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Gems: A general data structure for d-dimensional triangulations

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Gems: A general data structure for d-dimensional triangulations*

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Abstract

We describe in detail a novel data structure for d-dimensional triangulations. In an arbitrary d-dimension triangulation, there are $d!$ ways in which a specific facet of an simplex can be glued to a specific facet of another simplex. Therefore, in data structures for general d-dimensional triangulations, this information must be encoded using $\lceil \log_2(d!) \rceil$ bits for each adjacent pair of simplices. We study a special class of triangulations, called the *colored triangulations*, in which there is a only one way two simplices can share a specific facet. The *gem data structure*, described here, makes use of this fact to greatly simplify the repertoire of elementary topological operators.

1 Introduction

We describe a data structure capable of representing simplicial meshes, or triangulations, with any dimension ≥ 1.

There is a standard way to represent d-dimensional triangulations, which is to represent each d-dimensional simplex by one data record and encode the adjacency relations between these simplices. In an arbitrary d-dimension triangulation, there are $d!$ ways in which a specific facet of an simplex can be glued to a specific facet of another simplex. Therefore, in data structures for general d-dimensional triangulations, one must use $\lceil \log_2(d!) \rceil$ bits for each adjacent pair, in order to encode this information. This approach is used in Shewchuk's Triangle code [10] and in the CGAL 2D and 3D triangulation data structures [1].

In our work, we study a special class of triangulations, called the *colored triangulations*, in which there is a only one way two simplices can share a specific facet. In these triangulations, there is no need to keep additional bits for the adjacency relations. The *gem data structure*, described here, makes use of this fact to greatly simplify the repertoire of elementary topological operators.

2 Triangulations

We give an abstract definition of triangulation that generalizes most of the known triangulation data structures. The gem data structure can represent only a subclass of these triangulations.

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Simplex Let $x_0, \ldots x_d$ be the unit vectors of the coordinate axes of $\mathbb{R}^{d+1}$. The *canonical d-simplex* $\mathcal{A}_d \subset \mathbb{R}^{d+1}$ is the convex hull of the points $x_0, \ldots x_d$. The convex hull of any $C \subseteq \{x_0, \ldots x_d\}$, with $|C| = k + 1$, is a *k-face* of $\mathcal{A}_d$. A *d-simplex a* is a closed d-ball B with a specific homeomorphism f from $\mathcal{A}_d$ to B. The image of a k-face of $\mathcal{A}_d$ under f is a k-face of a.

Triangulation A *d-triangulation T* is a set of d-simplices with disjoint interiors such that, for distinct a and $b \in T$, $a \cap b$ is a union of faces of a and b. Any k-face of a d-simplex $a \in T$ is called a k-simplex of T. The set of all simplices (of any dimension) of T will be denoted by $\hat{T}$.

The union of all simplices of a triangulation T, with the obvious topology, is a compact Hausdorff topological space, called the *underlying space* of T and denoted $|T|$.

Star of a simplex Let a be a k-simplex of a d-triangulation T. The *star* of a, denoted $St(a)$, is the union of the interiors of all simplices of $\hat{T}$ containing a.

3 Gems

The gem representation was introduced as a mathematical device by S. Lins in 1982 [7]. It extends Brisson's *cell-tuple* representation [2] and Lienhardt's *generalized maps* [6] to triangulations that are not barycentric subdivisions, to manifolds with borders, and to non-manifold (but triangulable) topological spaces. The name *gem* is an acronym for *Graph Encoded Map*.

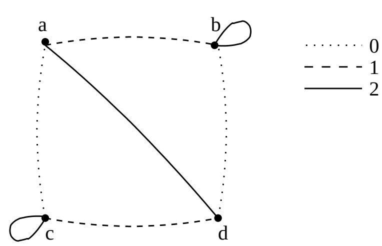
3.1 Definition

Let $C \subset \mathbb{N}$. A *C-gem* is a pair (V, ϕ) where V is a finite set of *gem nodes* and ϕ is a function that to each $i \in C$ associates an involution ϕ_i of V. If $|C| = d + 1$, a C-gem is also called a d-gem.

3.2 Gems as colored graphs

Node	ϕ_0	ϕ_1	ϕ_2
a	c	b	d
b	d	a	b
c	a	d	c
d	b	c	a

(a)



(b)

Figure 1: (a) A 2-gem with four nodes (b) the colored graph corresponding to this gem.

We can interpret a C-gem (V, ϕ) as a non-directed graph with C-colored edges, where V is the set of graph nodes and there is a i-colored edge between the nodes v and $w \in V$ if and only if $\phi_i(v) = w$. See figure 1. This interpretation provides implicitly the concepts of *walk*, *path*, *connectedness*, etc.

D-walk A D-*walk*, with $D \subseteq C$, is a walk whose edge colors belong to D (but may not use all colors of D).

3.3 Gems as triangulations

Every d-gem can be interpreted as a d-dimensional colored triangulation. We first define what is a colored simplex and a colored triangulation.

Colored simplex Let $C \subset \mathbb{N}$ with $|C| = d + 1$. A C-simplex S is a d-simplex whose vertices are uniquely labeled with *colors*, which are the elements of C. We say that S is a *colored d-simplex*.

Note that for each $D \subseteq C$ there is a D-simplex that is a face of S. We call it the D-face of S. In particular, the C-face of S is S itself, and the $\emptyset$-face of S is the empty set.

Colored triangulation A C-triangulation T, with $C \subset \mathbb{N}$ and $|C| = d + 1$, is a d-triangulation composed of C-simplices such that for every vertex v of T the d-simplices containing v agree about its color. We say that T is a *colored d-triangulation*.

Triangulation of a gem Let (V, ϕ) be a C-gem. A C-triangulation T is *represented* by (V, Φ) if these two conditions are satisfied:

(i) there is a bijection δ from V to T;

(ii) for any distinct v and $w \in V$, $\delta(v)$ and $\delta(w)$ share their D-face, for $D \subset C$, iff there is a $(C \setminus D)$-walk between v and w.

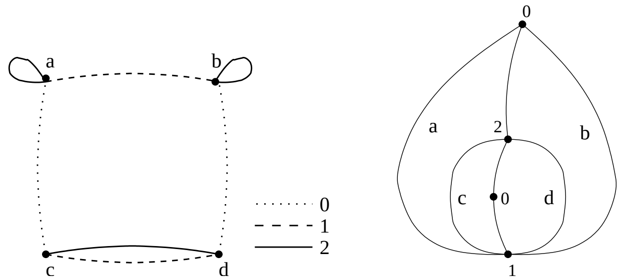
The triangulation T can be obtained by taking a set of disjoint C-simplices T' with a bijection δ from V to T' and, for every $i \in C$ and every pair of distinct a and $b \in T'$ with $\phi_i(a) = b$, identifying topologically the $(C \setminus \{i\})$-faces of $\delta(a)$ and $\delta(b)$, respecting the vertex colors. The purpose of condition (ii) is to require that two k-simplices, with $k < d - 1$, may only be identified implicitly, by the identification of $(d - 1)$-simplices.

There are certainly many distinct triangulations represented by a same gem, but they are all isomorphic, as expressed in the following theorem.

Theorem 1 *If T and T' are C-triangulations represented by the same C-gem, then there is a bijection $f : \hat{T} \to \hat{T}'$ mapping D-simplices of T into D-simplices of T', for $D \subseteq C$, such that, for any a and $b \in \hat{T}$, a is a face of b iff $f(a)$ is a face of $f(b)$.*

Hence, a gem fully represents the topology of its associated triangulations. The proof of theorem 1 is given in section 5.2.

Figure 2 shows a 2-gem and a triangulation represented by it.



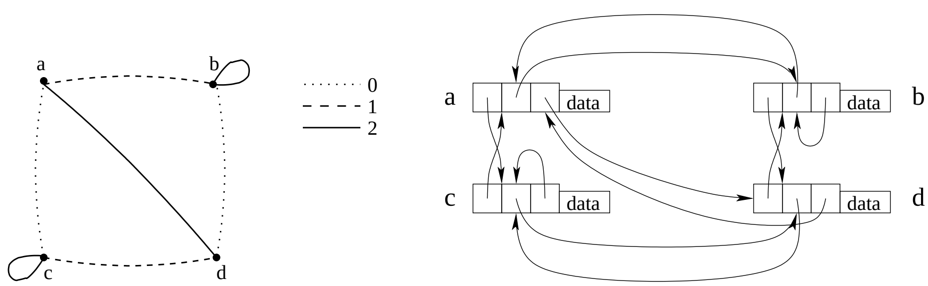
(a) (b)

Figure 2: (a) A 2-gem given as a graph (b) A colored 2-triangulation represented by this gem.

4 The gem data structure

In an arbitrary d-triangulation, there are $d!$ ways in which a specific facet of an d-simplex can be glued to a specific facet of another d-simplex. Therefore, in data structures for general d-triangulations, one must use $\lceil \log_2(d!) \rceil$ bits for each adjacent pair, in order to encode this information [3, 4]; or, alternatively, this information must be recomputed at each step when the structure is traversed [5]. In a colored triangulation, however, the constraints on vertex colors remove the need for that information, since two simplices can share a specific facet in only one way. The gem data structure makes use of this constraint to greatly simplify the repertoire of elementary topological operators.

The gem data structure represents a C-gem where $C \subseteq \{0, 1, \ldots, d\}$, for a given constant d. Its records correspond to the gem nodes. Each record r contains $d + 1$ pointers to other records, representing the functions $\phi_0, \ldots, \phi_d$ (see figure 3). We will write $\phi_i(r)$ to mean the record referenced by pointer i of record r. Each record r may also contain application-specific non-topological information $data(r)$.



(a) (b)

Figure 3: (a) A 2-gem given by a colored graph (b) the registers of the gem data structure.

The only constraint on this structure is that, if the pointer i of record r points to record r', then the pointer i of r' points to r.

4.1 Restrictions

The fact that gems can represent only colored triangulations is a major restriction. Take for instance an 1-triangulation consisting of a cycle of lines intercalated by vertices. This triangulation is colorable only if the number of vertices is even. In the case of 2-triangulations, if the link of any vertex is an odd cycle, it cannot be colored.

For this reason, gems cannot be used in problems that require *specific* triangulations, e.g. finding the Delaunay triangulation of a set of points.

4.2 Applications

Still, colored triangulations are suited for many applications. One of them is the representation of barycentric subdivision of general maps, studied by Brisson [2] and Lienhardt [6], and employed in our convex hull algorithm [9]. That algorithm uses the gem data structure to represent the barycentric subdivision of the convex hull of a set of points in $\mathbb{R}^d$. This approach makes possible the representation of hulls with non-simplicial faces.

Gems can also be used as adaptive triangulations, for example, to approximate surfaces by simplicial meshes with prescribed accuracy. Note that it doesn't matter if such a mesh is a colored triangulation, as long as it is a good approximation to the surface. Even though colored triangulations are not as easily subdivided as general ones, they do admit local k-simplex refinement schemes for any $k \in \{1, \dots, d\}$. When making a local refinement on a colored triangulation one must guarantee that the newly created vertices can be colored. In general, a local refinement on a colored triangulation requires more simplices than would be necessary on a general triangulation. More precisely, a local piecewise-linear refinement of a d-simplex that preserves its boundary requires subdivision into $2^{d+1} - 1$ simplices. In figure 4 (a), for example, a triangle is divided into three, but the vertices cannot be colored. Figure 4 (b) shows a colored refinement of the same triangle, composed of seven triangles.

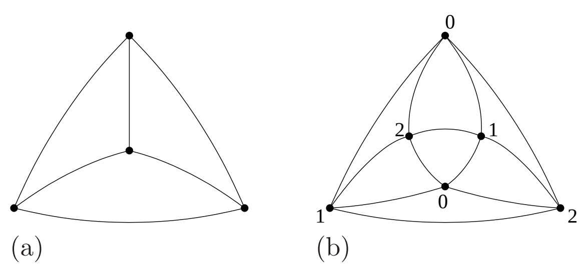


Figure 4: (a) The local refinement of a triangle on a general triangulation (b) the local refinement of a triangle on a colored triangulation.

5 Residues

In order to prove theorem 1, we need to define the concept of residue of a gem and understand its relation the incidence relations between simplices.

5.1 Definition

Let $G = (V, \phi)$ be a C-gem, and let $D \subseteq C$. A D-*residue* of G is a D-gem that is a connected component of $(V, \phi|_D)$, where $\phi|_D$ is the restriction of ϕ to the color set D. We denote by $\hat{G}$ the set of all residues of G.

5.2 Relation between residues and simplices

Let T be a C-triangulation represented by the C-gem $G = (V, \Phi)$.

We define now the function $\Delta : \hat{G} \to \hat{T}$. Let $R = (V', \phi')$ be a D-residue of G. The simplex $\Delta(R)$ is the $(C \setminus D)$-face of a C-simplex $\delta(v)$, for any $v \in V'$.

This definition is consistent, since any u and $v \in V_R$ are connected by a D-walk, thus $\delta(u)$ and $\delta(v)$ share their $(C \setminus D)$-face.

Lemma 1 *If T is a C-triangulation represented by the C-gem $G = (V, \phi)$, then the function $\Delta : \hat{G} \to \hat{T}$ defined above is a bijection.*

Proof: First we show that for any D-simplex $a \in \hat{T}$, $D \subseteq C$, there is a residue R of G such that $\Delta(R) = a$. Let b be a C-simplex of T incident on a and let $v \in V$ such that $\delta(v) = b$. Then, if R is the $(C \setminus D)$-residue of G containing v, $\Delta(R) = a$.

Now we prove that for distinct R and $S \in \hat{G}$, $\Delta(R) \neq \Delta(S)$. Let R be a D-residue and S a D'-residue of G, with D and $D' \subseteq C$ and $R \neq S$. If $D \neq D'$, $\Delta(R)$ and $\Delta(S)$ are clearly different, so let $D = D'$. If v and w are nodes of R and S respectively, then there is no walk with colors in D connecting them, and thus $\delta(v)$ and $\delta(w)$ do not share their $(C \setminus D)$-face. So $\Delta(R) \neq \Delta(S)$. $\square$

Lemma 2 *If R and S are residues of a C-gem G, then $\Delta(R)$ is a face of $\Delta(S)$ iff S is a residue of R.*

Proof: Let R be a D-residue of G and S be a E-residue of G, with D and $E \subseteq C$.

First, suppose that S is a residue of R, which implies $E \subseteq D$. If v is a node of S, then it is also a node of R. So $\Delta(S)$ is the $(C \setminus E)$-face of $\delta(v)$ and R is the $(C \setminus D)$-face of the same C-simplex $\delta(v)$. Since $(C \setminus E) \supseteq (C \setminus D)$, it is easy to see that $\Delta(R)$ is a face of $\Delta(S)$.

Now suppose that $\Delta(R)$ is a face of $\Delta(S)$, which implies $(C \setminus E) \supseteq (C \setminus D)$, i.e., $E \subseteq D$. Since every C-simplex incident on $\Delta(S)$ is also incident on $\Delta(R)$, every node of S is also a node of R. So, S is an E-residue of R. $\square$

Proof of theorem 1 Theorem 1 is a direct consequence of lemmas 1 and 2. $\square$

5.3 An interpretation of the residues

Every residue of a gem is also a gem, so it represents some triangulation. We now give an informal description of how to build the triangulation of a gem given the triangulations represented by its residues.

Suppose that G is a C-gem and we have already built the D-triangulations corresponding to each D-residue of G, for some $D \subset C$. We describe a way to obtain the triangulations of the $(D \cup \{i\})$-residues of G, for some $i \in C \setminus D$. Refer to figure 5.

First, for every D-triangulation T_R represented by the D-residue R of G (c), we create a new i-colored vertex p and extend T_R radially into p, creating a $(D \cup \{i\})$-triangulation (d). Then, for every pair of nodes v and w of G such that $\phi_i(v) = w$, we identify the D-simplices $\delta(v)$ and $\delta(w)$, respecting the vertex colors (e).

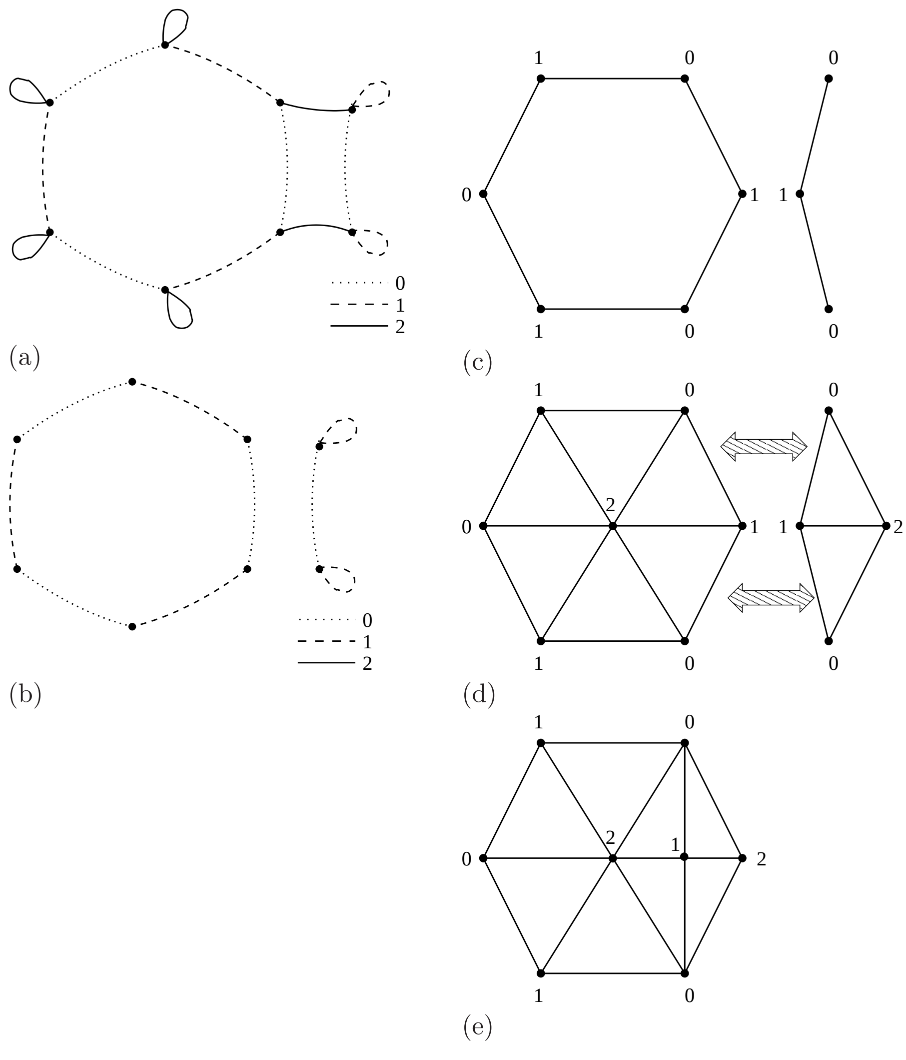


Figure 5: In (a) we have a $\{0, 1, 2\}$-gem G and in (b) the $\{0, 1\}$-residues of G. Figure (c) shows the $\{0, 1\}$-triangulations represented by the $\{0, 1\}$-residues of G. Figures (d) and (e) illustrate how one can obtain a triangulation represented by G: first, for each $\{0, 1\}$-residue R of G, a new 2-colored vertex p is created and the $\{0, 1\}$-triangulation $\Delta(R)$ is extended to p, forming a $\{0, 1, 2\}$-triangulation; then the $\{0, 1\}$-simplices of these triangulations are glued according to the 2-colored edges of G.

Proceeding like this, we can build a triangulation dimension by dimension.

6 Triangulations representable by gems

In this section we will determine the class of colored triangulations that are representable by gems. A triangulation belonging to this class will be called a *nice* triangulation.

Nice triangulations A *nice* triangulation is a colored d-triangulation T wherein every $(d-1)$-simplex is face of at most two d-simplices, and for every k-simplex a, with $0 \leq k \leq d-2$, $St(a) \setminus a$ is connected.

Theorem 2 *Every nice triangulation is represented by a gem.*

Theorem 3 *Every gem represents a nice triangulation.*

We need to define the concept of *k-star* to prove theorems 2 and 3.

The k-star of a simplex Let T be a d-triangulation and let a be a j-simplex of $\hat{T}$. The *k-star* of a, for $j \leq k \leq d$, is the union of the interiors of all simplices of $\hat{T}$ with dimension at least k and containing a. The k-star of a is denoted $St_k(a)$. Note that $St_k(a)$ is the union of the stars of all the k-simplices incident on a, and that, for $j < d$, $St_{j+1}(a) = St(a) \setminus a$.

k-star-sequence Let T be a d-triangulation and let a be a j-simplex of $\hat{T}$. A k-star-sequence of a, for $j \leq k \leq d$, is a finite sequence of k-simplices incident on a wherein the stars of consecutive k-simplices have nonempty intersection.

Lemma 3 *Let T be a d-triangulation and a be a j-simplex of $\hat{T}$. If, for $j \leq k \leq d$, a has a k-star-sequence connecting every pair of k-simplices incident on a, then a has a k-star-sequence containing all the k-simplices incident on a.*

Proof: If $l = (b_1, \ldots, b_n)$ is a k-star-sequence of a and c is a k-simplex incident on a and not contained in l, then we only have to obtain a k-star-sequence from b_n to c and concatenate it to l. By an induction on the number of k-simplices incident on a, we obtain the k-star-sequence containing all the k-simplices incident on a. □

Lemma 4 *Let T be a d-triangulation and a be a j-simplex of $\hat{T}$. The k-star of a, for $j \leq k \leq d$, is connected iff, for any pair of k-simplices b and c incident on a, there is k-star-sequence of a from b to c.*

Proof: Let a be a j-simplex and let b and c be a pair of k-simplices with no k-star-sequence connecting them, for $0 \leq j \leq k \leq d$. Let R be a binary relation on the set of k-simplices incident on a such that bRc iff $St(b) \cap St(c) \neq \emptyset$. The closure of R, denoted $R^\star$, is an equivalence relation. Since there is no k-star-sequence connecting b and c, then

R defines more than one equivalence class. Hence, $H = \bigcup_{bRe} St(e)$ and $St_k(a) \setminus H$ are two disjoint nonempty open sets partitioning $St_k(a)$, and thus $St_k(a)$ is disconnected.

Now, let a be a j-simplex such that for any pair of k-simplices b and c incident on a, for $k \geq j$, there is a k-star-sequence of a from b to c. Let then l be the k-star-sequence of a given by lemma 3. Since the star of any k-simplex in l is connected, the union of all these stars, which is $St_k(a)$, is connected. $\square$

Corollary 4 *Let T be a d-triangulation and a be a simplex of $\hat{T}$. The $(d-1)$-star of a is connected iff, for any pair of d-simplices b and c containing a, there is a sequence of d-simplices from b to c wherein every pair of consecutive d-simplices share a $(d-1)$-face incident on a.*

Proof: It follows from the fact that every such sequence of d-simplices can be obtained from a $(d-1)$-star-sequence of a, and vice-versa. $\square$

Lemma 5 *If T is a d-triangulation where every k-simplex with $0 \leq k < d-1$ has a connected $(k+1)$-star, then such simplices also have a connected $(d-1)$-star.*

Proof: Let a be a k-simplex of T with $0 \leq k < d-1$. We prove by an induction that a has a connected j-star for every j from $k+1$ to $d-1$. The base case is that a has a connected $(k+1)$-star. Then, we suppose that, for some j between $k+1$ and $d-2$ (limits included), $St_j(a)$ is connected. Now we must prove that $St_{j+1}(a)$ is connected.

Let $(b_1, \ldots, b_n)$ be a j-star-sequence containing all the j-simplices incident on a. Since, for any $1 \leq i \leq n$, $St_{j+1}(b_i)$ is connected, the set $\bigcup_{1 \leq i \leq n} b_i$, which is $St_{j+1}(a)$, is connected. $\square$

Proof of theorem 2 Let T be a nice C-triangulation. We will construct a C-gem $G = (V, \phi)$ that represents T. Let V be the set of C-simplices T, and, for all $i \in C$, let ϕ_i be an involution of V such that $\phi_i(a) = b$ for a and $b \in V$ iff (i) a and b are distinct C-simplices sharing their $(C \setminus \{i\})$-face or (ii) a and b are the same C-simplex whose $(C \setminus \{i\})$-face is contained in only one C-simplex. This definition is consistent, since every $(d-1)$-simplex of T is a face of one or two d-simplices.

Now let the bijection δ between V and T be the identity. We must show that for distinct v and $w \in V$, $\delta(v)$ and $\delta(w)$ share their D-face, for any $D \subset C$, iff there is a $(C \setminus D)$-walk between v and w in G.

Let c be a D-face of $\delta(v)$ and $\delta(w)$. If $|D| = d-1$, $St_{d-1}(c)$ is $St(c)$, which is connected; if $|D| < d-1$, lemma 5 states that St_{d-1} is connected. So, by lemma 4, there is a sequence of C-simplices containing c, from $\delta(v)$ to $\delta(w)$, wherein every pair of consecutive C-simplices share a $(d-1)$-face containing c. This sequence provides a $(C \setminus D)$-walk between v and w in G.

If there is a $(C \setminus D)$-walk between v and w in G, for every pair x and y of consecutive nodes in this walk, $\delta(x)$ and $\delta(y)$ share their D-face, so $\delta(v)$ and $\delta(w)$ also share their D-face. $\square$

Proof of theorem 3 Let G be a C-gem and let T be a triangulation represented by G.

First we show that the $(d-1)$-star of any k-simplex a of T, for $0 \leq k \leq d-2$, is connected, and consequently, $St(a) \setminus a$ is also connected. Let a be a D-simplex of T with $D \subset C$ and $|D| \leq |C|-2$, and let R be the D-residue of G corresponding to a. The residue R is connected, so there is a $(C \setminus D)$-walk between any pair of nodes in R. This walks provide the sequences of d-simplices required by corollary 4 to guarantee that $St_{d-1}(a)$ is connected.

Now we prove that every $(d-1)$-simplex is face of at most two d-simplices. Every $(C \setminus \{i\})$-simplex c of T, for any $i \in C$, is face of a C-simplex $\delta(v)$, for some node v of G. If $\phi_i(v) = v$ then c is face of only one C-simplex, else c is face of two C-simplices. $\square$

It is easy to see that every colored triangulation over a manifold, with or without border, is a nice triangulation. However, for $d \geq 3$, there are nice d-triangulations whose underlying spaces are not manifolds [8].

7 Basic operations

7.1 Manipulating the gems

Any gem data structure can be built using only two elementary operations.

`MakeNode():` creates a new record v and makes $\phi_i(v) = v$ for $0 \leq i \leq d$. This operation can be interpreted as adding to the triangulation a new d-simplex not adjacent to any other d-simplex.

`Swap(`v`,`w`,`i`):` where v and w are records and $i \in \{0, \ldots, d\}$. The precondition for `Swap` is that $\{\phi_i(v)\} \bigcup \{\phi_i(w)\} = \{v, w\}$. This operation exchanges the values of $\phi_i(v)$ and $\phi_i(w)$. Thus, if the nodes are not adjacent ($\phi_i(v) = v$ and $\phi_i(w) = w$), they become adjacent ($\phi_i(v) = w$ and $\phi_i(w) = v$), and vice-versa. Note that this operation is its own inverse, and it is a no-op if $v = w$.

For an example of use of these operations, see the convex hull algorithm in [9].

7.2 Traversing the gems

Let T be a C-triangulation represented by a gem G. Every C-simplex of T is referred as a pointer to the corresponding gem node. More generally, a D-simplex a of $\hat{T}$, for any $D \subseteq C$, is referred by a pair (v, D), where v is a node of the $(C \setminus D)$-residue corresponding to a.

Given this representation, we describe two operations to traverse the gem. The first one consists in, given a D-simplex a of T referred by (v, D), obtain the E-face of a for some $E \subseteq D$. The result of this operation is simply the pair (v, E).

The second operation consists in, given an E-simplex b of T referred by (w, E), enumerate the D-simplices of T, for some $D \supseteq E$, that have b as a face. This is the same as enumerating the $(C \setminus D)$-residues contained in the $(C \setminus E)$-residue corresponding to b, which can be done by a depth first search. If this operation is applied with $D = C$, the result is the set of C-simplices incident to the E-simplex b. Alternatively, if $E = \emptyset$, the result is the set of all D-simplices of the triangulation.

8 Conclusion

The gem data structure is a very simple and general way to represent the topology triangulations. This simplicity yields concise topological operators and elegant algorithms.

In spite of its restriction to colored triangulations, it is suitable for a range of applications, as the representation of barycentric subdivisions and the approximation of surfaces by affine triangulations.

References

[1] J.-D. Boissonnat, O. Devillers, S. Pion, M. Teillaud, and M. Yvinec. Triangulations in CGAL. *Computational Geometry*, 22(1–3):5–19, 2002.

[2] Erik Brisson. Representing geometric structures in d dimensions: Topology and order. *Proc. 5th Annual ACM Symp. on Computational Geometry*, pages 218–227, June 1989.

[3] David P. Dobkin and Michel J. Laszlo. Primitives for the manipulations of three-dimensional subdivisions. In *Proc. 3rd ACM Symp. on Comp. Geometry*, pages 86–99. ACM Press, June 1987.

[4] Leonidas J. Guibas and Jorge Stolfi. Primitives for the manipulations of general subdivisions and the computation of Voronoi diagrams. *ACM Transactions on Graphics*, 4(2):74–123, April 1985.

[5] D.-T. Lee and B. J. Schachter. Two algorithms for constructing a Delaunay triangulation. *Int. J. Computer Information Science*, 9:219–242, 1980.

[6] Pascal Lienhardt. Subdivisions of n-dimensional spaces and n-dimensional generalized maps. *Proc. 5th Annual ACM Symp. on Computational Geometry*, pages 228–236, June 1989.

[7] Sóstenes Lins. Graph-encoded maps. *Journal of Combinatory Theory (B)*, 32:171–181, 1982.

[8] Arnaldo Mandel and Sóstenes Lins. Graph-encoded 3-manifolds. *Discrete Mathematics*, 57:261–284, 1985.

[9] Arnaldo J. Montagner and Jorge Stolfi. General convex hull using the gem data structure. Technical Report IC-06-11, Institute of Computing, Univ. of Campinas, May 2006.

[10] Jonathan R. Shewchuk. Triangle: Engineering a 2D quality mesh generator and Delaunay triangulator. In *Proc. First Workshop on Applied Computational Geometry*, pages 124–133, Philadelphia, PA, USA, may 1996.