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Conversion formulas for simploidal Bernstein polynomials

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Abstract

In this report, we define simploidal polynomial functions and simploidal Bernstein bases, which are a generalization of the polynomials and Bernstein bases used in simplicial and tensorial Bézier patches. We then provide formulas for converting between simploidal polynomials expressed in various kinds of simploidal Bernstein bases.

Symbol	Description	Section
z	*real numbers*	
x, y	*real vectors*	
m, n	*rows and columns counts*	
i, j, k, p, q, rs	*natural numbers*	
$\kappa, \lambda, \mu, \nu, \rho, \sigma, \omega$	*multi-indices*	2.1
$\mathbb{I}, \mathbb{I}_d, \mathbb{I}_d^m$	*sets of multi-indices*	2.1
g, h	*degrees*	2.1
α, β, γ	*multi-degrees*	2.1
M, T	*irregular matrices*	2.1
$\mathbb{V}_m, \mathbb{V}_{m,\delta}$	*sets of irregular matrices*	2.1
Λ, Ω	*hyper-indices*	2.1
$\mathbb{H}, \mathbb{H}_d, \mathbb{H}_d^m, \mathbb{H}_d^{\alpha,\kappa}$	*sets of hyper-indices*	2.1
$\Lambda!, \kappa!$	*(multi-),hyper-factorial*	2.1
d, e	*dimensions*	2.2
δ, ε	*multi-dimensions*	2.2
$\mathbb{A}^n, \mathbb{A}^\delta$	*(multi-)affine spaces*	2.2
η, ψ, φ	*(multi)affine transformation*	2.2
u, v	*points of affine spaces*	2.2
U, V	*points multi-affine spaces*	2.2
$\mathbb{K}^n, \mathbb{K}^\delta$	*canonical simplices and simploids*	2.3
B	*Univariate Bernstein polynomials*	2.5
$\mathcal{B}$	*Multivariate Bernstein polynomials*	2.5
η_d, ψ_d	*embeddings of (multi-)affine spaces*	3

1 Introduction

1.1 Problem and motivation

Two approaches for describing multivariate polynomial functions are predominant. The *tensor* approach, introduced by C. de Boor [3], uses a basis of products of univariate polynomials on a Cartesian coordinate system. The *barycentric* approach, due to P. de Casteljau (and formalized by G. Farin [6]) uses a basis of multivariate homogeneous polynomials defined on the space of barycentric coordinates.

In geometric modelers one often has to convert functions from one representation to the other, e.g. when imposing continuity constraints between a spline defined over a rectangular mesh and a spline defined on a triangular mesh.

In this work, we consider this problem in the framework of *simploidal polynomial functions*[5], which generalize barycentric and tensor polynomial functions to arbitrary multi-affine domain spaces. We then give explicit general formulas for conversion between simploidal functions of arbitrary domain dimensions and degrees. Our formulas generalize those of Brueckner [1], Goldman [10], Feng and Peng [9] and Lasser [11, 12].

1.2 Previous work

In 1980 Brueckner[1] gave formulas for a conversion from 2D barycentric to 2D tensor functions. Some years later, Goldman[10] presented a result in opposite direction.

These conversions are special cases of composition of polynomial functions, where the inner function is of first degree. Explicit formulas for composing arbitrary barycentric functions were given in 1988 by T. de Rose [4]. In 1993 he described a general approach [5] for composing tensor and barycentric functions using Ramshaw's *blossoming operator* [13].

In 1999, Feng and Peng [9] described alternative composition formulas for 2D using the *shifting operator* of Geng-Zhe [2], restricting one of the functions to be linear (or bilinear). Lasser then gave explicit composition formulas for barycentric and tensor functions [11] (2002) and tensor and barycentric functions [12] (2008), for the special case of two dimensional domains.

2 Notation and Basic concepts

2.1 Multi-indices and hyper-indices

2.1.1 Multi-indices

A *multi-index* is a finite tuple $\kappa = (\kappa_0, \cdots, \kappa_d)$ of natural numbers [5]. We denote by $\mathbb{I}$ the set of all multi-indices, and by $\mathbb{I}_d$ the set of all multi-indices with $d+1$ components. For any $d \in \mathbb{N}$, any multi-indices $\kappa, \lambda \in \mathbb{I}_d$, and any vector $x \in \mathbb{R}^{d+1}$, we define the following operations:

sum and difference:	$\kappa \pm \lambda \;=\; (\kappa_0 \pm \lambda_0, \kappa_1 \pm \lambda_1, \cdots, \kappa_d \pm \lambda_d)$
comparison:	$\kappa \le \lambda \;\Leftrightarrow\; \kappa_0 \le \lambda_0 \wedge \kappa_1 \le \lambda_1 \wedge \cdots \wedge \kappa_d \le \lambda_d$
total:	$\lvert\kappa\rvert \;=\; \kappa_0 + \kappa_1 + \cdots + \kappa_d$
factorial:	$\kappa! \;=\; \kappa_0!\kappa_1! \cdots \kappa_d!$
combination:	$\binom{\kappa}{\lambda} \;=\; \frac{\kappa!}{(\kappa-\lambda)!\lambda!} = \binom{\kappa_0}{\lambda_0} \cdots \binom{\kappa_d}{\lambda_d}$
power:	$x^\kappa \;=\; x_0^{\kappa_0} x_1^{\kappa_1} \cdots x_d^{\kappa_d}$

When a multi-index α is used as an exponent, as in the power formula above, we call it a *multi-degree.*

Moreover, for any $g, d \in \mathbb{N}$ we denote by $\mathbb{I}_d^g$ the set of multi-indices $\kappa \in \mathbb{I}_d$ such that $|\kappa| = g$. We also denote by g^{*d} the tuple $(g, g, \cdots, g) \in \mathbb{I}_{d-1}$ with d elements equal to g.

The use of multi-indices simplifies many formulas of multivariable algebra and calculus, partial differential equations, probability, etc. For instance, the multinomial formula becomes

$$(x_0 + x_1 + \cdots + x_d)^g = \sum_{\kappa \in \mathbb{I}_d^g} \frac{g!}{\kappa!} x^\kappa \tag{2.1}$$

for any $d, g \in \mathbb{N}$ and any $x = (x_0, x_1, \cdots, x_d) \in \mathbb{R}^d$.

2.1.2 Irregular matrices

An *irregular matrix* is a tuple $M = (M_0, M_1, \cdots, M_m)$ of real tuples (its rows). Note that, unlike an ordinary matrix, an irregular matrix may have rows with different lengths.

For any $m \in \mathbb{N}$ we denote by $\mathbb{V}_m$ the set of all irregular matrices with $m+1$ rows. For any multi-index $\delta \in \mathbb{I}_m$, we denote by $\mathbb{V}_{m,\delta}$ the set of all irregular matrices $A \in \mathbb{V}_m$ such that its i-th row A_i has $\delta_i + 1$ components.

2.1.3 Hyper-indices

A *hyper-index* is an irregular matrix of natural numbers; that is a tuple $\Lambda = (\Lambda_0, \Lambda_1, \cdots, \Lambda_m)$ of multi-indices (its *rows*). Note that a multi-index can be viewed as a hyper-index with a single row.

For any $m \in \mathbb{N}$, we denote by $\mathbb{H}_m$ the set of all hyper-indices consisting of m multi-indices. For any multi-index $\delta \in \mathbb{I}_m$, we also denote by $\mathbb{H}_{m,\delta}$ the set of all hyper-indices $\Lambda \in \mathbb{H}_m$ such that each row Λ_i belongs to $\mathbb{I}_{\delta_i}$. In the specific case of $\delta = n^{*m+1}$, that is, when all rows Λ_i have the same number of components $\delta_i + 1 = n + 1$, we abbreviate $\mathbb{H}_{m,\delta}$ by $\mathbb{H}_{m,n}$. In this case, Λ is an ordinary (rectangular) matrix of natural numbers with $m+1$ rows and $n+1$ columns.

For any $m \in \mathbb{N}$, any multi-index $\kappa, \lambda, \delta \in \mathbb{I}_m$, any hyper-indices $\Lambda, \Omega \in \mathbb{H}_{m,\delta}$, and any irregular matrix $T \in \mathbb{V}_{m,\delta}$, we define the operations

comparison:	$\Lambda \leq \Omega \;\Leftrightarrow\; \Lambda_0 \leq \Omega_0 \wedge \Lambda_1 \leq \Omega_1 \wedge \cdots \wedge \Lambda_m \leq \Omega_m$
factorial:	$\Lambda! \;=\; \Lambda_0!\Lambda_1! \cdots \Lambda_m!$
combination:	$\binom{\Lambda}{\Omega} \;=\; \frac{\Lambda!}{\Omega!(\Lambda-\Omega)!} \;=\; \binom{\Lambda_0}{\Omega_0} \cdots \binom{\Lambda_d}{\Omega_d}$
power:	$T^{\Lambda} \;=\; \prod_{i=0}^{m} \prod_{j=0}^{\delta_i} T_{i,j}^{\Lambda_{i,j}}$

Moreover, if Λ is a hyper-index in $\mathbb{H}_m$ where each row has at least m elements, we denote by $\operatorname{diag}\Lambda$ the diagonal of Λ, that is the multi-index $(\Lambda_{00}, \Lambda_{11}, \cdots, \Lambda_{mm}) \in \mathbb{I}_m$.

For any $m, n \in \mathbb{N}$, any multi-index $\delta \in \mathbb{I}_m$, and any multi-degree $\alpha \in \mathbb{I}_m$, we denote by $\mathbb{H}^{\alpha}_{m,\delta}$ the set of all hyper-indices Λ such that the total $|\Lambda_i|$ of each row Λ_i is α_i. Furthermore, we denote by $\mathbb{H}^{\alpha,\beta}_{m,n}$ the set of all (rectangular) hyper-indices $\Omega \in \mathbb{H}^{\alpha}_{m,n}$ such that the sum of each column j is β_j. Note that these definitions are more general than DeRose's [5].

2.2 Affine and multi-affine spaces

2.2.1 Canonical affine spaces

We define $\mathbb{A}^d$, the *canonical affine space of dimension* $d \in \mathbb{N}$, as

$$\mathbb{A}^d = \{u \in \mathbb{R}^{d+1} \mid \sum_{i=0}^{d} u_i = 1\}. \tag{2.2}$$

Note that $\mathbb{A}^0$ has a single point, the 1-tuple (1); and that $\mathbb{A}^d \times \mathbb{A}^e$ is *not* the same as $\mathbb{A}^{d+e}$.

2.2.2 Affine transformation

An *affine transformation* φ of $\mathbb{A}^d$ is a function from $\mathbb{A}^d$ to $\mathbb{A}^d$ such that each coordinate of the result is a linear function of the coordinates of the arguments. It follows that an affine transformation of $\mathbb{A}^d$ corresponds to a unique $(d+1) \times (d+1)$ matrix of real numbers T where the sum of any row equals one. In matricial form

$$\varphi(u) = uT$$

where the argument $u \in \mathbb{A}^d$ is assumed to be a row vector with $d+1$ components.

2.2.3 Multi-affine spaces

If δ is a multi-index in $\mathbb{I}_m$, we define the *multi-affine space of dimension* δ as

$$\mathbb{A}^{\delta} = \mathbb{A}^{\delta_0} \times \cdots \times \mathbb{A}^{\delta_m}$$

Note that $\mathbb{A}^{\delta}$ is a subset of $\mathbb{V}_{m,\delta}$. That is, each point $U \in \mathbb{A}^{\delta}$ is an irregular matrix with m rows, where each row U_i has $\delta_i + 1$ real elements whose sum is 1. When a multi-index δ is used for denoting the dimension of a multi-affine space, as in the definition above, we call it a *multi-dimension*.

2.2.4 Cartesian power

We denote by $X^{\times m}$ the m-fold Cartesian product of any set X by itself, that is:

$$\mathbb{X}^{\times m} = \underbrace{\mathbb{X} \times \cdots \times \mathbb{X}}_{m}$$

Note that $\mathbb{R}^{\times m}$ as a linear vector space is isomorphic to $\mathbb{R}^m$, and, in general $(\mathbb{R}^{\times m})^{\times n}$ is isomorphic to $\mathbb{R}^{m \cdot n}$. However, $(\mathbb{A}^m)^{\times n}$ is *not* isomorphic to $\mathbb{A}^{m \cdot n}$ in general.

2.3 Simplicies and simploids

2.3.1 Canonical simplices

We define the *canonical simplex of dimension* $d \geq 0$ as the subset $\mathbb{K}^d$ of $\mathbb{A}^d$ with non-negative coordinates; that is

$$\mathbb{K}^d = \{(u_0, \cdots, u_d) \in \mathbb{R}^{d+1} \mid u_i \geq 0 \ \wedge \ \sum_{i=0}^{d} u_i = 1\} \tag{2.3}$$

The canonical simplices of dimension 1,2 and 3 are a segment in $\mathbb{R}^2$, an equilateral triangle in $\mathbb{R}^3$ and a regular tetrahedron in in $\mathbb{R}^4$, illustrated in figure 1. Note that the coordinates

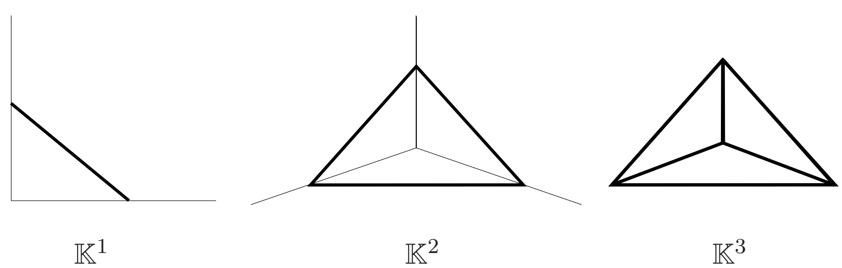


Figure 1: Canonical simplices.

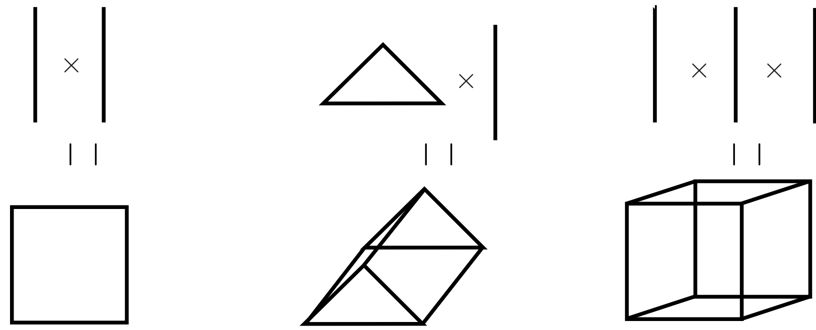
of a point $u \in \mathbb{A}^d$ are its barycentric coordinates relative to the corners of the canonical simplex $\mathbb{K}^d$.

2.3.2 Canonical simploids

For any prescribed $m \in \mathbb{N}$ and $\delta \in \mathbb{I}_m$, we define the *canonical simploid of dimension* δ as

$$\mathbb{K}^\delta = \mathbb{K}^{\delta_0} \times \cdots \times \mathbb{K}^{\delta_m}$$

The canonical simploids of dimension $(1, 1)$, $(2, 1)$ and $(1, 1, 1)$ are illustrated in figure 2.



$\mathbb{K}^{(1,1)}$ $\mathbb{K}^{(2,1)}$ $\mathbb{K}^{(1,1,1)}$

Figure 2: Canonical simploids

2.4 Simplicial and simploidal polynomials

2.4.1 Simplicial polynomials

A *simplicial polynomial of dimension* d *and degree* g is a function f from the canonical affine space $\mathbb{A}^d$ to the reals, such that $f(u)$ for any $u \in \mathbb{A}^d$ can be expressed as a homogeneous polynomial of degree g on the $d+1$ coordinates of u.

2.4.2 Simploidal polynomials

Likewise, a *simploidal polynomial of multi-dimension* $\delta \in \mathbb{I}_d$ *and multi-degree* α is a real-valued function f of the multi-affine space $\mathbb{A}^\delta$, such that $f(U)$ can be written as a polynomial on the elements of the matrix $U \in \mathbb{A}^\delta$; and, for each $i \in \{0, \cdots, d\}$, $f(U)$ is a simplicial polynomial function of row U_i with degree α_i (whose coefficients generally depend on the rows of U).

2.4.3 Tensorial polynomials

An important class of simploidal polynomials are those with domain dimension $\delta = 1^{*d}$, for some $d \in \mathbb{N}$; that is, when the domain $\mathbb{A}^\delta$ is $(\mathbb{A}^1)^{\times d}$. Note that each point $U \in (\mathbb{A}^1)^{\times d}$ is a matrix with d rows and 2 columns

$$U = \begin{bmatrix} z_0 & 1 - z_0 \\ z_1 & 1 - z_1 \\ \vdots & \vdots \\ z_{d-1} & 1 - z_{d-1} \end{bmatrix}$$

Therefore $(\mathbb{A}^1)^{\times d}$ is isomorphic to $\mathbb{R}^d$. In this case, a simploidal polynomial of multi-dimension δ is also called a *tensorial polynomial of dimension* d.

2.5 Bernstein polynomials

2.5.1 Univariate Bernstein polynomials

For any $g, i \in \mathbb{N}$, the *univariate Bernstein polynomial of degree g and index i* is the real polynomial B_i^g defined by the formula

$$B_i^g(z) = \binom{g}{i} z^i (1-z)^{g-i} \tag{2.4}$$

for all $z \in \mathbb{R}$ [8]. It is well known that the univariate Bernstein polynomials B_i^g for $i \in \{0, \cdots, g\}$ form a basis for the space of all univariate polynomials of degree g.

2.5.2 Simplicial Bernstein polynomials

For any $g \in \mathbb{N}$ and any multi-index $\kappa \in \mathbb{I}_d^g$, the *simplicial Bernstein polynomial of index κ and degree g*, $\mathcal{B}_\kappa^g(u)$ is the function from $\mathbb{A}^d$ to $\mathbb{R}$ defined by

$$\mathcal{B}_\kappa^g(u) = \frac{g!}{\kappa!} u^\kappa \tag{2.5}$$

for all $u \in \mathbb{A}^d$. Note that, in the specific case of $d = 1$, formula (2.5) is equivalent to formula (2.4), with $u_0 = z$, $u_1 = 1 - z$, $\kappa_0 = i$ and $\kappa_1 = g - i$.

It can be shown that the simplicial Bernstein polynomials $\mathcal{B}_\kappa^g$ for $\kappa \in \mathbb{I}_d^g$ are a basis for the space of all simplicial polynomials of degree g on $\mathbb{A}^d$.

2.5.3 Simploidal Bernstein polynomials

For any $m \in \mathbb{N}$, any multi-dimension $\delta \in \mathbb{I}_m$, any multi-degree $\alpha \in \mathbb{I}_m$, and any hyper-index $\Lambda \in \mathbb{H}_{m,\delta}^\alpha$, we define the *simploidal Bernstein polynomial of index Λ, degree α, and domain dimension δ* as the function from $\mathbb{A}^\delta$ to $\mathbb{R}$ such that

$$\mathcal{B}_\Lambda^\alpha(U) = \prod_{i=0}^{m} \mathcal{B}_{\Lambda_i}^{\alpha_i}(U_i) \tag{2.6}$$

for all $U \in \mathbb{A}^\delta$. These polynomials are a basis for the space of all simploidal polynomials of dimension δ and multi-degree α.

Note that the Bézier curves, patches and blocks (barycentric, tensorial and mixed) commonly used in geometric modeling and computer graphics [3, 7, 5] consist of three simploidal polynomials — the geometric X, Y, and Z coordinates— restricted to the corresponding canonical simploid. In these applications, the Bézier control points specify the coefficients of these polynomials in the Bernstein basis.

2.5.4 Tensorial Bernstein Polynomials

An important special case of formula (2.6) occurs when every dimension δ_i is 1. In this case, formula (2.6) can be recast into a product of univariate Bernstein polynomials. Namely, for any $d \in \mathbb{N}$, any multi-degree $\alpha \in \mathbb{I}_{d-1}$, and any multi-index $\kappa \in \mathbb{I}_{d-1}$ with $\kappa \leq \alpha$, we define the *tensorial Bernstein polynomial of dimension d, degree α and index κ* as the function from $(\mathbb{A}^1)^{\times d}$ to $\mathbb{R}$ defined by

$$\mathcal{B}^{\alpha}_{\kappa}(U) \quad = \quad \prod_{i=0}^{d-1} B^{\alpha_i}_{\kappa_i}(U_{i,0}) \tag{2.7}$$

for all $U \in (\mathbb{A}^1)^{\times d}$.

3 Conversion formulas for Bernstein bases

We will now provide conversion formulas between various simploidal Bernstein bases. In particular we give formulas for conversion of simplicial bases of dimension d to and from tensorial bases of dimension d (sections 3.3 and 3.4), between simploidal bases of same domain but different degrees (section 3.1), and affine domain change of a simplicial basis (section 3.2). Note that the conversion between any other classes of simploidal Bernstein bases can be obtained by composition of these special cases.

In general, each conversion formula expresses a simploidal Bernstein polynomial $\mathcal{B}^{\alpha}_{\Lambda}$ of some multi-degree α and hyper-index Λ, defined over some multi-affine domain $\mathbb{A}^{\delta}$, as a linear combination of the simploidal Bernstein polynomials $\mathcal{B}^{\beta}_{\Omega}$ of some other multi-degree β and a arbitrary subset of hyper-indices Ω, defined over another domain $\mathbb{A}^{\varepsilon}$. That is,

$$\mathcal{B}^{\alpha}_{\Lambda}(U) \quad = \sum_{\Omega \in \mathbb{H}(\alpha,\kappa,\lambda)} G(\alpha,\Lambda,\beta,\Omega)\; \mathcal{B}^{\beta}_{\Omega}(U) \tag{3.1}$$

where $\mathbb{H}(\alpha,\kappa,\beta)$ is some set of hyper-indices and $G(\alpha,\Lambda,\beta,\Omega)$ is a real coefficient.

3.1 Degree raising

3.1.1 Univariate polynomials

Consider the univariate Bernstein polynomial $\mathbb{B}^{g}_{i}(z)$. The *degree-raising formula* (3.3) below allow us to express $\mathbb{B}^{g}_{i}(z)$ as a linear combination of polynomials $\mathbb{B}^{h}_{j}(z)$ of any prescribed degree $h \geq g$.

$$
\begin{aligned}
B_i^g(z) &= \binom{g}{i} z^i (1-z)^{g-i} \\
&= \binom{g}{i} z^i (1-z)^{g-i} (1)^{h-g} \\
&= \binom{g}{i} z^i (1-z)^{g-i} (z + (1-z))^{h-g} \\
&= \binom{g}{i} z^i (1-z)^{g-i} \sum_{j=0}^{h-g} \binom{h-g}{j} z^j (1-z)^{h-g-j} \\
&= \sum_{j=0}^{h-g} \binom{g}{i} \binom{h-g}{j} z^{i+j} (1-z)^{h-i-j}
\end{aligned}
\tag{3.2}
$$

Introducing the variable $k = i + j$ and replacing j by $k - i$, we get

$$
\begin{aligned}
B_i^g(z) &= \sum_{k=i}^{i+h-g} \binom{g}{i} \binom{h-g}{k-i} z^k (1-z)^{h-k} \\
&= \sum_{k=i}^{i+h-g} \binom{g}{i} \binom{h-g}{k-i} \binom{h}{k}^{-1} \binom{h}{k} z^k (1-z)^{h-k} \\
&= \binom{h}{g}^{-1} \sum_{k=i}^{i+h-g} \binom{k}{i} \binom{h-k}{g-i} B_k^h(z)
\end{aligned}
\tag{3.3}
$$

3.1.2 Simplicial polynomials

Formula (3.3) can be generalized to simplicial Bernstein polynomials. Namely, for any degrees $g, h \in \mathbb{N}$ with $g \leq h$, any dimension $d \in \mathbb{N}$ and any multi-index $\kappa \in \mathbb{I}_d^g$, we have

$$
\begin{aligned}
\mathcal{B}_\kappa^g(u) &= \frac{g!}{\kappa!} u^\kappa \\
&= \frac{g!}{\kappa!} u^\kappa (1)^{h-g} \\
&= \frac{g!}{\kappa!} {u_0}^{\kappa_0} \cdots {u_d}^{\kappa_d} (u_0 + \cdots + u_d)^{h-g}
\end{aligned}
$$

Using the multinomial power formula (2.1) we get

$$
\begin{aligned}
\mathcal{B}^g_\kappa(u) &= \frac{g!}{\kappa!}{u_0}^{\kappa_0}\cdots{u_d}^{\kappa_d}\sum_{\lambda\in\mathbb{I}^{h-g}_d}\frac{(h-g)!}{\lambda!}{u_0}^{\lambda_0}\cdots{u_d}^{\lambda_d} \\
&= \sum_{\lambda\in\mathbb{I}^{h-g}_d}\frac{g!}{\kappa!}\,\frac{(h-g)!}{\lambda!}{u_0}^{\kappa_0+\lambda_0}\cdots{u_d}^{\kappa_d+\lambda_d} \\
&= \sum_{\lambda\in\mathbb{I}^{h-g}_d}\frac{g!}{\kappa!}\,\frac{(h-g)!}{\lambda!}\,\frac{(\kappa+\lambda)!}{h!}\,\frac{h!}{(\kappa+\lambda)!}{u_0}^{\kappa_0+\lambda_0}\cdots{u_p}^{\kappa_d+\lambda_d} \\
&= \sum_{\lambda\in\mathbb{I}^{h-g}_d}\frac{g!}{\kappa!}\,\frac{(h-g)!}{\lambda!}\,\frac{(\kappa+\lambda)!}{h!}\mathcal{B}^h_{\kappa+\lambda}(u)
\end{aligned}
$$

Introducing $\mu=\kappa+\lambda$, we get

$$
\begin{aligned}
\mathcal{B}^g_\kappa(u) &= \sum_{\substack{\mu\in\mathbb{I}^h_d\\ \kappa\le\mu}}\frac{\mu!}{(\mu-\kappa)!\kappa!}\,\frac{g!(h-g)!}{h!}\mathcal{B}^h_\mu(u) \\
&= \binom{h}{g}^{-1}\sum_{\substack{\mu\in\mathbb{I}^h_d\\ \kappa\le\mu}}\binom{\mu}{\kappa}\mathcal{B}^h_\mu(u) \qquad (3.4)
\end{aligned}
$$

Note that equations (3.3) and (3.4) generalize the formulas for raising a simplicial polynomial from degree g to $g+1$ given by Farin [6].

3.1.3 Simploidal polynomials

Formula (3.4) can be generalized further to a degree raising formula for general simploidal polynomials. Namely, for any $m\in\mathbb{N}$, any multi-dimension $\delta\in\mathbb{I}_m$, any multi-degrees $\alpha,\beta\in\mathbb{I}_m$ with $\alpha\le\beta$, and any hyper-index $\Lambda\in\mathbb{H}^\alpha_{m,\delta}$, we have

$$
\begin{aligned}
\mathcal{B}^\alpha_\Lambda(U) &= \prod_{i=0}^{m}\mathcal{B}^{\alpha_i}_{\Lambda_i}(U) \\
&= \left(\binom{\beta_0}{\alpha_0}^{-1}\sum_{\substack{\kappa_0\in\mathbb{I}^{\beta_0}_{\delta_0}\\ \Lambda_0\le\kappa_0}}\binom{\kappa_0}{\Lambda_0}\mathcal{B}^{\beta_0}_{\kappa_0}(U)\right)\cdots\left(\binom{\beta_m}{\alpha_m}^{-1}\sum_{\substack{\kappa_m\in\mathbb{I}^{\beta_m}_{\delta_m}\\ \Lambda_m\le\kappa_m}}\binom{\kappa_m}{\Lambda_m}\mathcal{B}^{\beta_m}{\kappa_m}(U)\right) \\
&= \binom{\beta_0}{\alpha_0}^{-1}\cdots\binom{\beta_m}{\alpha_m}^{-1}\sum_{\substack{\kappa_0\in\mathbb{I}^{\beta_0}_{\delta_0}\\ \Lambda_0\le\kappa_0}}\cdots\sum_{\substack{\kappa_m\in\mathbb{I}^{\beta_m}_{\delta_m}\\ \Lambda_m\le\kappa_m}}\binom{\kappa_0}{\Lambda_0}\cdots\binom{\kappa_m}{\Lambda_m}\mathcal{B}^{\beta_0}_{\kappa_0}(U)\cdots\mathcal{B}^{\beta_m}_{\kappa_m}(U)
\end{aligned}
$$

Introducing the hyper-index Ω whose rows are $\kappa_0, \kappa_1, \cdots, \kappa_m$ we get

$$\begin{aligned}\mathcal{B}^{\alpha}_{\Lambda}(U) &= \binom{\beta}{\alpha}^{-1} \sum_{\substack{\Omega \in \mathbb{H}^{\beta}_{m,\delta} \\ \Lambda \leq \Omega}} \binom{\Omega_0}{\Lambda_0} \cdots \binom{\Omega_m}{\Lambda_m} \mathcal{B}^{\beta}_{\Omega}(U) \\ &= \binom{\beta}{\alpha}^{-1} \sum_{\substack{\Omega \in \mathbb{H}^{\beta}_{m,\delta} \\ \Lambda \leq \Omega}} \binom{\Omega}{\Lambda} \mathcal{B}^{\beta}_{\Omega}(U) \qquad (3.5)\end{aligned}$$

3.2 Affine domain change

Let T be the matrix of an affine transformation φ from $\mathbb{A}^d$ to $\mathbb{A}^d$. The following formula expresses the composition of φ and an arbitrary simplicial Bernstein polynomial $\mathcal{B}^g_{\kappa}$, applied in that order, as a linear combination of simplicial Bernstein polynomials $\mathcal{B}^g_{\lambda}$ of the same degree:

$$\begin{aligned}\mathcal{B}^g_{\kappa}(\varphi(u)) &= \frac{g!}{\kappa!} v^{\kappa} \\ &= \frac{g!}{\kappa!} {v_0}^{\kappa_0} \cdots {v_d}^{\kappa_d} \\ &= \frac{g!}{\kappa!} \left(\sum_{i=0}^{d} T_{0,i} u_i \right)^{\kappa_0} \cdots \left(\sum_{i=0}^{d} T_{d,i} u_i \right)^{\kappa_d} \\ &= \frac{g!}{\kappa!} \prod_{j=0}^{d} \left(\sum_{i=0}^{d} T_{j,i} u_i \right)^{\kappa_j} \\ &= \frac{g!}{\kappa!} \prod_{j=0}^{d} \sum_{\lambda \in \mathbb{I}^{\kappa_j}_d} \frac{\kappa_j!}{\lambda!} (T_{j,0} u_0)^{\lambda_0} \cdots (T_{j,d} u_d)^{\lambda_d} \\ &= \frac{g!}{\kappa!} \sum_{\Lambda \in \mathbb{H}^{\kappa}_{d,d}} \prod_{j=0}^{d} \frac{\kappa_j!}{\Lambda_j!} (T_{j,0} u_0)^{\Lambda_{j,0}} \cdots (T_{j,d} u_d)^{\Lambda_{j,d}} \\ &= \sum_{\Lambda \in \mathbb{H}^{\kappa}_{d,d}} \frac{g!}{\Lambda!} \left(\prod_{j=0}^{d} \prod_{i=0}^{d} T_{j,i}^{\Lambda_{j,i}} \right) \left(\prod_{k=0}^{d} u_k^{(\Lambda_{0,k} + \cdots + \Lambda_{d,k})} \right)\end{aligned}$$

We can rearrange the summation over Λ separating terms according to the column totals μ of Λ:

$$\begin{aligned}
\mathcal{B}^g_\kappa(\varphi(u)) &= \sum_{\mu\in\mathbb{I}^g_d}\sum_{\Lambda\in\mathbb{H}^{\kappa,\mu}_{d,d}} \frac{g!}{\Lambda!} T^\Lambda u_0^{\mu_0}\cdots u_d^{\mu_d} \\
&= \sum_{\mu\in\mathbb{I}^g_d}\sum_{\Lambda\in\mathbb{H}^{\kappa,\mu}_{d,d}} \frac{g!}{\Lambda!} T^\Lambda \frac{\mu!}{g!}\frac{g!}{\mu!} u^\mu \\
&= \sum_{\mu\in\mathbb{I}^g_d}\left(\sum_{\Lambda\in\mathbb{H}^{\kappa,\mu}_{d,d}} \frac{\mu!}{\Lambda!} T^\Lambda\right) \mathcal{B}^g_\mu(u) \qquad (3.6)
\end{aligned}$$

As one should expect, in the special case where the affine transformation matrix is the identity matrix I we obtain the trivial result. This is due to the fact that I^Λ(and consequently the inner summation) is 1 if Λ is a diagonal matrix, and 0 otherwise. Note that the set $\mathbb{H}^{\kappa,\lambda}_{d,d}$ contains a diagonal matrix if and only if $\kappa = \lambda$; and, even then, the only diagonal matrix Λ in that set is the one which has $\operatorname{diag}\Lambda = \kappa$.

3.3 Conversion from simplicial to tensorial

We define the *canonical embedding of d-dimensional affine space* $\mathbb{A}^d$ *onto d-dimensional multi-affine space* $(\mathbb{A}^1)^{\times d}$ as the function η_d such that

$$\eta_d(u_0,\cdots,u_d) = \begin{bmatrix} u_0 & 1-u_0 \\ \vdots & \vdots \\ u_{d-1} & 1-u_{d-1} \end{bmatrix}$$

The mapping η_2 is shown in figure 3

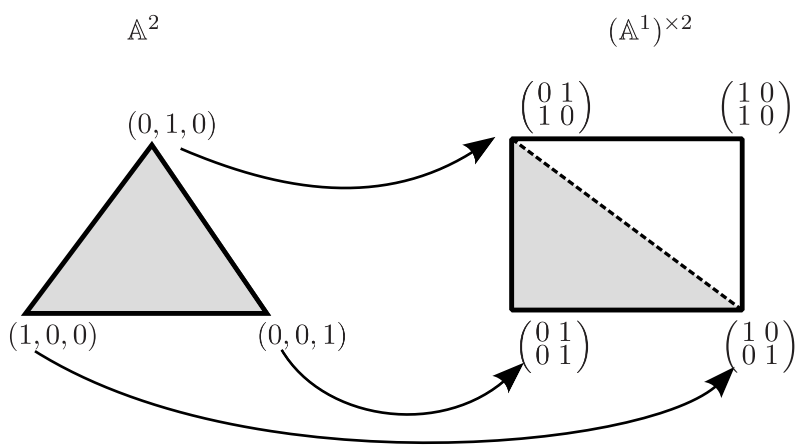


Figure 3: The canonical embedding η_2 of $\mathbb{A}^2$ onto $(\mathbb{A}^1)^{\times 2}$.

The formulas given below express a simplicial Bernstein polynomial $\mathcal{B}^g_\kappa(u)$ defined over $\mathbb{A}^d$ as a linear combination of the simploidal Bernstein polynomials $\mathcal{B}^\alpha_\sigma(\eta(u))$, where $\alpha = g^{*d} = (g, \cdots, g) \in \mathbb{I}_{d-1}$. For these formulas we denote by i the last element κ_d of κ, and by λ the multi-index in $\mathbb{I}^{g-i}_{d-1}$ that consists of all elements of κ except the last one. Observe that $\kappa! = \lambda!\, i!$ and $u^\kappa = u_d^i \prod_{j=0}^{d-1} u_j^{\lambda_j}$.

With these notations, equation (2.5) can be recast into

$$\begin{aligned}
\mathcal{B}^g_\kappa(u) &= \frac{g!}{\kappa!} u_0^{\lambda_0} u_1^{\lambda_1} \cdots u_{d-1}^{\lambda_{d-1}} (1 - u_0 - \cdots u_{d-1})^i \\
&= \frac{g!}{\kappa!} u_0^{\lambda_0} u_1^{\lambda_1} \cdots u_{d-1}^{\lambda_{d-1}} \sum_{\mu \in \mathbb{I}^i_d} \frac{i!}{\mu!} (1-d)^{\mu_d} \prod_{k=0}^{d-1} (1-u_k)^{\mu_k} \\
&= \sum_{\mu \in \mathbb{I}^i_d} (1-d)^{\mu_d} \frac{g!}{\kappa!} \frac{i!}{\mu!} \prod_{k=0}^{d-1} u_k^{\lambda_k} (1-u_k)^{\mu_k} \\
&= \sum_{\mu \in \mathbb{I}^i_d} (1-d)^{\mu_d} \frac{g!}{\kappa!} \frac{i!}{\mu!} \frac{(\mu+\kappa)!}{(\mu+\kappa)!} \prod_{k=0}^{d-1} u_k^{\lambda_k} (1-u_k)^{\mu_k} \\
&= \sum_{\mu \in \mathbb{I}^i_d} (1-d)^{\mu_d} \frac{g! i!}{(\mu+\kappa)!} \frac{(\mu_d+i)!}{i! \mu_d!} \prod_{k=0}^{d-1} \frac{(\mu_k+\lambda_k)!}{\mu_k! \lambda_k!} u_k^{\lambda_k} (1-u_k)^{\mu_k} \\
&= \sum_{\mu \in \mathbb{I}^i_d} (1-d)^{\mu_d} \frac{g!}{(\mu+\kappa)!} \frac{(\mu_d+i)!}{\mu_d!} \prod_{k=0}^{d-1} B^{\lambda_k+\mu_k}_{\lambda_k}(U_k)
\end{aligned} \tag{3.7}$$

We now introduce the variable $j = i - \mu_d$ and replace μ_d by $i - j$. Equation (3.7) becomes

$$\begin{aligned}
\mathcal{B}^g_\kappa(u) &= \sum_{j=0}^{i} \sum_{\nu \in \mathbb{I}^j_{d-1}} (1-d)^{i-j} \frac{g!(i-j+i)!}{(\nu+\lambda)!(i-j+i)!(i-j)!} \prod_{k=0}^{d-1} B^{\lambda_k+\nu_k}_{\lambda_k}(U_k) \\
&= \sum_{j=0}^{i} \sum_{\nu \in \mathbb{I}^j_{d-1}} (1-d)^{i-j} \frac{g!}{(\nu+\lambda)!(i-j)!} \prod_{k=0}^{d-1} B^{\lambda_k+\nu_k}_{\lambda_k}(U_k)
\end{aligned} \tag{3.8}$$

Substituting ρ for $\nu + \lambda$ and q for $i - j$, equation (3.8) becomes

$$\mathcal{B}^g_\kappa(u) = \sum_{q=0}^{i} \sum_{\substack{\rho \in \mathbb{I}^{g-q}_{d-1} \\ \rho \geq \lambda}} (1-d)^q \frac{g!}{\rho! q!} \prod_{k=0}^{d-1} B^{\rho_k}_{\lambda_k}(U_k) \tag{3.9}$$

In order to reduce formula (3.9) to a linear combination of Bernstein polynomials with the same multi-degree α, it is necessary to apply the degree raising formula for univariate Bernstein polynomials (3.3) into formula (3.9). We then get

$$
\begin{aligned}
\mathcal{B}^{g}_{\kappa}(u) &= \sum_{q=0}^{i} \sum_{\substack{\rho\in\mathbb{I}^{g-q}_{d-1}\\ \rho\geq\lambda}} (1-d)^{q}\frac{g!}{\rho!q!} \sum_{\substack{\sigma\in\mathbb{I}_{d-1}\\ \lambda\leq\sigma\\ \sigma\leq\lambda+\alpha-\rho}} \binom{\alpha}{\rho}^{-1}\binom{\sigma}{\lambda}\binom{\alpha-\sigma}{\rho-\lambda}\prod_{k=0}^{d-1} B^{g}_{\sigma_k}(U_k) \\
&= \sum_{\substack{\rho\in\mathbb{I}_{d-1}\\ \lambda\leq\rho\\ |\rho|\leq g}} \sum_{\substack{\sigma\in\mathbb{I}_{d-1}\\ \lambda\leq\sigma\\ \sigma\leq\lambda+\alpha-\rho}} (1-d)^{g-|\rho|}\frac{g!(\alpha-\rho)!}{\alpha!(g-|\rho|)!}\binom{\sigma}{\lambda}\binom{\alpha-\sigma}{\rho-\lambda}\mathcal{B}^{\alpha}_{\sigma}(U) \\
&= \sum_{\substack{\sigma\in\mathbb{I}_{d-1}\\ \sigma\geq\lambda\\ \sigma\leq\alpha}} \left(\sum_{\substack{\rho\in\mathbb{I}_{d-1}\\ \rho\geq\lambda\\ |\rho|\leq g\\ \rho\leq\alpha+\lambda-\sigma}} (1-d)^{g-|\rho|}\frac{g!(\alpha-\rho)!}{\alpha!(g-|\rho|)!}\binom{\sigma}{\lambda}\binom{\alpha-\sigma}{\rho-\lambda} \right) \mathcal{B}^{\alpha}_{\sigma}(U) \qquad (3.10)
\end{aligned}
$$

3.4 Conversion from tensorial to simplicial

We define the *canonical embedding* of $(\mathbb{A}^1)^{\times d}$ into $\mathbb{A}^d$ as $\psi = \eta^{-1}$, that is, the mapping ψ such that $\psi_d((U_{0,0}, U_{0,1}), (U_{1,0}, U_{1,1}), \cdots, (U_{d-1,0}, U_{d-1,1})) = (U_{0,0}, U_{1,0}, \cdots, U_{d-1,0}, 1 - \sum_{i=0}^{d-1} U_{i,0})$.

By definition (2.7), for $U \in (\mathbb{A}^1)^{\times d}$, any multi-degree $\alpha \in \mathbb{I}_{d-1}$, and any multi-index $\mu \in \mathbb{I}_{d-1}$ with $\mu \leq \alpha$, the tensorial Bernstein polynomial $\mathcal{B}^{\alpha}_{\mu}(U)$ can be rewritten as

$$
\mathcal{B}^{\alpha}_{\mu}(U) = \prod_{j=0}^{d-1} B^{\alpha_j}_{\mu_j}(u_j) \qquad (3.11)
$$

where $u = \varphi(U) \in \mathbb{A}^d$.

Moreover, each univariate Bernstein polynomial $B^{g}_{i}(u_j)$ of the product (3.11) can be described as a linear combination of simplicial Bernstein polynomials defined over $\mathbb{A}^d$, namely

$$\begin{aligned}
B_i^g(u_j) &= \frac{g!}{i!(g-i)!} u_j^i (1-u_j)^k \\
&= \frac{g!}{i!(g-i)!} u_j^i \left(1 - \left(\sum_{\substack{p=0 \\ j \neq j}}^{d} u_p \right) + \left(\sum_{\substack{p=0 \\ j \neq j}}^{d} u_p \right) - u_j \right)^k \\
&= \frac{g!}{i!k!} u_j^i (u_0 + \cdots + u_{j-1} + u_{j+1} + \cdots + u_d)^k \\
&= \frac{g!}{i!k!} u_j^i \sum_{\substack{\kappa \in \mathbb{I}_d^k \\ \kappa_j = 0}} \frac{k!}{\kappa!} u_0^{\kappa_0} \cdots u_{j-1}^{\kappa_{j-1}} u_{j+1}^{\kappa_{j+1}} \cdots u_d^{\kappa_d} \\
&= \sum_{\substack{\kappa \in \mathbb{I}_d^k \\ \kappa_j = 0}} \frac{g!}{\kappa! i!} u_0^{\kappa_0} \cdots u_{j-1}^{\kappa_{j-1}} u_j^i u_{j+1}^{\kappa_{j+1}} \cdots u_d^{\kappa_d} \\
&= \sum_{\substack{\lambda \in \mathbb{I}_d^g \\ \lambda_j = i}} \frac{g!}{\lambda!} u_0^{\lambda_0} \cdots u_d^{\lambda_d} \\
&= \sum_{\substack{\lambda \in \mathbb{I}_d^g \\ \lambda_j = i}} \mathcal{B}_\lambda^g(u)
\end{aligned} \tag{3.12}$$

Therefore

$$\begin{aligned}
\mathcal{B}_\mu^\alpha(U) &= \prod_{j=0}^{d-1} B_{\mu_j}^{\alpha_j}(u_j) \\
&= B_{\mu_0}^{\alpha_0}(u_0) \cdots B_{\mu_{d-1}}^{\alpha_{d-1}}(u_{d-1}) \\
&= \sum_{\nu \in \mathbb{I}_d^{|\alpha|}} \sum_{\substack{\Lambda \in \mathbb{H}_{d-1,d}^{\Lambda,\nu} \\ \mu = \operatorname{diag} \Lambda}} \mathcal{B}_{\Lambda_0}^{\alpha_0}(u) \cdots \mathcal{B}_{\Lambda_{d-1}}^{\alpha_{d-1}}(u)
\end{aligned} \tag{3.13}$$

We now use the fact that the product of $m+1$ simplicial Bernstein polynomials with degrees $\alpha_0, \alpha_1, \cdots, \alpha_m$ can be reduced to a single simplicial Bernstein polynomial of degree $g = \alpha_0 + \alpha_1 + \cdots + \alpha_m$ [5]. Namely, for any $m, d \in \mathbb{N}$, any multi-degree $\alpha \in \mathbb{I}_m$, any multi-index $\kappa \in \mathbb{I}_d$ and any hyper-index $\Lambda \in \mathbb{H}_{m,d}^{\alpha,\kappa}$, the product $\mathcal{B}_{\Lambda_0}^{\alpha_0}(u) \cdots \mathcal{B}_{\Lambda_m}^{\alpha_m}(u)$ can be rewritten as

$$\mathcal{B}^{\alpha_0}\Lambda_0(u) \cdots \mathcal{B}^{\alpha_m}\Lambda_m(u) = \frac{\alpha!}{\Lambda!} \frac{\kappa!}{g!} \mathcal{B}_\kappa^g(u) \tag{3.14}$$

for all $u \in \mathbb{A}^d$. Therefore,

$$\mathcal{B}^{\alpha}_{\mu}(U) = \sum_{\nu \in \mathbb{I}^{|\alpha|}_{d}} \left(\sum_{\substack{\Lambda \in \mathbb{H}^{\Lambda,\nu}_{d-1,d} \\ \operatorname{diag} \Lambda = \mu}} \frac{\alpha!}{\Lambda!} \frac{\nu!}{|\alpha|!} \right) \mathcal{B}^{|\alpha|}_{\nu}(u) \tag{3.15}$$

4 Conclusions

The formulas in section 3 are sufficient for conversion between any two kinds of simploidal Bernstein polynomial functions. Together with formulas for restriction to a subspace and differentiation (to be given in a future report), they allow to express any sort of continuity constraints between arbitrary polynomial patches.

References

[1] I. Brueckner. Construction of Bézier points of quadrilaterals from those of triangles. *Computer Aided Design*, 12(1):21–24, 1980.

[2] G.-Z. Chang. Bernstein polynomials via the shifting operator. *The American Mathematical Monthly*, 91(10):634–638, 1984.

[3] C. deBoor. Bicubic spline interpolation. *J. Math and Physics*, 41(3):212–218, 1962. as cited by [8].

[4] T.D. DeRose. Composing Bézier simplexes. *ACM Trans. Graph.*, 7(3):198–221, 1988.

[5] T.D. DeRose, R.N. Goldman, H. Hagen, and S. Mann. Functional composition algorithms via blossoming. *ACM Trans. Graph.*, 12(2):113–135, 1993.

[6] G. Farin. Bézier polynomials over triangles and the construction of piecewise CR polynomials. Technical Report TR91, Brunel University, 1980.

[7] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design*. Academic Press Professional, Inc., San Diego, CA, USA, 1990.

[8] G. Farin, J. Hoschek, and M.-S. Kim, editors. *Handbook of Computer Aided Geometric Design*, chapter 1. North Holland, 2002.

[9] J.Q. Feng and Q.S. Peng. Functional compositions via shifting operators for Bézier patches and their applications. *Chinese Journal of Advanced Software Research*, 10(12):1316–1321, 1999.

[10] R.N. Goldman and D.J. Filip. Conversion from Bézier rectangles to Bézier triangles. *Computer Aided Design*, 19(1):25–27, 1987.

[11] D. Lasser. Tensor product Bézier surfaces on triangle Bézier surfaces. *Computer Aided Geometric Design*, 19(8):625–643, 2002.

[12] D. Lasser. Triangular subpatches of rectangular Bézier surfaces. *Comput. Math. Appl.*, 55(8):1706–1719, 2008.

[13] L. Ramshaw. Blossoms are polar forms. Technical Report SRC-RR-34, HP Labs, 1989.