of E_i^{in} receive the following colours:

$$\begin{array}{rclcl} \pi_i(t_{i-2}s_{i-1}) & = & \pi_i(t_i s_1) & = & 1; \\ \pi_i(y_{i-2}x_{i-1}) & = & \pi_i(y_i x_1) & = & 1; \\ \pi_i(u_{i-2}u_{i-1}) & = & \pi_i(u_i u_1) & = & 2. \end{array}$$

Now, we show that π_i is a 4-total colouring for G_i. Since π_{i-2} and π are 4-total colourings for G_{i-2} and L_i, respectively, we conclude that the restriction of π_i to the elements of subgraph $G_i[E(G_i) \setminus E_i^{in}]$ is a 4-total colouring for it. Moreover, by the construction of π_{i-2} and π, edges of E_i^{in} do not join vertices with the same colour.

In order to conclude the proof we show that the colours of the edges of E_i^{in} do not add any conflicts in π_i. Considering $E_i^{out} = \{t_{i-2}s_1, y_{i-2}x_1, u_{i-2}u_1\}$, we remember that $\pi_{i-2}(t_{i-2}s_1) = 1$, $\pi_{i-2}(y_{i-2}x_1) = 1$, and $\pi_{i-2}(u_{i-2}u_1) = 2$. Therefore, for graph $G_i[E(G_i) \setminus E_i^{in}]$, colour 1 does not occur in vertices $t_{i-2}, s_1, y_{i-2}, x_1$, and colour 2 does not occur in vertices u_{i-2}, u_1. Similarly, considering L_i, we conclude that colour 1 does not occur in vertices s_{i-1}, x_{i-1}, t_i, y_i, and colour 2 does not occur in u_{i-1} and u_i. □

We end this section determining the total chromatic number of Twisted Goldberg Snarks. We define the *Twisted Goldberg Snarks*, TG_i, i odd, $i \geq 3$, from G_i, by replacing edges s_1t_i and x_1y_i by edges s_1y_i and x_1t_i, respectively [14].

Corollary 3. *Each Twisted Goldberg Snark TG_i, i odd and $i \geq 3$, is a type 1 graph.*

Proof. Let G_i be a Goldberg Snark and let π_i be the 4-total colouring defined in the proof of Theorem 2. By construction, $\pi_i(s_1t_i) = \pi_i(x_1y_i)$. Moreover, $\pi_i(s_1) = \pi_i(x_1)$ and $\pi_i(t_i) = \pi_i(y_i)$. Therefore, we define ϕ_i, a 4-total colouring for TG_i as follows: $\phi_i(q) = \pi_i(q)$ if q is an element of G_i and of TG_i; $\phi_i(s_1y_i) = \pi_i(s_1t_i)$ and $\phi_i(x_1t_i) = \pi_i(s_1t_i)$. □

3 Conclusion

We have shown that the total chromatic number of Flower Snarks, Goldberg Snarks, and Twisted Goldberg Snarks is type 1. These results constitute one more piece of supporting evidence to the conjecture that all snarks are type 1.

Finally, the technique presented here could be adapted to colour other families of graphs that have recursive constructions; in particular, it could be adapted to other families of snarks. The colourings obtained with this technique can be used to determine the total chromatic number or, to settle the total colouring conjecture for a given class.

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