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Abstract

In this paper we present a new approach to solve the Graph Reconstruction Conjecture. We reduce the problem of reconstructing connected graphs to the problem of finding a special system of linear equations with a unique solution. We also show how this method can be applied to some simple cases and propose some extensions that possibly could be used for a proof of the conjecture.

1 Introduction

The Graph Reconstruction Conjecture was first proposed by S.M. Ulam and P.J. Kelly in 1941. Since then, several researchers in the area have tried to answer this apparently simple question. Many techniques have been created and partial results have been obtained. In spite of this, after almost seventy years of research, the question remains an open problem.

First, we define formally the problem.

Definition. *A* **primal subgraph** *of a simple graph G is a graph obtained from G by deleting one of its vertices. The* **deck** *of a graph G is the family of all primal subgraphs of G (without naming the vertices), called the* **cards** *of the deck. A* **reconstruction** *of a graph G is a graph H with the same deck as G. A graph G is* **reconstructible** *if every reconstruction of G is isomorphic to G.*

Informally, we would like to know if a valid deck uniquely determines the isomorphism class of the original graph. Using these definitions, we can state the problem:

Graph Reconstruction Conjecture. *All finite simple graphs on at least three vertices are reconstructible.*

The conjecture has been proved for many particular classes of graphs such as trees and disconnected graphs (Bondy[1]). The general case remains an open question. Because the literature is very rich, we present here just the relevant results necessary to expose our ideas.

A very useful result was obtained by Kelly in 1957 and it will be extensively applied in this paper. First we need some definitions:

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Definition. *A* **class** *of graphs is a family of graphs closed under isomorphism. A class $\mathcal{G}$ of graphs is* **reconstructible** *if every graph in $\mathcal{G}$ is reconstructible.*

Definition. *A function defined over a class $\mathcal{G}$ of graphs is* **reconstructible** *if, for every graph G in $\mathcal{G}$, it has the same value for all reconstructions of G.*

Definition. *Given a graph G, the number of vertices and edges of G is denoted by $v(G)$ and $e(G)$, respectively. For graphs F and G, the number of subgraphs of G isomorphic to F is denoted by $s(F, G)$.*

Kelly's Lemma. *Given graphs F and G such that $v(F) < v(G)$, the parameter $s(F, G)$ is reconstructible.*

Proof. Each subgraph of G isomorphic to F occurs exactly in $v(G) - v(F)$ primal subgraphs of G. Therefore:

$$s(F, G) = \frac{1}{v(G) - v(F)} \sum_{v \in V} s(F, G - v).$$

Since the right-hand side of this identity is reconstructible, so too is the left-hand side. □

By Kelly's Lemma we can obtain some basic results:

Corollary. *Given graphs F and G such that $v(F) < v(G)$, the number os subgraphs of G isomorphic to F containing a given vertex v is reconstructible.*

Proof. This number is exactly $s(F, G)$-$s(F, G - v)$, so is reconstructible. □

Corollary. *The number of edges and the sequence of vertices degree is reconstructible.*

Proof. Just take $F = K_2$ in Kelly's Lemma and in the previous corollary. □

Later we show how Kelly's Lemma can be used to prove that the class of disconnected graphs is reconstructible.

W.L. Kocay, in 1981, found an elementary and extremely powerful technique that allow to overcome the limitation imposed by Kelly's Lemma on the number of vertices of the subgraph considered. To present this result we introduce new definitions:

Definition. *Let G be a graph and $\mathcal{F} := (F_1, F_2, ..., F_m)$ be a sequence of graphs(not necessarily distinct). A* **cover** *of G by $\mathcal{F}$ is a sequence $(G_1, G_2, ..., G_m)$ of subgraphs of G such that for each $1 \leq i \leq m$, we have $G_i \cong F_i$ and $\bigcup G_i = G$. The number of covers of G by $\mathcal{F}$ is denoted by $c(\mathcal{F}, G)$.*

Kocay's Lemma. *For each graph G and every sequence $\mathcal{F} := (F_1, F_2, \ldots, F_m)$ of graphs such that $v(F_i) < v(G), 1 \leq i \leq m$, the parameter*

$$\sum_X c(\mathcal{F}, X) s(X, G)$$

is reconstructible, where the sum extends over the classes X of graphs such that $v(X) = v(G)$.

Proof. Counting in two ways the sequence $(G_1, G_2, ..., G_m)$ of subgraphs of G such that $G_i \cong F_i, 1 \leq i \leq m$, we have

$$\prod_{i=1}^{m} s(F_i, G) = \sum_{X} c(\mathcal{F}, X) s(X, G) \tag{1}$$

where the sum is taken over all the isomorphism classes X of graphs. By Kelly's Lemma, the left-hand side is reconstructible and hence we have that the right-hand side has the same value for every reconstruction of G. Because the terms $c(\mathcal{F}, X)s(X, G)$ are all reconstructible by Kelly's Lemma if $v(X) < v(G)$ and trivially if v(X) > v(G), the result follows. □

Kocay's Lemma allows us to obtain some interesting results. By means of a convenient choice of the sequence $\mathcal{F}$, it is possible to prove that the number of perfect matchings, spanning trees and Hamiltonian cycles is reconstructible. All the proofs that appeared so far and other important consequences can be found in Bondy[1].

Using a particular case of Kocay's Lemma, Tutte[2] reduced the number of classes X in the sum of the expression stated in Kocay's Lemma.

Modified Kocay's Lemma. *Kocay's Lemma remains valid when we restrict the sum to be taken only over the class of connected graphs.*

In the next section we give a proof of a classic result related to the Reconstruction Conjecture. This will allow us to state our main theorems in a more general setting.

2 The Reconstruction of Disconnected Graphs

In this section we use Kelly's Lemma to present a simple proof that the deck of a disconnected graph uniquely determines the original graph. The proof follows a method that is well-known when one tries to show that a particular class of graphs is reconstructible.

Definition. *Let $\mathcal{G}$ be a class of graphs. A $\mathcal{G}$-graph is a graph that belongs to $\mathcal{G}$. The class $\mathcal{G}$ is said to be* **recognizable** *if, for each $\mathcal{G}$-graph G, every reconstruction of G is in $\mathcal{G}$. The class $\mathcal{G}$ is* **weakly reconstructible** *if, for each $\mathcal{G}$-graph G, every reconstruction of G that belongs to $\mathcal{G}$ is isomorphic to the graph G.*

It follows easily from these definitions that a class of graphs is reconstructible if and only if it is both recognizable and weakly reconstructible.

Definition. *Denote by $\alpha(G)$ the number of components of a graph G.*

Lemma 1. *If F is a reconstruction of G, then $\alpha(F) = \alpha(G)$, that is, the function α is reconstructible.*

Proof. Recall that the sequence of vertex degrees of every reconstruction F of G is the same. We consider two cases:

(i) The sequence has at least one vertex of degree zero. Let $v \in V(G)$ be a vertex of degree zero. In this case, the card $G - v$ of the deck of G has $e(G)$ edges. Because F has the same card in its deck and $e(F) = e(G)$, there is one vertex $v' \in F$ of degree zero such that $F - v' \cong G - v$. It follows that $G \cong F$ and therefore $\alpha(G) = \alpha(F)$.

(ii) Every component of a reconstruction F of G has at least two vertices. We claim that $\alpha(F) = \alpha(C)$, where C is the card of the deck with the least number of components. Since removing a vertex in F does not reduce the number of its components, we have $\alpha(C) \geq \alpha(F)$. Let v be a leaf in a spanning tree of a connected component in F. Clearly, $\alpha(C) \leq \alpha(F - v) = \alpha(F)$. Therefore, $\alpha(F) = \alpha(C)$. Because the right-hand side of this identity is reconstructible, the proof is complete. □

This last result tell us that disconnected graphs are recognizable. Next we prove that a disconnect graph G and a reconstruction F of G have the same components.

Definition. *Denote by $m(C, G)$ the number of components of a graph G isomorphic to a connected graph C.*

To prove that $m(C, G)$ is reconstructible, we could try to use Kelly's Lemma. However, since $s(C, G)$ counts subgraphs C of G that are not components of G, that is, subgraphs that are properly included in another component of G, we may not have $s(C, G) = m(C, G)$. We can overcome this if we subtract from $s(C, G)$ the subgraphs C of G that are not components of G.

Lemma 2. *If F is a reconstruction of a disconnected graph G, then $m(C, G) = m(C, F)$ for every connected graph C.*

Proof. We know that $v(F) = v(G) = n$ and that these two graphs are disconnected by Lema 1. Hence $m(K_{n-1}, G) = s(K_{n-1}, G) = s(K_{n-1}, F) = m(K_{n-1}, F)$, because K_{n-1} cannot be properly included in any other subgraph of G or F. We also have that if $e(C') > e(K_{n-1})$ for some connected graph C', $m(C', G) = m(C', F) = 0$.

(Reversed Induction Hypothesis) For every connected graph C such that $e(C) \geq k$, we have that $m(C, G) = m(C, F)$.

Now consider a connected graph C such that $e(C) = k - 1$. Then the following identity holds:

$$m(C, F) = s(C, F) - \sum_{X} s(C, X)m(X, F) \tag{2}$$

where the sum is taken over all isomorphism class X of connected graphs such that $e(X) \geq k$. This identity is valid because we obtain $m(C, F)$ when subtract from $s(C, F)$ the number of times that C occurs in F without being a component. That is, C occurs $s(C, X)$ times in a bigger component X that occurs $m(X, F)$ times in F.

As $e(X) > e(C)$, the induction hypothesis can be applied and we have that the right-hand side of identity (2) is the same for every reconstruction of G. Hence, $m(C, G)$ is reconstructible. □

We are now in condition to state the main result of this section.

Theorem 1. *Disconnected graphs can be reconstructed in a unique way.*

Proof. Immediately from lemmas 1 and 2. □

We believe that the exposition given here is clearer than the usual proofs in the literature. The same principle can be used to prove the well-known Greenwell-Hemminger's Lemma from the Reconstruction Theory (Bondy [1]).

3 Kocay's System of Equations

It sometimes happens in mathematics that a generalization of a difficult problem or a new way of stating it can help solving the original problem. Here we present a generalization, possibly harder than the Reconstruction Conjecture itself. It's interesting to work with this new problem because the concept of reconstruction is not necessary anymore.

In the problem of reconstructing a graph, we are interested in finding two distinct graphs with the same deck or show that such pair of graphs does not exist. The next result show that the existence of different sequences that satisfy some properties implies the Reconstruction Conjecture. As we will see, there is a lot of freedom when choosing these sequences, which give us some evidence that the conjecture may be true.

Definition. *Let $\psi(n)$ be the number of distinct connected graphs of n vertices and $\Psi(n)$ be the set of these graphs.*

The Modified Kocay's Lemma states that, given a graph G such that $v(G) = n$ and a sequence $\mathcal{F} := (F_1, F_2, ..., F_m)$ of graphs such that $v(F_i) < v(G)$, $1 \leq i \leq m$, there is an integer k such that

$$c(\mathcal{F}, X_1)s(X_1, G) + c(\mathcal{F}, X_2)s(X_2, G) + ... + c(\mathcal{F}, X_{\psi(n)})s(X_{\psi(n)}, G) = k \qquad (3)$$

is a valid identity for each reconstruction F of G, that is, we can replace $s(X_i, G)$ by $s(X_i, F)$ in identity (3). The sum is taken over all connected graphs of n vertices, that is, $X_i \in \Psi(n)$.

For each graph X_i, the function $c(\mathcal{F}, X_i)$ can be computed and is just a constant in equation (3). Since X_i has $v(G)$ vertices, Kocay's Lemma does not say anything about the value of $s(X_i, G)$ for different reconstructions of G. Therefore, the identity (3) can be viewed as a linear equation on $\psi(n)$ unknowns, in which each of the terms $s(X_i, G)$ are the variables. The coefficients of these variables are uniquely determined by the sequence $\mathcal{F}$ selected.

Selecting a connected graph G on n vertices and $\psi(n)$ valid sequences and applying the Modified Kocay's Lemma in each of these sequences, we get a system of linear equations that we call a Kocay's System of Equations of order n. If the sequences chosen are the same for each reconstruction F of G, it is the case that we always end up with the same linear system, since the right-hand side is the same by Modified Kocay's Lemma and also the coefficients of the variables on the left-hand side. For convenience and to explicit the independence of the system related to the reconstruction of G in question, the variable

$s(X_i, G)$ will be denoted by x_i. If F_i is the i-th sequence, then the constant $c(\mathcal{F}_i, X_j)$ will be denoted by c_{ij}. In this way, the related Kocay's System of Equations is associated to a set of graph sequences and a deck (the system is uniquely determined by the deck and the graph sequences). Thus we have the following system of equations:

$$\begin{array}{ccccccccc} c_{11}x_1 & + & c_{12}x_2 & + & ... & + & c_{1\psi(n)}x_{\psi(n)} & = & k_1 \\ c_{21}x_1 & + & c_{22}x_2 & + & ... & + & c_{2\psi(n)}x_{\psi(n)} & = & k_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ c_{\psi(n)1}x_1 & + & c_{\psi(n)2}x_2 & + & ... & + & c_{\psi(n)\psi(n)}x_{\psi(n)} & = & k_\psi(n) \end{array}$$

Now we can state an interesting result about this system.

Theorem 2. *If a Kocay's System of Equations of order n has an unique solution, then the graph G that originates the deck of the system is reconstructible.*

Proof. As we can think of the variable x_i as $s(X_i, G)$, we know that the system has at least one solution. If the system has unique solution, since it is the same for every reconstruction F of, we have $x_i = s(X_i, G) = s(X_i, F)$, that is, the solution is the same for both systems. By Lemma 1, $F \in \Psi(n)$, and clearly $G \in \Psi(n)$. Hence

$$s(G, F) = s(G, G) = 1 = s(F, F) = s(F, G)$$

From this we have $F \cong G$. □

Actually, finding such system for a particular deck can be as difficult as to prove the reconstructibility of the original graph. The next result give us a better reason to find such system.

Theorem 3. *If a Kocay's System of Equations of order n has an unique solution, then every connected graph of order n is reconstructible.*

Proof. Let $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_{\psi(n)}$ be the set of graph sequences that originates the system, $k_1, k_2, ..., k_{\psi(n)}$ be the constants of the system and G the graph that gives rise to the deck of the system. The constants k_i are uniquely determined by the deck and the graph sequences. The coefficients of the variables on the left-hand side of the system are uniquely determined by the sequences and does not depend on the deck of the system. Hence, the matrix of coefficients of the system is:

$$\mathbf{A} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1\psi(n)} \\ c_{21} & c_{22} & \cdots & c_{2\psi(n)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\psi(n)1} & c_{\psi(n)2} & \cdots & c_{\psi(n)\psi(n)} \end{pmatrix}$$

This system of equations has an unique solution if and only if the matrix A has determinant different from zero. Hence, if we use the same graph sequences for any other connected graph H on n vertices, the system obtained has the same matrix and its solution is also unique. From now on we can apply the same reasoning as in the proof of theorem 2 to show that the graph H is reconstructible. □

Corollary. *If a Kocay's System of Equations of order n has unique solution, then all graphs on n vertices are reconstructible.*

Proof. Immediately from theorems 1 and 3. □

So the reconstruction problem can be reduced to an algebraic and combinatorial problem: determine the right graph sequences and show that the related system has a unique solution. In the next section we present some systems of small order. We will be concerned just with the coefficients of the variables of the system, since this is enough to determine the uniqueness of its solution.

4 Applications of the Method

In this section we present some Kocay's System of Equations with unique solution. (determinant different from zero). A strategy for doing this will become apparent after the first examples.

First we deal with the simple case $n = 3$ (Figure 1). The element b_{ij} in the matrix is the value of $c(\mathcal{F}, G)$, where $\mathcal{F}$ is the sequence on the left of the line and G is the graph above the column.

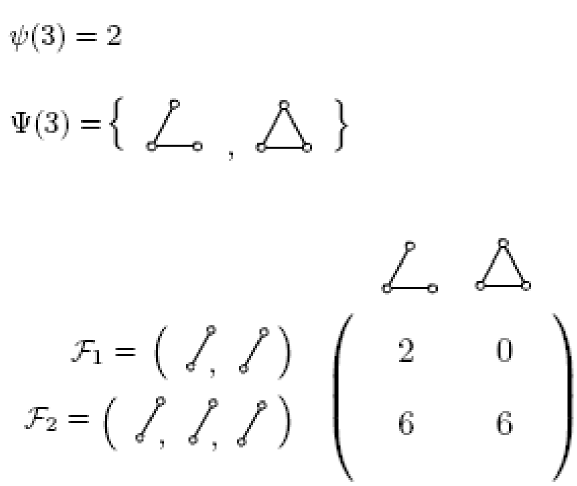

Figure 1: Kocay's System of Equation of order three with unique solution.

Working a little harder we can get the case $n = 4$ (Figure 2). Note that the values ? on this figure don't need to be computed, they are given by a well-known combinatorial function, though.

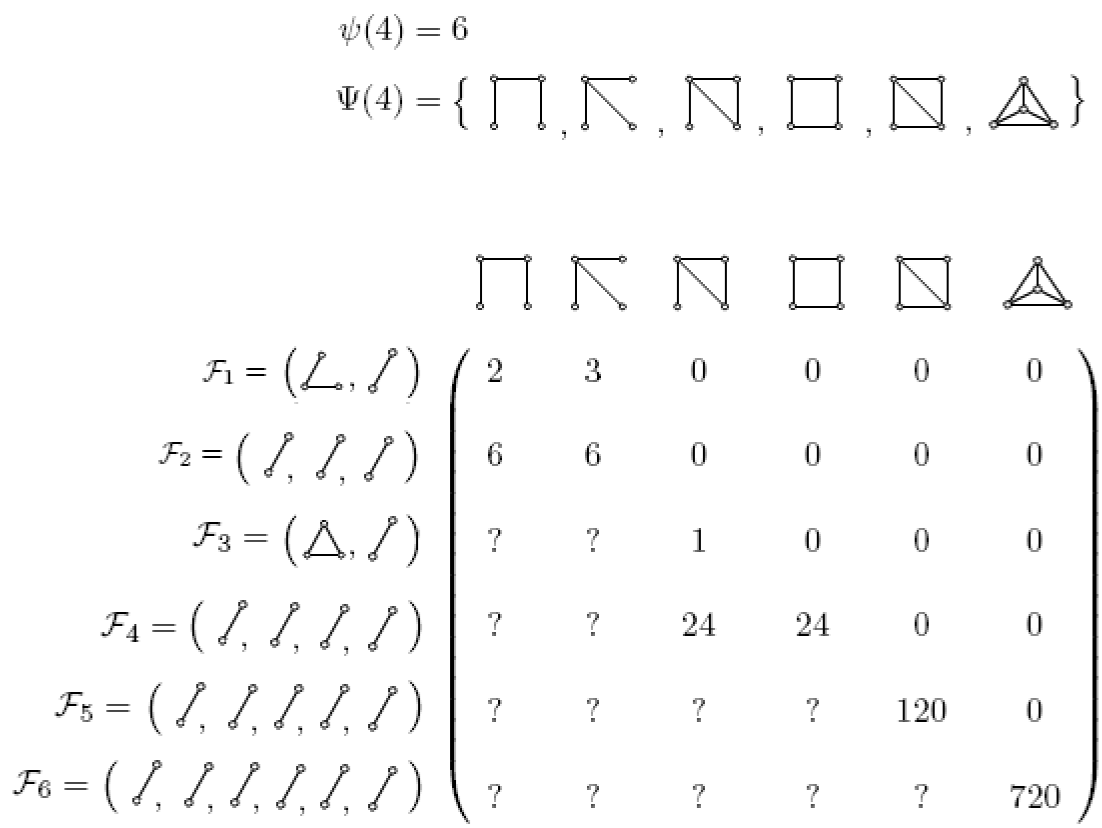

Figure 2: Kocay's System of Equation of order four with unique solution.

From the corollary of theorem 3 we can assert that all graphs on three and four vertices are reconstructible. The simple idea presented below simplifies the process of find these systems.

Definition. *Denote by $\psi(n, m)$ the number of distinct connected graphs on n vertices and m edges and by $\Psi(n, m)$ the set of these graphs.*

Definition. *If $\mathcal{F}$ is a graph sequence, then the* **size** *$e(\mathcal{F})$ of the sequence is the total sum of the number of edges of the graphs in the sequence.*

If G is a graph on m edges and $\mathcal{F}$ is a sequence such that $e(\mathcal{F}) > m$, then $c(\mathcal{F}, G) = 0$. Due to this fact, when constructing the system, for each class of connected graphs on n vertices and m edges, we choose correspondent sequences of size m. For example, to find a Kocay's system of order n with unique solution, first isolate for each m the set $\Psi(n, m)$. Afterwards we select $\psi(n, m)$ graph sequences of size m in such a way that the submatrix in the matrix of coefficients of the original system (that takes into account all connected graphs on n vertices) determined by these sequences and by the graphs in $\Psi(n, m)$ has a non null determinant. Doing this for each value of m, the variables of the system that correspond to the connected graphs of n vertices with the least number of edges are uniquely determined, later on the variables correspondent to the graphs with one more edge are determined, and so forth. The same idea was applied in the two examples given. Therefore, to create the desired system it is enough to focus on the graphs on n vertices and m edges and find the sequences of size m related.

5 Open Problems and Some Remarks

Let us formalize the problem presented in the previous section.

Conjecture. *For each $n \geq 3$, there exists a Kocay's System of Equation of order n that has a unique solution.*

As shown before, this result implies the Graph Reconstruction Conjecture. It would be interesting if the converse of this statement was also true, but this doesn't seem obvious to the authors.

The techniques illustrated in this paper can also be used with other versions of Kocay's Lemma (it is enough that the graphs in the sum of the lemma determine a recognizable class of graphs, see Tutte[2] for these modified versions). Another approach could be to modify the lemma and try to reduce the reconstruction problem to one of these modified versions.

The method presented can be explored by applying powerful results of algebraic graph theory as well. The new proposed conjecture seems to be more compatible with the techniques from this branch of graph theory. Since algebraic methods have obtained relative success when applied to reconstruction problems, this approach seems promising.

Last and most important, we have restricted ourselves to the vertex reconstruction conjecture, but the methods presented here are general enough to be used on the edge reconstruction conjecture as well. In this direction, other powerful results can be used and also another version of Kocay's Lemma, one that give us much more information about the covers of the graphs (see again Bondy[1]).

6 References

[1] J. A. Bondy, A Graph Reconstructor's Manual, Surveys in Combinatorics, LMS-Lecture Note Series 166, 1991.

[2] W. T. Tutte, Graph Theory, Cambridge, University Press, 2001.