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# Edge-Coloring of Split Graphs

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**Abstract**

The *Classification Problem* is the problem of deciding whether a simple graph has chromatic index equals to $\Delta$ or $\Delta + 1$, where $\Delta$ is the maximum degree of the graph. It is known that to decide if a graph has chromatic index equals to $\Delta$ is NP-complete. A *split graph* is a graph whose vertex set admits a partition into a stable set and a clique. The chromatic indexes for some subsets of split graphs, such as split graphs with odd maximum degree and split-indifference graphs, are known. However, for the general class, the problem remains unsolved. In this paper we exhibit a new subset of split graphs with even maximum degree that have chromatic index equal to $\Delta$. Moreover, we present polynomial time algorithms to perform an edge-coloring and to recognize these graphs.

## 1 Introduction

A *k-edge-coloring* of a graph $G$ is an assignment of one of $k$ colors to each edge of $G$ such that there are no two edges with the same color incident to a common vertex. In the discussion below, a "coloring" of a graph always means an edge-coloring, while a "$k$-coloring" is a coloring that uses only $k$ colors. The *chromatic index* of $G$, $\chi'(G)$, is the minimum $k$ such that $G$ has a $k$-coloring. By definition, $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of $G$. In 1964, Vizing [17] showed that for any simple graph $G$, $\chi'(G) \leq \Delta(G) + 1$. It was the origin of the *Classification Problem*, that consists of deciding whether a given graph $G$ has $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. In the first case, we say that $G$ is *Class 1*, otherwise, we say that $G$ is *Class 2*. Despite the powerful restriction imposed by Vizing, it is very hard to compute the chromatic index in general. In fact, it is NP-complete to decide if a graph is Class 1 whereas Class 2 recognition is co-NP-complete [11]. In 1991, Cai and Ellis [1] proved that this holds also when the problem is restricted to some classes of graphs such as perfect graphs. However, the classification problem is entirely solved for a few known set of graphs that includes the complete graphs, bipartite graphs [13], complete multipartite graphs [12], and graphs with universal vertices [15].

Efforts have been made to give partial solutions. Considering the class of doubly chordal graphs, a superclass of interval graphs, we know the solution of the classification problem for doubly chordal graphs with odd maximum degree [4]. In the class of split graphs, the

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classification problem is solved for split-indifference graphs [14], complete split graphs, and split graphs with odd maximum degree [2].

In this work, we focus our attention on the class of split graphs. A *split graph* is a graph whose vertex set admits a partition into a stable set and a clique. This paper presents a new result about the classification problem for split graphs as a contribution in the direction of solving the entire problem for this class.

In Section 2 we give some definitions and preliminary results. In Section 3, we discuss the classification problem of split graphs and present the solution of this problem for a subset of these graphs. We also prove that this subset can be recognized in polynomial time. Finally, Section 4 presents some conclusions about our work.

## 2 Definitions and necessary background

In this paper, $G$ denotes a simple, finite, undirected and connected graph; $V(G)$ and $E(G)$ are the vertex and edge sets of $G$. Write $n = |V(G)|$ and $m = |E(G)|$. The *degree* of a vertex $v$ in a graph $G$ is denoted by $d_G(v)$. The *maximum degree* of $G$, denoted $\Delta(G)$, is the maximum vertex degree in $G$. A $\Delta(G)$-vertex is a vertex of a graph $G$ with degree $\Delta(G)$. A universal vertex is a vertex with degree $n-1$. Let $v$ be a vertex of $G$. The set of vertices which are adjacent to $v$ in $G$ is denoted by $N(v)$, and $N[v] = \{v\} \cup N(v)$. The set $N[v]$ is called *neighborhood* of $v$. A *clique* is a set of pairwise adjacent vertices of a graph. A *maximal clique* is a clique that is not properly contained in any other clique. A *stable set* is a set of pairwise non-adjacent vertices. A subgraph of $G$ is a graph $H$ with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, denote by $G[X]$ the subgraph induced by $X$, that is, $V(G[X]) = X$ and $E(G[X])$ consists of those edges of $E(G)$ having both ends in $X$. Let $D \subseteq E(G)$. The subgraph induced by $D$ is the subgraph $H$ with $E(H) = D$ and $V(H)$ is the set of vertices $v$ having at least one edge of $D$ incident to $v$. The notation $G \setminus D$ denotes the subgraph of $G$ with $V(G \setminus D) = V(G)$ and $E(G \setminus D) = E(G) \setminus D$. We denote by $K_n$ a complete graph with $n$ vertices.

The following lemmas are used in our discussion about the coloring of split graphs. The results of lemmas 1, 2, and 5 have been known for a long time and can be found in [7, 13].

**Lemma 1** *The complete graph $K_n$ is Class 1 if, and only if, $n$ is even.*

**Lemma 2** *[13] Every bipartite graph is Class 1.*

A graph $G$ is *overfull* ($\mathcal{O}$) when $n$ is odd and $\Delta(G) \lfloor \frac{n}{2} \rfloor < m$ [10]. A graph $G$ is *subgraph-overfull* ($\mathcal{SO}$) when it has an overfull subgraph $H$ with $\Delta(H) = \Delta(G)$ [10]. If the overfull subgraph $H$ is a subgraph induced by the neighborhood of a $\Delta(G)$-vertex, then $G$ is *neighborhood-overfull* ($\mathcal{NO}$) [4]. These classes are related as follows: $\mathcal{O} \subset \mathcal{SO} \subset$ Class 2 and $\mathcal{NO} \subset \mathcal{SO} \subset$ Class 2 and $\mathcal{O}$ and $\mathcal{NO}$ are incomparable.

**Lemma 3** *[15] Let $G$ be a graph which contains a universal vertex. Then $G$ is Class 1 if, and only if, $G$ is not overfull.*

A $k$-coloring partitions the set of edges of $G$ into $k$ color classes. An *equitable $k$-coloring* of a graph $G$ is a $k$-coloring of $G$ such that the sizes of any two color classes differ by at most one.

We say that a vertex $v$ *misses* a color $c$ (or that a color $c$ misses a vertex $v$) when there is no edge with color $c$ incident to $v$. Otherwise, we say that the color $c$ *appears* in $v$. For equitable colorings the following lemmas hold.

**Lemma 4** *[15] Let $G$ be a graph. If $G$ has a $k$-coloring, then there exists an equitable $k$-coloring of $G$.*

**Lemma 5** *Let $n$ be an odd integer. If $K_n$ is colored with $n$ colors, then each one of these $n$ colors misses exactly one vertex and each vertex misses exactly one color.*

**Lemma 6** *Let $n$ be an even integer. Then $K_n$ has an equitable $n$-coloring such that each vertex misses one color, each one of $\frac{n}{2}$ colors misses two vertices, and each one of the other $\frac{n}{2}$ colors appears in every vertex of $K_n$.*

*Proof.* Consider a graph $K_n$ with even $n$. Since $K_n$ has maximum degree $n-1$, by Vizing [17], $K_n$ has an $n$-coloring. Since each vertex has degree equal to $n-1$, each vertex misses exactly one color. By Lemma 4, $K_n$ has an equitable $n$-coloring. Since $K_n$ has $\frac{n(n-1)}{2}$ edges, an equitable $n$-coloring of $K_n$ corresponds to $\frac{n}{2}$ classes of color with size $\left\lfloor \frac{n-1}{2} \right\rfloor = \frac{n}{2} - 1$ each, and $\frac{n}{2}$ classes of color with size $\left\lceil \frac{n-1}{2} \right\rceil = \frac{n}{2}$ each. By definition of coloring, each color class determines a matching of $K_n$. Since there are $n$ vertices, each one of $\frac{n}{2}$ color classes with size $\frac{n}{2} - 1$ covers $n-2$ vertices, so each one of these colors misses two vertices of $K_n$. Each one of $\frac{n}{2}$ color classes with size $\frac{n}{2}$ covers $n$ vertices of $K_n$ and, therefore, each one of these colors appears in every vertex of $K_n$. ■

**Lemma 7** *Let $n$ be an even integer and $G = K_n \setminus F$, where $F$ is a subset of $E(K_n)$ with $|F| = k$. Then $G$ has an equitable $(n-1)$-coloring, such that there are $k' = \min\{k, n-1\}$ colors missing at least two vertices of $G$.*

*Proof.* Let $F$ be a subset of $E(K_n)$ with $|F| = k$ and $G = K_n \setminus F$. Since $n$ is even, $K_n$ has an $(n-1)$-coloring. Then $G$ also has an $(n-1)$-coloring. So, by Lemma 4, $G$ has an equitable $(n-1)$-coloring. Since $|E(G)| = \frac{n(n-1)}{2} - k$, each color class of an equitable $(n-1)$-coloring either has size $\frac{n}{2} - \left\lfloor \frac{k}{n-1} \right\rfloor$, or it has size $\frac{n}{2} - \left\lceil \frac{k}{n-1} \right\rceil$.

If $k \geq n-1$, each one of the $n-1$ classes of color has size at most $\frac{n}{2} - 1$ and it corresponds to a color that misses at least two vertices. Otherwise, if $k < n-1$, there are $k$ classes of color with size $\frac{n}{2} - 1$ and $n-1-k$ classes of color with size $\frac{n}{2}$. In this case, each one of the $k$ classes of color with size $\frac{n}{2} - 1$ corresponds to a color that misses two vertices of $G$. ■

**Lemma 8** *Let $n$ be an odd integer and $G = K_n \setminus F$, where $F$ is a subset of $E(K_n)$ with $|F| = k$, $k \geq \frac{n-1}{2}$. Then $G$ has an equitable $(n-1)$-coloring. Moreover, if $\frac{n-1}{2} \leq k \leq \frac{3(n-1)}{2}$ each one of $k - \frac{n-1}{2}$ colors misses exactly three vertices of $G$, and each one of the remaining $\frac{3(n-1)}{2} - k$ colors misses exactly one vertex of $G$. If $k \geq \frac{3(n-1)}{2}$, every color misses at least three vertices of $G$.*

*Proof.* The graph $G$ described above has an equitable $(n-1)$-coloring. In fact, if $G$ has maximum degree less than $n-1$, by Vizing [17] and by the definition of coloring, $G$ has an $(n-1)$-coloring and, by Lemma 4, $G$ has an equitable $(n-1)$-coloring. If $G$ has degree $n-1$, then $G$ has a universal vertex. Since, by hypothesis, $|E(\overline{G})| = |F| = k \geq \frac{n-1}{2}$, then $G$ is not overfull. So, by Lemma 3, $G$ has an $(n-1)$-coloring and, by Lemma 4, $G$ has an equitable $(n-1)$-coloring.

If $k = \frac{n-1}{2}$, then $G$ has $\frac{(n-1)(n-1)}{2}$ edges. In this case, an equitable $(n-1)$-coloring corresponds to $n-1$ color classes with size $\frac{n-1}{2}$. Each one of these color classes represents a color used on $\frac{n-1}{2}$ edges of a matching of $G$. So, there are $n-1$ disjoint matchings and each one of them covers $n-1$ vertices. Therefore, each color misses one vertex.

If $\frac{n-1}{2} < k \leq \frac{3(n-1)}{2}$, then an equitable $(n-1)$-coloring of $G$ corresponds to $k - \frac{n-1}{2}$ color classes with size $\frac{n-1}{2} - 1 = \frac{n-3}{2}$ and $n-1-(k-\frac{n-1}{2}) = \frac{3(n-1)}{2} - k$ color classes with size $\frac{n-1}{2}$. Therefore, each one of $k - \frac{n-1}{2}$ colors misses exactly three vertices and each one of the other $\frac{3(n-1)}{2} - k$ colors misses exactly one vertex.

Note that when $k = \frac{3(n-1)}{2}$, every color misses exactly three vertices. Therefore, if $k > \frac{3(n-1)}{2}$, every color of an equitable $(n-1)$-coloring of $G$ misses at least three vertices of $G$. ■

# 3 Coloring some split graphs with even maximum degree

Some classes have the chromatic index of the odd maximum degree graphs determined. This is the case of known classes of graphs such as interval graphs, dually chordal graphs [5], and split graphs [2]. Much less is known about the chromatic index when graphs with even maximum degree are considered. In this section, we consider a split graph with even maximum degree. As a consequence of the work of Chen et.al. [2], the Class 2 split graphs have even maximum degree. We exhibit a new subclass of these graphs that is Class 1 and we present polynomial time algorithms to perform an edge-coloring and to recognize this subclass.

Let $G$ be a split graph with $\Delta(G)$ even. Let $\{Q, S\}$ be a partition of $V(G)$, where $Q$ is a clique and $S$ is a stable set. From now on, we consider $Q$ as a maximal clique. Note that, in this case, each $\Delta(G)$-vertex belongs to $Q$.

Now we give a sketch of our approach to obtain a $\Delta(G)$-edge-coloring for a split graph $G$ satisfying the conditions of Theorem 9. We partition this split graph $G$ into two subgraphs such that each one can be colored with $\frac{\Delta}{2}$ colors. To do this, we find a special vertex $v$ with maximum degree and we partition $V(G)$ into $N[v]$ and $P = V(G) \setminus N[v]$. Note that $P \subset S$. Then, we choose a set $R$ with $\frac{\Delta}{2}$ vertices of $Q$ containing every vertex that is adjacent to some vertex of $P$. The subgraph of $G$ induced by the edges with a vertex in $R$ and another in $V(G) \setminus R$ is a bipartite graph with maximum degree at most $\frac{\Delta}{2}+1$. The remaining edges induce a subgraph $H$ with maximum degree $\frac{\Delta}{2}$. We choose a set $F$ of edges of the bipartite graph that are incident to distinct vertices with degree $\frac{\Delta}{2}+1$ in this bipartite graph. We color the edges of $E(H) \cup F$ with $\frac{\Delta}{2}$ colors. The bipartite graph without the edges of $F$ has degree at most $\frac{\Delta}{2}$ and we use $\frac{\Delta}{2}$ new colors obtaining a $\Delta$-coloring of $G$.

**Theorem 9** *Let $G$ be a split graph with even maximum degree and partition $\{Q, S\}$. The graph $G$ is Class 1 if there exists a $\Delta(G)$-vertex $v$ such that $N[v]$ admits a partition $\{L, R\}$ where:*

1. *$R \subset Q$,*
2. *$|R| = \frac{\Delta(G)}{2}$,*
3. *the vertices in $L$ are not adjacent to vertices in $V(G) \setminus (L \cup R)$,*
4. *$\overline{G[L]}$ has $k$ edges, $k \geq \frac{\Delta(G)}{4}$, and*
5. *$R$ has at most $k'$ vertices with degree $\Delta(G)$, where $k' = \min\{k, \frac{\Delta(G)}{2}\}$.*

*Proof.* Let $G$ be a split graph with partition $\{Q, S\}$ and $\Delta = \Delta(G)$ even. Suppose that there exists a $\Delta$-vertex $v$ of $G$ as described above. Let $P = V(G) \setminus \{L \cup R\}$. (See Fig. 1.) Since, by hypothesis, $|R| = \frac{\Delta}{2}$, then $|L| = \frac{\Delta}{2} + 1$. Hence, the maximum degree of $G[L]$ is at most $\frac{\Delta}{2}$. We consider two cases: $\frac{\Delta}{2}$ is odd and $\frac{\Delta}{2}$ is even.

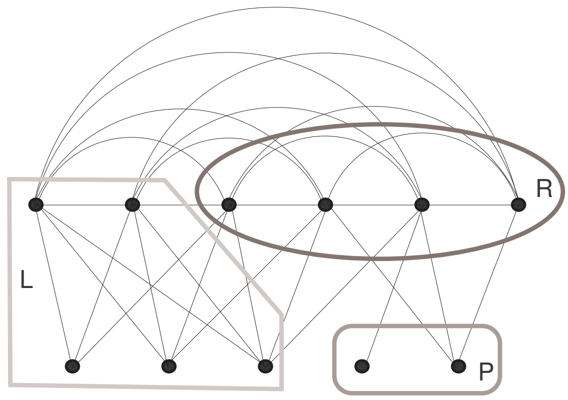


Figure 1: A split graph $G$ and the subsets $L$, $R$ and $P$ of $V(G)$.

**Case 1:** $\frac{\Delta}{2}$ *is odd*

The graph $G[L]$ is isomorphic to a subgraph of $K_{\frac{\Delta}{2}+1}$ and, by hypothesis, $\overline{G[L]}$ has $k \geq \frac{\Delta}{4}$ edges. By Lemma 7, $G[L]$ has an equitable $\frac{\Delta}{2}$-coloring where each color $c_i$ misses at least two vertices in $L$, $1 \leq i \leq k' = \min\{k, \frac{\Delta}{2}\}$.

Let $R = \{v_1, v_2, \ldots, v_{\frac{\Delta}{2}}\}$, and let $J = \{v_1, v_2, \ldots v_{|J|}\}$ be the subset of vertices of $R$ that are adjacent to every vertex of $L$. The vertices in $J$ are adjacent to $\frac{\Delta}{2} - 1$ vertices of $R$ and $\frac{\Delta}{2} + 1$ vertices of $L$, therefore these vertices have degree $\Delta$. By hypothesis, there are at most $k'$ $\Delta$-vertices in $R$, so $|J| \leq k'$. The graph $G[R]$ is isomorphic to $K_{\frac{\Delta}{2}}$ and $\frac{\Delta}{2}$ is odd, hence, by lemmas 1 and 5, $G[R]$ can be colored with $\frac{\Delta}{2}$ colors such that each color misses exactly one vertex and each vertex misses one color. By the symmetry of $G[R]$, we can perform the coloring of $G[R]$ such that the color missed by vertex $v_i$ is $c_i$, $1 \leq i \leq \frac{\Delta}{2}$.

Since $|J| \leq k'$ and the vertices in $J$ are adjacent to every vertex of $L$, each vertex $v_i$ in $J$ is adjacent to a vertex $u$ of $L$ that misses the color $c_i$. Assign the color $c_i$ to the edge $\{v_i, u\}$. For each vertex $v$ in $R\backslash J$ which is adjacent to $\frac{\Delta}{2}+1$ vertices of $L \cup P$, there is a vertex $w$ in $P$ such that $w$ is adjacent to $v$. So, assign the color $c$, missed by $v$ in the coloring of $G[R]$, to the edge $\{v, w\}$. This process can be repeated for every $\Delta$-vertex in $R\backslash J$ because the color missed by each vertex in $R$ is distinct from the other ones.

By hypothesis, the vertices in $L$ are not adjacent to vertices of $P$. Thus, each vertex of $L$ is adjacent to at most $\frac{\Delta}{2}$ vertices of $R$. Moreover, each vertex of $R$ is adjacent to at most $\frac{\Delta}{2}+1$ vertices of $L \cup P$. Note that, when the vertex of $R$ is adjacent to exactly $\frac{\Delta}{2}+1$ vertices of $L \cup P$, one of the edges incident to it is already colored. Hence the graph induced by the uncolored edges of $G$ is a bipartite graph with partition $\{L \cup P, R\}$ and its maximum degree is at most $\frac{\Delta}{2}$. Therefore, by Lemma 2, we can color this subgraph with $\frac{\Delta}{2}$ new colors.

**Case 2:** $\frac{\Delta}{2}$ *is even*

In this case, $G[L]$ is isomorphic to a subgraph of $K_{\frac{\Delta}{2}+1}$. By condition (4), the size of $E(\overline{G[L]})$ is equal to $k$, $k \geq \frac{\Delta}{4}$. So, by Lemma 8, $G[L]$ has an equitable $\frac{\Delta}{2}$-coloring such that each one of $p = \min\{k - \frac{\Delta}{4}, \frac{\Delta}{2}\}$ colors misses at least three vertices in $L$ and each one of the other $\frac{\Delta}{2} - p$ colors misses at least one vertex in $L$. Note that $p \in [0, \frac{\Delta}{2}]$, $p = k - \frac{\Delta}{4}$ when $\frac{\Delta}{4} \leq k \leq 3\frac{\Delta}{4}$, and $p = \frac{\Delta}{2}$ when $k \geq 3\frac{\Delta}{4}$. Let $c_1, \ldots, c_p$ be the colors missed by at least three vertices in $L$.

The graph $G[R]$ is isomorphic to $K_{\frac{\Delta}{2}}$ and $\frac{\Delta}{2}$ is even. So, by Lemma 6, $G[R]$ has an equitable $\frac{\Delta}{2}$-coloring such that each one of $\frac{\Delta}{4}$ colors misses two vertices, each one of the other $\frac{\Delta}{4}$ colors does not miss any vertex in $R$, and each vertex in $R$ misses exactly one color. We order the vertices of $R$ such that the first vertices are those $\Delta$-vertices adjacent to every vertex of $L$, next the $\Delta$-vertices adjacent to at least one vertex of $P = V(G)\backslash\{L \cup R\}$, and, finally, the remaining vertices of $R$.

The symmetry of $G[R]$ allows us to choose which vertex in $R$ misses a specific color in the $\frac{\Delta}{2}$-coloring of $G[R]$. Let $p' = \min\{p, \frac{\Delta}{4}\}$. Since $p \in [0, \frac{\Delta}{2}]$, then $p' \in [0, \frac{\Delta}{4}]$ and $p' \leq p$. Let $X = \{v_1, \ldots, v_{2p'}\}$ be the set of the first $2p'$ vertices of $R$. We perform the $\frac{\Delta}{2}$-coloring of $G[R]$ forcing each pair of vertices of $X$, $v_{2i-1}$ and $v_{2i}$, to miss the color $c_i$. Note that $p'$ denotes the number of colors that miss at least three vertices in $L$ and two vertices in $R$. Note also that when $k = \frac{\Delta}{4}$, the set $X$ is empty.

Remember that $p = \min\{k - \frac{\Delta}{4}, \frac{\Delta}{2}\}$ is the number of colors that miss at least three vertices of $L$; $p' = \min\{p, \frac{\Delta}{4}\}$ is the number of colors that miss two vertices of $R$ and at least three vertices of $L$; and $k' = \min\{k, \frac{\Delta}{2}\}$ is the maximum number of $\Delta$-vertices of $R$.

If $k \geq \frac{\Delta}{2}$, then $p \in \left[\frac{\Delta}{4}, \frac{\Delta}{2}\right]$, $p' = \frac{\Delta}{4}$, $|X| = 2p' = \frac{\Delta}{2}$ and, therefore, $X = R$.

If $\frac{\Delta}{4} \leq k < \frac{\Delta}{2}$, then $p = k - \frac{\Delta}{4}$, $p' = p$, $k' = k$. In this case, $X$ is a proper subset of $R$. Then, there are $\frac{\Delta}{2} - 2p' = 2(\frac{\Delta}{2} - k)$ vertices in $R \setminus X$.

By construction, each one of the colors $c_1, \ldots, c_{p'}$ misses two vertices of $X$ (if $|X| \neq \emptyset$) and $p' < \frac{\Delta}{4}$. Moreover, by Lemma 6, there are $\frac{\Delta}{4}$ colors that miss two vertices of $R$ and each vertex of $R$ has to miss one color. Then each color of the set $\alpha = \{c_{p'+1}, \ldots, c_{\frac{\Delta}{4}}\}$ has to miss two vertices in $R \setminus X$. Let $Y$ be the set of $\Delta$-vertices in $R \setminus X$. The cardinality of $Y$ is at most $|\alpha|$. In fact, by condition (5), the number of vertices that are not $\Delta$-vertices in

$R$ is at least $\frac{\Delta}{2} - k$. Hence, there are at most $\frac{\Delta}{2} - k = |\alpha|$ vertices with degree $\Delta$ in $R \setminus X$. So, we force each vertex of $Y$ to miss a distinct color of $\alpha$ in the coloring of $G[R]$.

Figure 2 shows the set $R$ and the subsets $X$ and $Y$ with the color missed by each vertex of $X \cup Y$. Now, for each $\Delta$-vertex of $R$, we use the color missed by this vertex to color one edge of $G$ incident to it. By the ordering of the vertices of $R$, the $\Delta$-vertices that are in $R$ belong to $X \cup Y$.

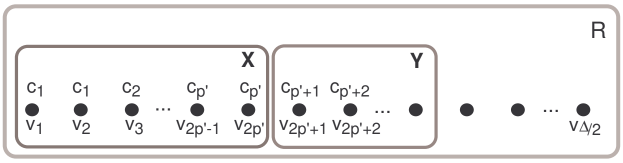


Figure 2: The set $R$ and the subsets $X$ and $Y$ with the color missed by each vertex of $X \cup Y$.

If $X$ is nonempty, then there are $p'$ pairs of vertices, $v_{2i-1}$ and $v_{2i}$, that miss the color $c_i$, $1 \leq i \leq p'$. Let $v$ be a $\Delta$-vertex in $X$. Thus, $v$ is adjacent to $\frac{\Delta}{2} + 1$ vertices of $L \cup P$. Since $|L| = \frac{\Delta}{2} + 1$, then for each vertex of $L$ which is not adjacent to $v$, there is a vertex in $P$ which is adjacent to $v$. Remember that there are at least three vertices in $L$ that miss the color $c_i$, $1 \leq i \leq p'$. Considering that every color misses every vertex of $P$, thus for each vertex of $L$ which misses the color $c_i$ and is not adjacent to $v$, there is a vertex in $P$ that is adjacent to $v$ and misses the color $c_i$. Therefore, each $\Delta$-vertex $v$ of $R$ is adjacent to at least three vertices of $L \cup P$ which miss the same color missed by $v$. Hence, for a pair of $\Delta$-vertices of $X$, $v_{2i-1}$ and $v_{2i}$, which miss a color $c_i$, it is possible to choose two distinct vertices, $x_1$ and $x_2$, belonging to $L \cup P$ which miss the color $c_i$ and such that $x_1$ is adjacent to $v_{2i-1}$ and $x_2$ is adjacent to $v_{2i}$, $1 \leq i \leq p'$. Now, we color the edges $\{x_1, v_{2i-1}\}$ and $\{x_2, v_{2i}\}$ with the color $c_i$, $1 \leq i \leq p'$. Remember that the vertices of $R$ are ordered such that the $\Delta$-vertices belonging to $R$ are the first vertices of $R$. So, when the number of $\Delta$-vertices of $R$ is odd and less than $|X|$, we have a pair $v_{2i-1}$ and $v_{2i}$ such that $v_{2i-1}$ is a $\Delta$-vertex and $v_{2i}$ is not a $\Delta$-vertex. In this case, the vertex $v_{2i-1}$ is adjacent to at least three vertices of $L \cup P$ that miss the color $c_i$ and we can choose any one of them (let $w$ be this vertex) to color the edge $\{v_{2i-1}, w\}$ with the color $c_i$.

If $Y$ is nonempty, we can color one edge incident to each vertex that is in $Y$ in the following way. For each vertex $v$ in $Y$ adjacent to all vertices of $L$, there is a vertex $u$ in $L$ that miss the same color missed by $v$. Then, we assign the color missed by $v$ to the edge $\{v, u\}$. For each vertex $v$ in $Y$ adjacent to some vertex $w$ of $P$, we assign the color missed by $v$ to the edge $\{v, w\}$. (Remember that the colors missed by the vertices of $Y$ are pairwise distinct.)

Note that there are no $\Delta$-vertices in $R \setminus (X \cup Y)$, so these vertices have at most $\frac{\Delta}{2}$ uncolored incident edges.

Now, each $\Delta$-vertex in $R$ has at most $\frac{\Delta}{2}$ uncolored incident edges. By hypothesis, the vertices in $L$ are not adjacent to vertices of $P$. Moreover, each vertex of $L \cup P$ is adjacent to at most $\frac{\Delta}{2}$ vertices of $R$. Thus, there are at most $\frac{\Delta}{2}$ uncolored edges incident to each vertex of $G$. The graph induced by the uncolored edges of $G$ is a bipartite graph with a partition $\{L \cup P, R\}$ and maximum degree $\frac{\Delta}{2}$. So, by Lemma 2, we can color this subgraph with $\frac{\Delta}{2}$ new colors.

Therefore, by cases 1 and 2, we conclude that $G$ is Class 1. ■

Given a split graph $G$ and a partition $\{L, R\}$ of $N[v]$ where $v$ is a $\Delta$-vertex of $G$, the $\Delta$-edge-coloring described in Theorem 9 is of polynomial time complexity. Moreover, if there exists a partition $\{L, R\}$ of $N[v]$ satisfying the conditions presented in Theorem 9, it also can be found in polynomial time. Now we show how to construct this partition if it exists.

Consider a split graph $G$ with a maximal clique $Q$, a stable set $S$, and a $\Delta$-vertex $v$. Let $P = V(G) \setminus N[v]$. Now, we partition the set $Q$ into four subsets as follows.

- $P_l$ is the set of vertices of $Q$ with degree less than $\Delta$ and adjacent to some vertex in $P$.

- $P_\Delta$ is the set of vertices of $Q$ with degree equal to $\Delta$ and adjacent to some vertex in $P$.

- $L_l$ is the set of vertices of $Q$ with degree less than $\Delta$ and without neighbors in $P$.

- $L_\Delta$ is the set of vertices of $Q$ with degree equal to $\Delta$ and without neighbors in $P$.

We consider the following ordering of the vertices of $Q$: first, the vertices of $L_l$ in non-decreasing order of degree, after the vertices of $L_\Delta$, after the vertices of $P_\Delta$, and then the vertices of $P_l$.

Let $R$ be the set of the last $\frac{\Delta}{2}$ vertices of $Q$, then conditions (1) and (2) of Theorem 9 hold. Let $L$ be the set $(Q \setminus R) \cup (N[v] \cap S)$.

If $|P_l| + |P_\Delta| > \frac{\Delta}{2}$, then every vertex of $R$ is adjacent to some vertex of $P$ and there is at least one vertex of $P_\Delta \cup P_l$ in $L$. Thus, condition (3) of Theorem 9 does not hold. Therefore, there is not a partition $\{L, R\}$ for $N[v]$.

If $|P_l| + |P_\Delta| \leq \frac{\Delta}{2}$, condition (3) is satisfied and we verify the number of edges in $\overline{G[L]}$ (condition (4)). If $|E(\overline{G[L]})| < \frac{\Delta}{4}$, a partition $\{L, R\}$ of $N[v]$ does not exist. In fact, the ordering of the vertices of $Q$ guarantees the maximum number of edges in $\overline{G[L]}$. Note that the vertices in $R$ either are adjacent to vertices of $P$ or have degree equal to or greater than the degree of the vertices in $L$. If $|E(\overline{G[L]})| \geq \frac{\Delta}{4}$, condition (4) is satisfied.

Now, we verify the number of $\Delta$-vertices in $R$ (condition 5). If this number is greater than the number of edges in $\overline{G[L]}$, a partition of $N[v]$ into $\{L, R\}$ does not exist. In fact, the ordering imposed to the vertices of $Q$ forces the maximum number of edges in $\overline{G[L]}$. So, we need to decrease the number of $\Delta$-vertices in $R$. Then, we replace a $\Delta$-vertex of $R$ with a vertex in $L$ with degree less than $\Delta$. If we do this, condition (5) remains unsatisfied. This replacement decreases by at least one the number of edges in $\overline{G[L]}$ and decreases by at most one the number of $\Delta$-vertices in $R$. Note that, if the $\Delta$-vertex chosen for the replacement belongs to $P_\Delta$, also condition (3) does not hold. Hence, there is not a partition $\{L, R\}$ of $N[v]$.

Therefore, if the number of $\Delta$-vertices in $R$ is less than or equal to the number of edges in $\overline{G[L]}$, $|P_l| + |P_\Delta| \leq \frac{\Delta}{2}$, and the number of edges in $\overline{G[L]}$ is greater than or equal to $\frac{\Delta}{4}$, then $\{L, R\}$ is a partition of $N[v]$ that satisfies the conditions of Theorem 9.

We recall that the algorithm presented in [9] recognize a split graph in linear time and returns also a partition $\{Q, S\}$ where $Q$ is a maximum clique. The conditions of Theorem 9 for a fixed $\Delta$-vertex of $G$ can be verified in polynomial time. Since there are at most $|Q|$ vertices with degree $\Delta$ in $G$, the subclass of split graphs considered in Theorem 9 can be recognized in polynomial time.

# 4 Conclusions

Split graphs is a well-studied class of graphs for which most combinatorial problems are solved [3, 8, 14, 16]. It has been shown that every odd maximum degree split graph is Class 1 [2] and that every subgraph-overfull split graph is in fact neighbourhood-overfull [6]. It has been conjectured that every Class 2 chordal graph is neighbourhood-overfull [6]. (Chordal graphs is a superclass of split graphs.) The validity of this conjecture for split graphs implies that the edge-coloring problem for these graphs is in P. In Theorem 9 we described a new subset of split graphs with even maximum degree that is Class 1, therefore, this subclass is not neighborhood-ovefull. We also showed how to recognize this subset in polynomial time. This result gives another positive evidence for the validity of the above conjecture for split graphs.

The $\Delta$-coloring presented in Theorem 9 can also be used to other subsets of split graphs. As an example, if a split graph $G$ with $\Delta$ even has a set of $\Delta + 1$ vertices which admits a partition $\{L, R\}$ satisfying the conditions of Theorem 9, then $G$ is Class 1. Note that this set of $\Delta + 1$ vertices is not necessarily a neighborhood of a $\Delta$-vertex of $G$ (see Figure 3).

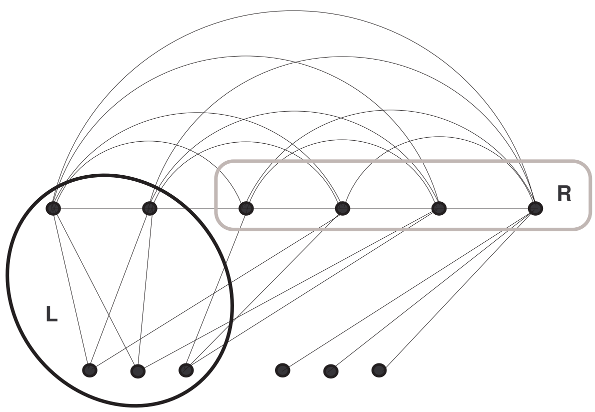


Figure 3: A split graph $G$ with a subset of vertices that is not the neighborhood of a $\Delta$-vertex, but satisfies the conditions of Theorem 9.

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