INSTITUTO DE COMPUTAÇÃO
UNIVERSIDADE ESTADUAL DE CAMPINAS

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C. N. Campos   *S. Dantas*   *C. P. de Mello*

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C. N. Campos†   S. Dantas‡   C. P. de Mello§

Abstract

A clique-colouring of a graph G is a colouring of the vertices of G so that no maximal clique of size at least two is monochromatic. The clique-hypergraph, $\mathcal{H}(G)$, of a graph G has $V(G)$ as its set of vertices and the maximal cliques of G as its hyperedges. A vertex-colouring of $\mathcal{H}(G)$ is a clique-colouring of G. Determining the clique-chromatic number, the least number of colours for which a graph G admits a clique-colouring, is known to be *NP*-hard. By determining some structural properties of powers of cycles, we establish that the clique-chromatic number of these graphs is equal to two, except for odd cycles of size at least five, that need three colours. For odd-seq circulant graphs, we show that their clique-chromatic number is at most four, and determine the cases when it is equal to two. Similar bounds for the chromatic number of these graphs are also obtained.

1 Introduction

A *hypergraph* $\mathcal{H}$ is a pair $(V, \mathcal{E})$ where V is a finite set of vertices and $\mathcal{E}$ is a family of non-empty subsets of V called *hyperedges*. A *k-colouring* of $\mathcal{H}$ is a mapping $\phi : V \rightarrow \{1, 2, \ldots, k\}$ such that for each $S \in \mathcal{E}$, with $|S| \geq 2$, there exist $u, v \in S$ with $\phi(u) \neq \phi(v)$, that is, there is no monochromatic hyperedge of size at least two. The *chromatic number* $\chi(\mathcal{H})$ of $\mathcal{H}$ is the smallest k for which $\mathcal{H}$ admits a k-colouring.

Let G be an undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. A *clique* is a set of pairwise adjacent vertices of G. The *clique number* of a graph G, $\omega(G)$, is the greatest integer k for which there exists a clique Q with $|Q| = k$. A *maximal clique* of G is a clique not properly contained in any other clique.

Given a graph G, we define the *clique-hypergraph* $\mathcal{H}(G)$ of G as the hypergraph whose vertices are the vertices of G, and whose hyperedges are the maximal cliques of G. A k-colouring of $\mathcal{H}(G)$ is also called a *k-clique-colouring* of G, and the chromatic number $\chi(\mathcal{H}(G))$ of $\mathcal{H}(G)$ is the *clique-chromatic number* of G. If $\chi(\mathcal{H}(G)) = k$, then G is *k-clique-chromatic.* Note that if $\omega(G) = 2$, then $\mathcal{H}(G) = G$, which implies $\chi(\mathcal{H}(G)) = \chi(G)$.

The clique-hypergraph colouring problem was posed by Duffus et al. [11]. Kratochvíl and Tuza [14] have proved that determining the bicolourability of clique-hypergraphs of perfect

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†Institute of Computing, University of Campinas
‡Institute of Mathematics, Fluminense Federal University
§Institute of Computing, University of Campinas

graphs is NP-hard, but solvable in polynomial time for planar graphs. Additionally, the chromatic number of triangle-free graphs is known to be unbounded [21], and so is their clique-chromatic number. On the other hand, Bacsó et al. [1] proved that almost all perfect graphs are 3-clique-colourable. Other works considering the clique-hypergraph colouring problem in classes of graphs can be found in the literature [10, 12].

We study the clique-hypergraph colouring problem on *circulant graphs*, that are graphs whose adjacency matrix is circulant. This class of graphs has several applications in combinatorics and linear algebra, having been extensively studied over the years [5, 9, 20, 22, 23, 24, 25]. There are different characterizations of these graphs. For instance, circulant graphs are a particular case of Cayley graphs. We postpone the definition used in this work to the following section.

Determining the clique number and the chromatic number of circulant graphs in general is an NP-hard problem [9]. Here, we study two subclasses of circulant graphs. The first class considered is *powers of cycles*. The choice of this class was motivated by significant works that have been done in powers of certain classes of graphs [4, 6, 7] and, in particular, in powers of cycles [2, 3, 15, 17, 18, 26, 28].

Recently, powers of cycles were studied in the context of the Hadwiger's and Hajós' conjectures, which are classical and related conjectures. In 2005, Thomassen [27] showed that certain powers of cycles are counter-examples to the Hajós Conjecture. However, in 2007, Li and Liu [16] showed that powers of cycles satisfy the Hadwiger's Conjecture. Additionally, powers of cycles have important connections to the analysis of perfect graphs [8].

In this work we prove that the clique-chromatic number of powers of cycles is equal to two, except for cycle graphs C_n, n odd and $n \geq 5$, that needs three colours. The second class considered is *odd-seq circulant graphs*. For this class, we show that its clique-chromatic number is at most four, and determine the cases when it is equal to two. Also, we verify similar bounds for the chromatic number of these graphs.

2 Preliminaries

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For each $v \in V(G)$, $N(v)$ denotes the set of vertices which are adjacent to v and $N[v] = N(v) \cup \{v\}$. Also, $\overline{N}[v] = V(G) \setminus N[v]$.

Let $d_1, \ldots, d_k$ be a (nonempty) sequence of positive integers satisfying $d_1 < \cdots < d_k \leq \lfloor n/2 \rfloor$ for some integer $n \geq 2$. A *circulant graph* $C_n(d_1, \ldots, d_k)$ is a simple graph with $V(G) = \{v_0, \ldots, v_{n-1}\}$ and $E(G) = E^{d_1} \cup \cdots \cup E^{d_k}$, with $\{v_i, v_j\} \in E^{d_l}$ if, and only if, $d_l = \min\{(j-i) \bmod n, (i-j) \bmod n\}$. If $e \in E^{d_l}$, then edge e has *reach* d_l. Moreover, if the reach of e is even (odd), then e is called an *even (odd) edge*. We take $(v_0, \ldots, v_{n-1})$ to be a *cyclic order* on the vertex set of G and always perform arithmetic modulo n on vertex indexes. Let $v_i \in V(G)$, $v_j \in V(G)$, with $0 \leq i < j \leq n-1$. We define $Inf(v_i, v_j) = j - i$ and $Sup(v_i, v_j) = n - j + i$.

A circulant graph $G = C_n(d_1, \ldots, d_k)$ is an *odd (even)-seq circulant graph* when each d_i, $1 \leq i \leq k$, is odd (even).

A circulant graph $G = C_n(d_1, \ldots, d_k)$ is a *power of cycle* when $d_1 = 1$, $d_i = d_{i-1} + 1$,

$d_k < \lfloor n/2 \rfloor$, and it is denoted C_n^k. Let u, v be two vertices of G. We denote $d(u,v)$ as the length of the shortest path joining u and v in the subgraph $G[E^1] \cong C_n$. Let Q be a clique of C_n^k. If every vertex $v_i \in Q$ has even (odd) index, then Q is an *even (odd) clique*. A *maximal even (odd) clique* is an even (odd) clique not properly contained in any other even (odd) clique.

A structural property of circulant graphs [19] is stated next.

Lemma 1. *Let $G = C_n(d_1, \dots, d_k)$, $d_i \leq \lfloor n/2 \rfloor$. Then, for each d_i, the induced subgraph $C_n(d_i)$ is comprised by $\gcd(n, d_i)$ connected components, each one being a cycle of length $n/\gcd(n, d_i)$,* □

3 Powers of cycles

In this section, we show that the clique-chromatic number of C_n^k is equal to two, except for odd cycles with $n \geq 5$. Note that cycles graphs C_n, i.e., powers of cycles with $k = 1$, have $\chi(\mathcal{H}(C_n)) = 3$ if n is odd and $n \geq 5$; and $\chi(\mathcal{H}(C_n)) = 2$ otherwise.

We start with a useful property:

Property 2. *Let G be a graph and Q be a maximal clique of G. Let u and v be two adjacent vertices of G such that $u \in Q$ and $v \notin Q$. Then, there exists $w \in Q$ such that $w \in \overline{N}[v] \cap N(u)$.*

Proof. It follows from the fact that such a vertex w prevents vertex v from being included in Q because it is not adjacent to v. □

We say that vertex v in Property 2 is *forbidden by* w and that every vertex $w \in Q$ that belongs to $\overline{N}[v] \cap N(u)$ *forbids* vertex v to belong to clique Q.

The next three lemmas determine structural properties of powers of cycles concerning the existence of maximal even (odd) cliques in these graphs. These lemmas play an important role in the proof of Theorem 9, that determines the clique chromatic number of powers of cycles.

Lemma 3. *Let $G = C_n^k$, $k \geq 2$, n odd, be a power of cycle. If Q is a maximal clique in G that includes vertices v_0 and v_{n-1}, then Q includes at least one vertex of odd index.*

Proof. Let $G = C_n^k$, $k \geq 2$, n odd, such that $V(G) = \{v_0, \dots, v_{n-1}\}$. Let Q be a maximal clique of G such that $v_{n-1}, v_0 \in Q$.

If $k = 2$, then either $Q = \{v_{n-1}, v_0, v_1\}$, or $Q = \{v_{n-2}, v_{n-1}, v_0\}$, and both contain a vertex of odd index. If $k > 2$ and at least one of v_1, v_{n-2} belongs to Q, then the result follows. So, we can assume that $k > 2$, $v_1 \notin Q$, and $v_{n-2} \notin Q$.

Since $v_1 \notin Q$ but $v_0 \in Q$ and v_0 and v_1 are adjacent, then, by Property 2, we conclude that there exists $v_j \in Q$ such that $v_j \in \overline{N}[v_1] \cap N(v_0)$. By the definition of powers of cycles, $\overline{N}[v_1] \cap N(v_0) = \{v_{n-k}\}$. Therefore, $v_{n-k} \in Q$ (note that $v_{n-k} \in N(v_{n-1})$). Symmetrically, since $v_{n-2} \notin Q$ but $v_{n-1} \in Q$, there exists $v_j \in Q$ such that $v_j \in \overline{N}[v_{n-2}] \cap N(v_{n-1})$. We conclude that $\overline{N}[v_{n-2}] \cap N(v_{n-1}) = \{v_{k-1}\}$ and that $v_{k-1} \in Q$.

We have just concluded that v_{n-k} and v_{k-1} are both in Q. If k is even, then $k-1$ and $n-k$ are both odd and we are done. Therefore, we assume that k is odd. Since v_{n-k} and v_{k-1} are both in Q, edge $\{v_{k-1}, v_{n-k}\} \in E(G)$. Let d be the reach of this edge. Thus, $d = n-k-(k-1) = n-2k+1$. Since $d \leq k$, we have that $k > n/3$.

Consider now vertex v_{k-2}. If $v_{k-2} \in Q$, then the result follows. Thus, we assume that $v_{k-2} \notin Q$. We know that $v_{k-1} \in Q$. Additionally, $\overline{N}[v_{k-2}] \cap N(v_{k-1}) = \{v_{2k-1}\}$. By Property 2, we have that $v_{2k-1} \in Q$, concluding the proof. □

Lemma 4. *Let $G = C_n^k$, $k \geq 2$, n odd, be a power of cycle. Then, there does not exist a maximal even (odd) clique in G.*

Proof. Suppose G has a maximal even or odd clique Q. We consider four cases.

Case 1. *There exists $v_i \in Q$, $i \in (0, k-1]$.*

By hypothesis, $v_i \in Q$ and Q is even or odd and thus $v_{i+1} \notin Q$, $v_{i-1} \notin Q$.

Considering that $v_{i+1} \notin Q$, then, by Property 2, we conclude that there exists $v_j \in \overline{N}[v_{i+1}] \cap N(v_i)$ such that $v_j \in Q$. Therefore, $v_{i-k} \in Q$ (note that $\overline{N}[v_{i+1}] \cap N(v_i) = \{v_{i-k}\}$). By the hypothesis of this case, $i-k < 0$, which implies that $i-k \equiv n+i-k \pmod{n}$. Similarly, if $v_{i-1} \notin Q$, then there exists $v_j \in Q$, such that $v_j \in \overline{N}[v_{i-1}] \cap N(v_i)$. Since $\overline{N}[v_{i-1}] \cap N(v_i) = \{v_{i+k}\}$, we have that $v_{i+k} \in Q$. We conclude that if $v_i \in Q$, then $v_{n+i-k} \in Q$ and $v_{i+k} \in Q$.

If Q is even, then i is even since $v_i \in Q$. If k is even, then $n+i-k$ is odd. On the other hand, if k is odd, then $i+k$ is odd. Therefore, in both cases, we have a contradiction to the fact that $v_{n+i-k} \in Q$ and $v_{i+k} \in Q$. Analogously, when Q is odd, we conclude that either $n+i-k$ or $k+i$ is even, and again we have a contradiction.

Case 2. *There exists $v_i \in Q$, $i \in [n-k+1, n-1)$.*

This case is reduced to Case 1 by relabelling v_0 as v_{n-1}, v_1 as v_{n-2}, ..., v_{n-2} as v_1, and v_{n-1} as v_0.

Case 3. *There exists $v_i \in Q$, $i \in \{0, n-1\}$.*

In this case Q is even. If both $v_0 \in Q$ and $v_{n-1} \in Q$, then, by Lemma 3, there exists $v_j \in Q$ such that j is odd, a contradiction. Thus, we assume that exactly one of v_0, v_{n-1} belongs to Q.

Consider first that $v_0 \in Q$ and $v_{n-1} \notin Q$. Since Q is even, $v_1 \notin Q$. Note that $\overline{N}[v_{n-1}] \cap N(v_0) = \{v_k\}$ and $\overline{N}[v_1] \cap N(v_0) = \{v_{n-k}\}$. By Property 2, we conclude that $v_k \in Q$ and $v_{n-k} \in Q$. If k is odd, then v_k has odd index, a contradiction because Q is even. On the other hand, if k is even, then v_{n-k} has odd index because n is also odd. Again a contradiction.

Now, assume that $v_0 \notin Q$ and $v_{n-1} \in Q$. This case is analogous to the previous one. In order to see that, just relabel the vertices of G as it was done for Case 2.

Case 4. *For all $v_i \in Q$, $n-k \leq i \leq k$.*

Suppose that $k \geq \frac{n}{3}$. In this case, $\{v_k, v_{n-k}\} \in E(G)$ because $n - k - k = n - 2k \leq n - \frac{2n}{3} = \frac{n}{3} \leq k$. Thus, $\{v_k, \ldots, v_{n-k}\}$ is a clique. Since $Q \subseteq \{v_k, \ldots, v_{n-k}\}$ is even or odd, we conclude that Q is not a maximal clique, a contradiction.

Now, we consider $k < \frac{n}{3}$. Let $v_i \in Q$ such that i is minimum. Since $v_i \in Q$, vertex $v_{i+1} \notin Q$. By Property 2, there exists $v_j \in Q$ such that $v_j \in \overline{N}[v_{i+1}] \cap N(v_i)$. We conclude that $v_j = v_{i-k}$, which means that $v_{i-k} \in Q$. However, $i > i - k \geq 0$, contradicting the minimality of i. □

Lemma 5. *Let $G = C_n^k$, $k \geq 2$, n even, be a power of cycle. Graph G has a maximal even (odd) clique if, and only if, k is even and $k = n\left(\frac{i}{2i+1}\right)$, for some integer $i \geq 1$.*

Proof. Let $G = C_n^k$, $k \geq 2$, n even. Let Q be an even or odd clique of G. By the symmetry of powers of cycles, we consider only even cliques. Adjust notation so that $v_0 \in Q$.

First we show that $k \geq n/3$. Considering that Q is an even clique, $v_1 \notin Q$ and $v_{n-1} \notin Q$. Since $v_0 \in Q$, then $v_k \in Q$ and $v_{n-k} \in Q$, by Property 2. Also, since Q is a clique, vertices v_k and v_{n-k} are adjacent. Therefore, $(n - k) - k \leq k$ and we have $k \geq n/3$ as stated before.

Note that, if $k = n/3$, then $Q = \{v_0, v_k, v_{n-k}\}$ is a maximal clique since the vertices of set $[v_1, v_{k-1}]$ are forbidden by v_{n-k}, the vertices of set $[v_{k+1}, v_{n-k-1}]$ are forbidden by v_0, and the vertices of set $[v_{n-k+1}, v_{n-1}]$ are forbidden by v_k. Also, $k = n\left(\frac{i}{2i+1}\right)$ with $i = 1$ and k is even.

Now, we assume that $k > n/3$. We prove the assertion of Lemma 5 by showing the correctness of the algorithm BC(G,Q), defined below. This algorithm receives G and $Q = \{v_0, v_k, v_{n-k}\}$ as input and returns a set of vertices Q that is: either (i) empty when G does not have a nonempty maximal even clique Q, or (ii) a maximal even clique of G.

Algorithm 1 BC(G,Q)

INPUT: Graph G and set $Q := \{v_0, v_k, v_{n-k}\}$.
OUTPUT: Set Q, a maximal even clique of G.

1. $x := 0$; $y := 0$;
2. **if** $x = y$ **then** $i := x(n - 2k) + (n - 2k)$;
   2.1 $j := n - x(n - 2k) - (n - 2k)$;
   2.2 $x := x + 1$;
3. **else** $i := k - y(n - 2k) - (n - 2k)$;
   3.1 $j := n - k + y(n - 2k) + (n - 2k)$;
   3.2 $y := y + 1$;
4. $Q := Q \cup \{v_i, v_j\}$;
5. **if** $x(n - 2k) + (n - 2k) > k - y(n - 2k)$ **then return** $Q := \emptyset$;
6. **else if** $x(n - 2k) + (n - 2k) = k - y(n - 2k)$ **then return** Q;
   6.1 **else** Go to line 2.

First, we show that BC(G,Q) stops. By hypothesis, $k > n/3$. Therefore, $n - 2k < k$. In the beginning of the first iteration, $x(n - 2k) = 0$ and $k - y(n - 2k) = k$. Therefore, $x(n-2k)+(n-2k) < k-y(n-2k)$ and the stop conditions of lines 5 and 6 are not reached.

In each iteration, $(k-y(n-2k))-x(n-2k)$ is decreased by $n-2k$, either because $x(n-2k)$ is increased by $n-2k$, or because $k-y(n-2k)$ is decreased by the same amount. Since the decrement is an integer, we conclude that after some iterations, at least one of the two stop conditions is reached.

The variables x and y are control variables of the algorithm. It is not difficult to see that in each iteration either $x = y$, or $x = y + 1$. Moreover, in each pair of consecutive iterations, both step 2 and step 3 are executed, that is, these steps are executed alternately.

Next, we prove two important invariants of algorithm BC(G,Q).

Property 6. *In each iteration, before the execution of line 5*

$$\begin{aligned} Q = \ & \{v_0, v_{n-2k}, \ldots, v_{x(n-2k)}, v_{k-y(n-2k)}, \ldots, v_k, \\ & v_{n-k}, \ldots, v_{n-k+y(n-2k)}, v_{n-x(n-2k)}, \ldots, v_{2k}\}. \end{aligned}$$

Proof. We prove this invariant by induction. In the beginning of the first iteration, $x = y = 0$ and the test condition of line 2 is true. Therefore, after line 4, $Q = \{v_0, v_{n-2k}, v_k, v_{n-k}, v_{2k}\}$. Moreover, $x = 1$ and $y = 0$. Hence, the invariant holds.

Consider now set Q just before line 5, in some iteration. Let Q', x' and y' denote the set Q and the control variables in the beginning of this iteration, that is, just before line 2. By induction hypothesis,

$$\begin{aligned} Q' = \ & \{v_0, v_{n-2k}, \ldots, v_{x'(n-2k)}, v_{k-y'(n-2k)}, \ldots, v_k, \\ & v_{n-k}, \ldots, v_{n-k+y'(n-2k)}, v_{n-x'(n-2k)}, \ldots, v_{2k}\}. \end{aligned}$$

We consider two cases depending on the result of the test condition in line 2. If $x' = y'$, then the test condition of line 2 is true. Therefore, $Q = Q' \cup \{v_{(x'+1)(n-2k)}, v_{n-(x'+1)(n-2k)}\}$. Moreover, $x = x'+1$ and $y = y'$. We conclude that the invariant holds. On the other hand, if $x' = y' + 1$, then step 3 was executed and $Q = Q' \cup \{v_{k-(y'+1)(n-2k)}, v_{n-k+(y'+1)(n-2k)}\}$. Since $x = x'$ and $y = y' + 1$, we conclude that the invariant holds again. □

Figure 1 depicts the vertices of Q.

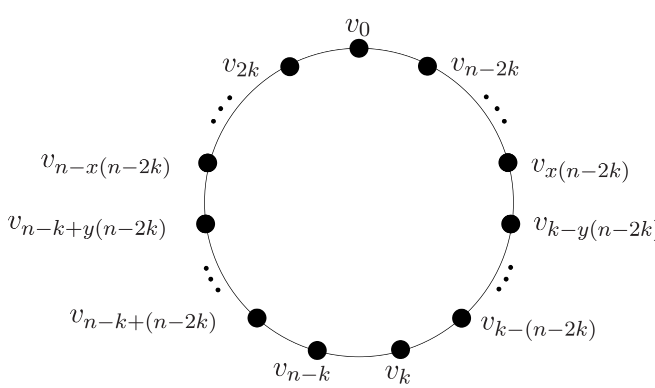

Figure 1: Sketch of the vertices of Q.

Property 7. *Let u, w be a pair of vertices included in Q in some iteration. Then, each of vertices u, w forbids a vertex of odd index from being included in Q.*

Proof. Consider algorithm BC(G,Q) just before line 5, in some iteration. As we have just proved, set Q is defined by Property 6. We have two cases to analyse depending on the result of the test condition of line 2.

If the test condition of line 2 is true, then step 2 was executed, $u = v_{x(n-2k)}$ and $w = v_{n-x(n-2k)}$. We consider vertices $v_{k-y(n-2k)} \in Q$ and one of its adjacent vertices of odd index, $v_{k-y(n-2k)-1}$. We also consider $v_{n-k+y(n-2k)} \in Q$ and one of its adjacent vertices of odd index, $v_{n-k+y(n-2k)+1}$. By the definition of powers of cycles,

$$\begin{aligned} N(v_{k-y(n-2k)}) &= \{v_{n-y(n-2k)}, \ldots, v_{k-y(n-2k)-1}\} \cup \\ &\quad\ \{v_{k-y(n-2k)+1}, \ldots, v_{2k-y(n-2k)}\}, \\ \overline{N}[v_{k-y(n-2k)-1}] &= V(G) \setminus N[v_{k-y(n-2k)-1}], \end{aligned}$$

where $N[v_{k-y(n-2k)-1}] = \{v_{n-y(n-2k)-1}, \ldots, v_{k-y(n-2k)-2}\} \cup \{v_{k-y(n-2k)-1}\} \cup \{v_{k-y(n-2k)}, \ldots, v_{2k-y(n-2k)-1}\}$. On the other hand,

$$\begin{aligned} N(v_{n-k+y(n-2k)}) &= \{v_{(y+1)(n-2k)}, \ldots, v_{n-k+y(n-2k)-1}\} \cup \\ &\quad\ \{v_{n-k+y(n-2k)+1}, \ldots, v_{n+y(n-2k)}\}, \\ \overline{N}[v_{n-k+y(n-2k)+1}] &= V(G) \setminus N[v_{n-k+y(n-2k)+1}], \end{aligned}$$

where $N[v_{n-k+y(n-2k)+1}] = \{v_{(y+1)(n-2k)+1}, \ldots, v_{n-k+y(n-2k)}\} \cup \{v_{n-k+y(n-2k)+1}\} \cup \{v_{n-k+y(n-2k)+2}, \ldots, v_{n+y(n-2k)+1}\}$. Therefore,

$$\begin{aligned} \overline{N}[v_{k-y(n-2k)-1}] \cap N(v_{k-y(n-2k)}) &= \{v_{2k-y(n-2k)}\}, \\ \overline{N}[v_{n-k+y(n-2k)+1}] \cap N(v_{n-k+y(n-2k)}) &= \{v_{(y+1)(n-2k)}\}. \end{aligned}$$

By Property 2, we conclude that $v_{(y+1)(n-2k)} \in Q$, $v_{2k-y(n-2k)} \in Q$. Considering that $x = y+1$, then $v_{(y+1)(n-2k)} = v_{x(n-2k)} = u$ and $v_{2k-y(n-2k)} = v_{n-x(n-2k)} = w$. We conclude that the property holds in this case.

If the test condition of line 2 is false, then step 3 was executed and $u = v_{k-y(n-2k)}$ and $w = v_{n-k+y(n-2k)}$. Using the same reasoning of the previous case, we conclude that $\overline{N}[v_{x(n-2k)+1}] \cap N(v_{x(n-2k)}) = \{w\}$, $\overline{N}[v_{n-x(n-2k)-1}] \cap N(v_{n-x(n-2k)}) = \{u\}$, and again $u \in Q$ and $w \in Q$. $\square$

Now, we analyse two cases: (i) BC(G,Q) stops with a nonempty Q, showing in this case that Q is a maximal even clique; (ii) BC(G,Q) stops with an empty Q, showing in this case that there is no nonempty maximal even clique.

Case 1. *BC(G,Q) stops with a nonempty Q.*

Algorithm BC(G,Q) stopped because the condition of line 6 is true and thus set Q is nonempty. By Property 6,

$$\begin{aligned} Q = \ &\{v_0, v_{n-2k}, \ldots, v_{x(n-2k)}, v_{k-y(n-2k)}, \ldots, v_k, \\ &v_{n-k}, \ldots, v_{n-k+y(n-2k)}, v_{n-x(n-2k)}, \ldots, v_{2k}\}. \end{aligned}$$

Additionally, the stop condition of BC(G,Q) is $x(n-2k)+(n-2k) = k-y(n-2k)$. Hence,

$$\begin{array}{lcl} k-y(n-2k) & = & x(n-2k)+(n-2k), \\ k-(y-1)(n-2k) & = & (x+2)(n-2k), \\ & \dots & \\ k-(y-y)(n-2k) & = & (x+y+1)(n-2k), \\ k & = & (x+y+1)(n-2k). \end{array}$$

Therefore, vertices $v_0,\dots,v_{x(n-2k)},v_{k-y(n-2k)},\dots,v_k$ can be rewritten as $v_0,v_{n-2k},\dots,v_{x(n-2k)},v_{(x+1)(n-2k)},\dots,v_{(x+y+1)(n-2k)}$, respectively. Still using $k=(x+y+1)(n-2k)$, vertices $v_{n-k},\dots,v_{n-k+y(n-2k)}$ can be written as $v_{(x+y+2)(n-2k)},\dots,$ $v_{(x+2y+2)(n-2k)}$. Since $x(n-2k)+(n-2k) = k-y(n-2k)$, we have $n-x(n-2k) = n-k+y(n-2k)+(n-2k)$. Thus, the remaining vertices in Q satisfy: $v_{n-x(n-2k)} = v_{(x+2y+3)(n-2k)};\dots;$ $v_{2k} = v_{(2x+2y+2)(n-2k)}$. Let $i=x+y+1$; we conclude that

$$Q=\{v_0,v_{n-2k},\dots,v_{i(n-2k)},v_{(i+1)(n-2k)},\dots,v_{2i(n-2k)}\}.$$

Also, $|Q| = 1+(x+y+1)+(x+y+1) = 2i+1$.

Now, we show that Q is a clique. By the definition of powers of cycles we know that $N(v_0)=\{v_{n-k},\dots,v_{n-1}\}\cup\{v_1,\dots,v_k\}$. Therefore, v_0 is adjacent to every vertex of $Q\setminus\{v_0\}$. Consider the vertices of Q in increasing order of their indexes and u,v two consecutive vertices in this cyclic order. Then $d(u,v)=n-2k$. This implies that any relabelling $v_j \rightarrow v_{(j+\alpha(n-2k)) \bmod n}$, $\alpha \geq 0$, is an automorphism in $G[Q]$. We conclude that all vertices of Q are pairwise adjacent. Moreover, all vertices of Q are even; thus, Q is an even clique.

In the following, we show that Q is a maximal clique. By definition, $\overline{N}[v_0]=\{v_{k+1},\dots,$ $v_{n-k-1}\}$. Therefore, none of these vertices can be included in Q while preserving the clique property. By the symmetry of the vertices in Q, we conclude that none of the vertices in $V(G)\setminus Q$ can be included in Q. Therefore, Q is maximal.

Finally, to conclude this case we observe that

$$k=(x+y+1)(n-2k)=i(n-2k) \quad\Rightarrow\quad k=n\left(\frac{i}{2i+1}\right).$$

Moreover, n even and $k=n\left(\frac{i}{2i+1}\right)$ imply that k is even.

Case 2. BC(G,Q) *stops with an empty* Q.

In this case, BC(G,Q) stops because the condition of line 5 is true. That is, $x(n-2k)+(n-2k) > k-y(n-2k)$.

Let u,w be the pair of vertices included in Q in the last iteration. By Property 7, these vertices are needed to prevent vertices of odd index from belonging to Q. In this case, we prove that u and w are not adjacent, which implies that u and w cannot belong to Q simultaneously. We conclude that some vertex of odd index must eventually belong to Q. Therefore, Q cannot be a maximal even clique.

Now, we prove that u and w are not adjacent. We have to consider two cases, depending on the result of the test condition of line 2.

Subcase 1. *The test condition of line 2 is true.*

In this case, step 2 was executed. Therefore, in line 6, $x = y+1$. Moreover, $u = x(n-2k)$ and $w = n - x(n - 2k)$. By the definition of powers of cycles, we have that

$$\begin{aligned} Sup(u, w) &= n - (n - x(n - 2k)) + x(n - 2k) \\ &= 2x(n - 2k). \end{aligned}$$

The algorithm stopped because the condition $x(n - 2k) + (n - 2k) > k - y(n - 2k)$ is true. Since $x = y + 1$, we conclude that $2x(n - 2k) > k$.

By construction, $x(n - 2k) = k - y(n - 2k) - z$, for $0 < z < n - 2k$. Moreover,

$$\begin{aligned} Inf(u, w) &= n - x(n - 2k) - x(n - 2k) \\ &= n - x(n - 2k) - (k - y(n - 2k) - z) \\ &= n - x(n - 2k) - (k - y(n - 2k) + n - n + k - k) + z \\ &= n - x(n - 2k) - (n - k - (y + 1)(n - 2k)) + z \\ &= n - x(n - 2k) - (n - k - x(n - 2k)) + z \\ &= k + z \\ &> k. \end{aligned}$$

We conclude that u and w are not adjacent.

Subcase 2. *The test condition of line 2 is false.*

In this case, step 3 was executed. Therefore, in line 6, $x = y$. Moreover, $u = k-y(n-2k)$ and $w = n - k + y(n - 2k)$. The argument of this case is analogous to the previous one.

This ends the proof of Lemma 5. □

As a consequence of Lemma 5, Corollary 8 determines the number of maximal even and odd cliques of powers of cycles when n and k are even.

Corollary 8. *Let $G = C_n^k$ with n, k even. If $k = n\left(\frac{i}{2i+1}\right)$, $i \geq 1$, integer, then G has $\frac{n-2k}{2}$ different maximal even cliques and $\frac{n-2k}{2}$ different maximal odd cliques.* □

Now we are ready to state the clique-chromatic number of a power of cycle.

Theorem 9. *Let $G = C_n^k$, $k \geq 2$, be a power of cycle. Then, $\chi(\mathcal{H}(G)) = 2$.*

Proof. Let $G = C_n^k$, $k \geq 2$, be a power of cycle. We consider two cases and in each case we construct a 2-clique colouring π for G.

Case 1. *n is even with $k \neq n\left(\frac{i}{2i+1}\right)$ or n is odd.*

In this case, we define $\pi(v_i) = i \bmod 2$.

By lemmas 4 and 5, graph G does not have maximal even (odd) cliques. Therefore, each maximal clique of G has at least one vertex of even index and at least one vertex of odd index. We conclude that π does not yield monochromatic maximal cliques and the result follows.

Case 2. *n is even and* $k = n\left(\frac{i}{2i+1}\right)$.

In this case, we have that k is even and $k \geq n/3$. Let π be defined as follows:

$$\pi(v_i) = \begin{cases} 0, & \text{if } 0 \leq i \leq \lfloor n/3 \rfloor - 1; \\ 1, & \text{if } \lfloor n/3 \rfloor \leq i \leq 2\lfloor n/3 \rfloor - 1; \\ i \bmod 2, & \text{if } 2\lfloor n/3 \rfloor \leq i \leq n-1. \end{cases}$$

Consider first maximal cliques whose vertices have consecutive indexes (modulo n). By the definition of powers of cycles, the size of these cliques is $k+1$. By construction of π, $\pi(v_{2\lfloor n/3 \rfloor - 1}) = 1$ and $\pi(v_{2\lfloor n/3 \rfloor}) = 0$, also, $\pi(v_{n-1}) = 1$ and $\pi(v_0) = 0$. Thus, the greatest number of consecutive vertices that receive the same colour is $\lfloor n/3 \rfloor$. Moreover, $\lfloor n/3 \rfloor \leq n/3 \leq k$. Therefore, there are no monochromatic maximal cliques with consecutive indexes.

Assume now that there exists a maximal clique Q that is monochromatic. Since Q cannot be comprised by consecutive vertices, there exists $v_i \in Q$ such that $2\lfloor n/3 \rfloor \leq i \leq n-1$. Moreover, since Q is maximal, there exists $v_j \in Q$ such that $0 \leq j \leq 2\lfloor n/3 \rfloor - 1$.

Subcase 1. *The vertices of Q received colour* 0.

Since n is even, $\pi(v_{n-1}) = 1$. Therefore, $v_{n-1} \notin Q$. It implies that there exists $v_j \in Q$ such that v_{n-1} and v_j are not adjacent. Moreover, $\pi(v_j) = 0$ since it belongs to Q.

By the definition of powers of cycles,

$$N(v_{n-1}) = \{v_{n-1-k}, \ldots, v_{n-2}\} \cup \{v_0, \ldots, v_{n-1+k}\}.$$

Since $n-1+k > n$, we have that $n-1+k \equiv k-1 \pmod{n}$ Moreover, $k-1 \geq n/3-1 \geq \lfloor n/3 \rfloor - 1$. Therefore, v_{n-1} is adjacent to each vertex from v_0 to $v_{\lfloor n/3 \rfloor - 1}$. It implies that $j \geq \lfloor n/3 \rfloor$.

Suppose that $n \not\equiv 2 \pmod{3}$. Then, $2(n/3) < 2\lfloor n/3 \rfloor + 1$. This implies that $n-1-k \leq n-1-n/3 \leq 2n/3-1 < 2\lfloor n/3 \rfloor$. Therefore, $\pi(v_{n-1-k}) = 1$ and it implies that $j \leq 2\lfloor n/3 \rfloor - 1$. Since we have already proved that $j \geq \lfloor n/3 \rfloor$, we conclude that $\pi(v_j) = 1$, a contradiction.

Assume now that $n \equiv 2 \pmod{3}$. In this case $2(n/3) < 2\lfloor n/3 \rfloor + 2$, implying that $n-1-k \leq n-1-n/3 = 2n/3-1 < 2\lfloor n/3 \rfloor + 1$. Note that $n-1-k \neq 2\lfloor n/3 \rfloor$ since $n-1-k$ is odd, because n and k are even and $2\lfloor n/3 \rfloor$ is even. Therefore, $n-1-k < 2\lfloor n/3 \rfloor$ and again we have a contradiction as in the previous case.

Subcase 2. *The vertices of Q received colour* 1.

Construct G' from G relabelling the vertices of G in the following way:

$$v'_i = v_j, \quad \text{where } v'_i \in V(G'), v_j \in V(G) \text{ and } j \equiv 2\lfloor n/3 \rfloor - (i+1) \pmod{n}.$$

For each $v'_i \in V(G')$ obtained from $v_j \in V(G)$ in this formula, we have that $\pi(v'_i) = 1$ if, and only if, $\pi(v_j) = 0$. Hence, the result follows from the previous case.

Therefore, there are no monochromatic maximal cliques with non-consecutive indexes, which completes the proof of Theorem 9. $\square$

4 Odd-Seq Circulant graphs

A circulant graph $C_n(d_1, \ldots, d_k)$ is an *odd-seq circulant graph* when each d_i, $1 \leq d_i < d_k$, is odd. We analyse two cases depending on the parity of n.

Let $G = C_n(d_1, \ldots, d_k)$ be an odd-seq circulant graph with n even. These graphs were shown bipartite by Heuberger [13]. Therefore, $\omega(G)$ is 2 and each maximal clique in G is maximum. We conclude that there exists a 2-clique colouring for odd-seq circulant graphs with n even.

Consider now $G = C_n(d_1, \ldots, d_k)$, an odd-seq circulant graph with n odd. We analyse some cases, according to the clique number of G. Lemma 11 establishes conditions for which a graph G has $\omega(G)$ equal to 2 or 3. The next result is a structural property of odd-seq circulant graphs.

Property 10. *Let $G = C_n(d_1, \ldots, d_k)$ be an odd-seq circulant graph with n odd. Then, each cycle of size 3 has at least one vertex v_i with $0 \leq i \leq \lfloor n/2 \rfloor$, and at least one vertex v_j with $\lceil n/2 \rceil \leq j \leq n-1$.*

Proof. The result follows from the fact that every edge of G is odd and n is odd. □

Lemma 11. *Let $G = C_n(d_1, \ldots, d_k)$ be an odd-seq circulant graph with n odd. Then, $\omega(G) = 3$ if, and only if, there exist $r_i, r_j, r_l \in \{d_1, \ldots, d_k\}$, not necessarily distinct, such that $r_i + r_j + r_l = n$. Otherwise, $\omega(G) = 2$.*

Proof. Let $V(G) = \{v_0, \ldots, v_{n-1}\}$. First, we prove that every clique of G has size at most 3. Suppose that Q is a clique in G and that $|Q| = 4$. Adjust notation so that $v_0 \in Q$. Let v_i, v_j, v_k be the other vertices of Q. Assume that $i < j < k$. Because set Q is a clique, $(v_0, v_i, v_j, v_k, v_0)$ is a cycle (not induced) in G. Since every edge of G is odd, we conclude that i and k are odd and j is even. However, for v_l adjacent to v_0, if l is odd, then $l \leq \lfloor n/2 \rfloor$; otherwise $l \geq \lceil n/2 \rceil$. Therefore, we conclude that $i, k \leq \lfloor n/2 \rfloor$ and $j \geq \lceil n/2 \rceil$, a contradiction, since $i < j < k$.

Now, assume that $Q = \{v_i, v_j, v_l\}$ is a clique in G. Adjust notation so that $0 \leq i < j < l \leq n-1$. By Property 10, we can assume that $i \leq \lfloor n/2 \rfloor$ and $l \geq \lceil n/2 \rceil$. Let r_i, r_j and r_l be the reaches of v_iv_j, v_jv_l, and v_lv_i, respectively. Thus, $r_i = j - i$, $r_j = l - j$ and $r_l = n - l + i$. Therefore, $r_i + r_j + r_l = n$.

Consider now that there exist $r_i, r_j, r_l \in \{d_1, \ldots, d_k\}$ such that $r_i + r_j + r_l = n$. Edges $v_0v_{r_i}$, $v_{r_i}v_{r_i+r_j}$ and $v_{r_i+r_j}v_{r_i+r_j+r_l}$ belong to $E(G)$. Since $r_i + r_j + r_l = n$, we have that $v_{r_i+r_j+r_l} = v_0$. Therefore, $(v_0, v_{r_i}, v_{r_i+r_j}, v_0)$ is a cycle and $\{v_0, v_{r_i}, v_{r_i+r_j}\}$ is a clique in G.

In order to conclude the proof note that we have already proved that $\omega(G) \leq 3$. However, if $\omega(G) \neq 3$, then $\omega(G) = 2$ because $E(G) \neq \emptyset$. □

As an example of Lemma 11, consider graph $C_{15}(3, 5, 7)$ with $\omega(C_{15}(3, 5, 7)) = 3$. It has a clique with $r_i = r_j = r_l = 5$ and another clique with $r_i = 3$, $r_j = 5$, $r_l = 7$.

We proceed considering first odd-seq circulant graphs G with $\omega(G) = 3$ for which every maximal clique is also maximum. Afterwards, we assume that G has maximal cliques of size two and establish bounds to the clique-chromatic number of G in this case. We close this section with Corollary 14, that extends the bounds of Theorem 13 to the chromatic number of odd-seq circulant graphs.

Theorem 12. *Let $G = C_n(d_1, \ldots, d_k)$ be an odd-seq circulant graph with n odd. If every maximal clique of G has size three, then $\chi(\mathcal{H}(G)) = 2$.* □

Proof. Let π be a colour assignment to $V(G)$ such that each v_i, $0 \leq i \leq \lfloor n/2 \rfloor$, has $\pi(v_i) = 0$; otherwise $\pi(v_i) = 1$. By Property 10, π is a 2-clique colouring for G. □

Now we can assume that G has maximal cliques of size two. The next lemma establishes bounds to the clique-chromatic number of odd-seq circulant graph in this case.

Theorem 13. *Let $G = C_n(d_1, \ldots, d_k)$ be an odd-seq circulant graph with n odd. If G has maximal cliques of size two, then $3 \leq \chi(\mathcal{H}(G)) \leq 4$.*

Proof. We start by showing that $\chi(\mathcal{H}(G)) \leq 4$. Consider the following 4-colour assignment π to the vertices of G: $\pi(v_i) = i \mod 2$, if $0 \leq i \leq \lfloor n/2 \rfloor$; and $\pi(v_i) = 2 + (i \bmod 2)$, if $\lceil n/2 \rceil \leq i \leq n-1$. The validity of π follows from the fact that vertices of same colour are non-adjacent.

It remains to show that $\chi(\mathcal{H}(G)) \geq 3$. Let $Q = \{u, v\}$ be a maximal clique. By the definition of odd-seq circulant graphs, edge uv belongs to an odd cycle C. By symmetry of circulant graphs, uv can be any edge of C. By maximality of Q, $|C| > 3$. Therefore, in order to construct a clique-colouring for this cycle at least three colours are needed. □

Corollary 14. *If G is an odd-seq circulant graph with n odd, then $3 \leq \chi(G) \leq 4$.*

Proof. Let $G = C_n(d_1, \ldots, d_k)$, with n odd and each d_i odd.

First we prove that $\chi(G) \geq 3$. By Lemma 1, $C_n(d_i)$ is comprised by $\gcd(n, d_i)$ connected components of size $n/\gcd(n, d_i)$. Since n and d_i are odd, $\gcd(n, d_i)$ is also odd. That is, $C_n(d_i)$ is comprised by some connected components that are odd cycles. Thus, $\chi(C_n(d_i)) = 3$. Since $C_n(d_i)$ is a subgraph of G, we conclude that $\chi(G) \geq 3$.

Consider the 4-colour assignment defined in the proof of Theorem 13. As we have already noticed π is a 4-colouring to the vertices of G. Therefore, $\chi G \leq 4$ and the result follows. □

It is important to note that the bounds obtained in Theorem 13 are tight. For instance, $G = C_{21}(1, 5, 9)$ has $\chi(\mathcal{H}(G)) = 4$ and $G = C_{21}(1, 3, 7)$ has $\chi(\mathcal{H}(G)) = 3$.

5 Concluding remarks

In this work we considered the clique-colouring problem in two classes of graphs: powers of cycles and odd-seq circulant graphs.

We completely determined the clique-chromatic number of powers of cycles. A power of cycle, C_n^k, has maximal cliques that are obtained taking any $k+1$ consecutive vertices. These cliques are a natural consequence of the definition of powers of cycles. However, there exist maximal cliques that do not have consecutive vertices. We establish structural properties related to its maximal cliques. Lemma 3 and Lemma 4 together prove that if n is odd, C_n^k does not have a maximal clique comprised only by vertices of same parity. Additionally, in Lemma 5, we characterize which powers of cycles with n even have maximal even (odd) cliques. More precisely, we exhibit the vertices of these cliques.

Next, we consider the class of odd-seq circulant graphs. For this class, we determined which graphs have clique-chromatic number 2. For the remaining graphs in this class, we show that the clique-chromatic number is 3 or 4. It is important to notice that these bounds are tight. For instance, $G = C_{21}(1, 5, 9)$, $\omega(G) = 2$, and $G = C_{21}(1, 5, 7, 9)$, $\omega(G) = 3$, have clique chromatic number 4; in contrast, $G = C_{21}(1, 5)$, $\omega(G) = 2$, and $G = C_{21}(1, 3, 7)$ $\omega(G) = 3$, both have clique chromatic number 3.

One approach to determine which graphs have clique-chromatic number 3 and which have clique-chromatic number 4 could be to partition $E(G)$ according to the size of the maximal cliques of G. That is, let G_i be the subgraph induced by the edges that belong to maximal cliques of size i, $i \in \{2, 3\}$. We know that $\chi(\mathcal{H}(G_2)) = \chi(G_2)$ and $3 \leq \chi(\mathcal{H}(G_2)) \leq 4$; moreover $\chi(\mathcal{H}(G_3)) = 2$. So, it is natural to ask whether it is possible to obtain a minimum clique-colouring for G from the clique-colouring of its subgraphs G_2 and G_3.

Another interesting problem is to consider the clique-colouring problem of even-seq circulant graphs. Let $G = C_n(d^1, \ldots, d^l)$ be an even-seq circulant graph. We know that if n is even, then G is not connected. In particular, $C_n(d_1, \ldots, d_l) = C_{\frac{n}{2}}(d_1/2, \ldots, d_l/2) \cup C_{\frac{n}{2}}(d_1/2, \ldots, d_l/2)$. This implies that if $d_1 = 2$ and $d_i = d_{i-1} + 2$, each $C_{\frac{n}{2}}(d_1/2, \ldots, d_l/2)$ is a power of cycle whose clique-chromatic number has just been determined. On the other hand, if $d_i \neq d_{i-1} + 2$ for some d_i, we go back to the problem of determining the clique-chromatic number of a general circulant graph.

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