INSTITUTO DE COMPUTAÇÃO
UNIVERSIDADE ESTADUAL DE CAMPINAS

How to Build a Brace

Marcelo H. de Carvalho *Cláudio L. Lucchesi*
U. S. R. Murty

Technical Report - IC-05-014 - Relatório Técnico

June - 2005 - Junho

How to Build a Brace

Marcelo H. de Carvalho*
Dept. of Computing and Statistics,
UFMS, Campo Grande, Brasil and
University of Waterloo, Waterloo, Canada

Cláudio L. Lucchesi†
Institute of Computing,
UNICAMP, Campinas, Brasil

U. S. R. Murty‡
University of Waterloo, Waterloo, Canada

Abstract

We prove in this paper that every brace may be obtained, by means of edge additions and suitable vertex-splittings, from the two basic braces K_2 and C_4. McCuaig [5] showed that this may be achieved, staying within the realm of simple graphs, starting with three infinite families of braces. While our theorem may be deduced from McCuaig's, our proof is simpler. (We presented an analogous procedure for generating bricks in [1].)

1 The Main Theorem

1.1 Braces

A graph G is *matching covered* if it is connected, has at least two vertices and each edge lies in a perfect matching. Matching covered graphs on four or more vertices are sometimes called *1-extendable* [3].

Let G be a graph, X a set of vertices of G. We denote by $\partial_G(X)$ the *cut* of G of *shore* X, that is, the set of those edges of G that have one end in X, the other end in $\overline{X}$. If G is understood then we omit the subscript G and write simply $\partial(X)$. If G is connected then every cut has precisely two shores. A cut is *trivial* if it has a shore consisting of exactly one vertex.

Let G be a matching covered graph. A cut of G is *tight* if it contains precisely one edge of each perfect matching of G. If G is free of nontrivial tight cuts then it is a *brace* if it is bipartite, a *brick* otherwise. The braces on six or more vertices are precisely the bipartite *2-extendable* graphs [3]. Figure 1 depicts C_4, the basic brace that is used to generate all braces on four or more vertices.

*Supported by CNPq, Brasil, by PRONEX/FAPESP/CNPq (2003/09925-5), by FUNDECT-MS(0284/01) and a Fellowship from the University of Waterloo, Canada.

†Supported by CNPq (301268/1997-5). Supported by PRONEX/FAPESP/CNPq (2003/09925-5).

‡Work done during this author's visit to UNICAMP during March-April 2001 and February-March 2002, with the support of PRONEX/CNPq, Brasil.

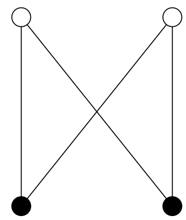

Figure 1: The basic brace C_4.

1.2 Vertex Splittings

For any nonnull set X of vertices of a graph G, we denote by $G\{X \to x\}$ the graph obtained from G by the contraction of X to a single new vertex x. Let G and H be two graphs, X a set of two or more vertices of G and let x be a vertex of H. We say that G is obtained from H by *splitting x into X* (denoted $G = H\{x \Rightarrow X\}$) if X is independent in G and $H = G\{X \to x\}$,

1.3 Brace Expansions

We now define the three operations on braces, called *brace expansions*, that can be used to generate all braces from the basic brace. In the description of each of the three brace expansions, we obtain a graph G from a brace H and we denote by $\{A, B\}$ the bipartition of H. Figure 2 illustrates the three operations.

Expansion of zero vertices: Let x and y be vertices of H, $x \in A$, $y \in B$. Obtain G from H by adding a new edge labelled e, joining x and y.

Expansion of one vertex: Let x and y be distinct vertices of A such that the degree of x is four or more. Let $H' := H\{x \Rightarrow \{x_1, x_2\}\}$ such that, in the underlying simple graph of H', both x_1 and x_2 have degree at least two. Now obtain G from H' by adding a new vertex x_0 and joining it to x_1, x_2 and y, where the edge x_0y is labelled e.

Expansion of two vertices: Let x and y be vertices of H, $x \in A$, $y \in B$, whose degrees are four or more. Let

$$H'' := H\{x \Rightarrow \{x_1, x_2\}\}\{y \Rightarrow \{y_1, y_2\}\}$$

such that, in the underlying simple graph of H'', each of x_1, x_2, y_1 and y_2 has degree two or more and has at least one neighbour in $V(G) - \{x_1, x_2, y_1, y_2\}$. Now obtain G from H'' by adding two new vertices x_0 and y_0, joining x_0 to x_1 and x_2, joining y_0 to y_1 and y_2, and joining x_0 and y_0 by an edge labelled e.

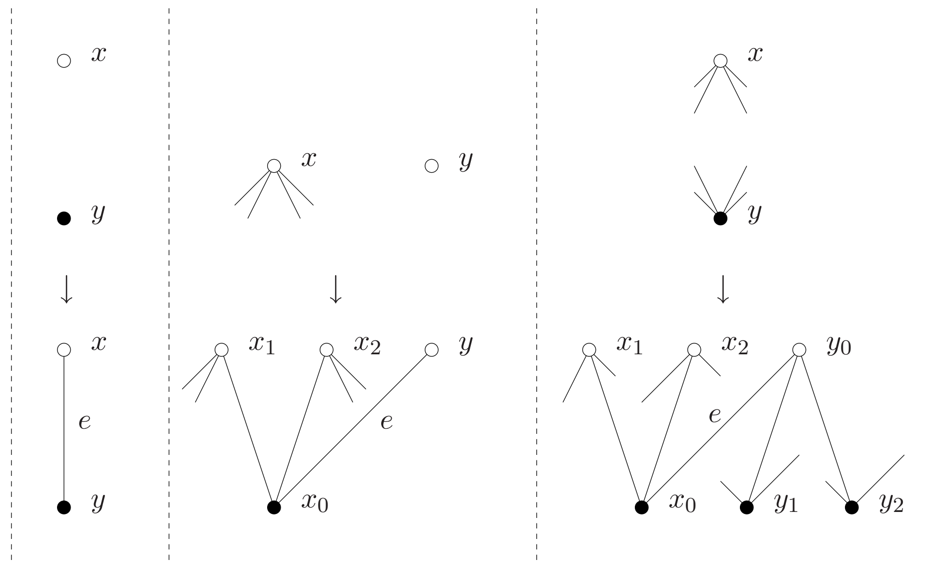

Figure 2: The three brace expansions: expansions of zero, one or two vertices.

1.4 The Main Theorem

We are now in position to state our main theorem.

Theorem 1.1 (Main Theorem)
A graph G on four vertices or more is a brace if and only if it may be obtained from C_4 by repeated applications of the three operations of brace expansion.

1.4.1 Examples

Figure 3 describes the generation of $K_{3,3}$ and Figure 4 depicts the generation of B_{10}, the biwheel on 10 vertices.

1.4.2 Necessity of the Three Operations

The three operations are necessary for building up all braces. The basic brace is 2-regular. Vertex expansions always create one or two vertices of degree three. Thus, braces whose minimum degree is four or more, such as $K_{4,4}$, need an edge addition as the last operation, not to mention braces having multiple edges. Cubic braces on 8 or more vertices, such as the cube (see Figure 4), need the expansion of two vertices as the last operation, since the other two operations would apply to a brace on six or more vertices and thus necessarily introduce a vertex of degree four or more.

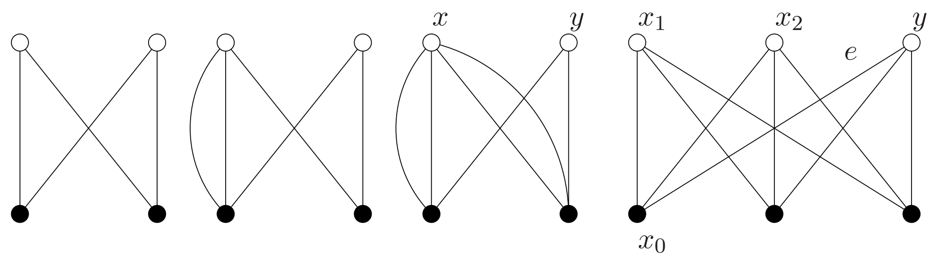

Figure 3: The generation of $K_{3,3}$.

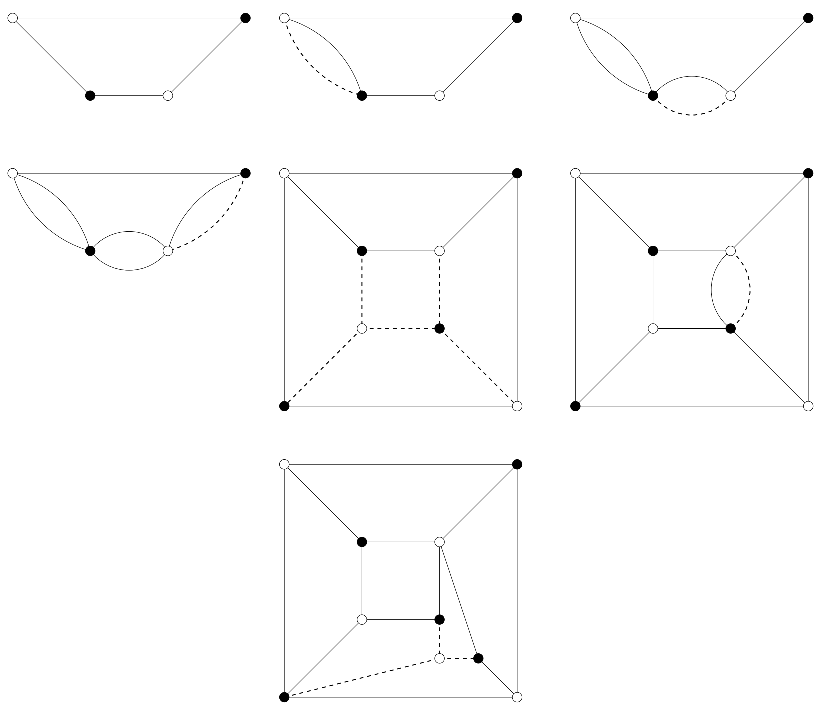

Figure 4: The generation of B_{10}, the biwheel on 10 vertices, where dashed lines indicate the edges added to the previous brace.

To see the necessity of the expansion of one vertex, it suffices to note that the operation of expanding zero vertices keeps the number of vertices of the graph and the operation of expanding two vertices increases the number of vertices by four. Thus, the generation of $K_{3,3}$ necessarily uses the expansion of one vertex. (It can be shown that biwheels also need that operation.)

2 Matching Covered Graphs

In this section we present some fundamental properties of matching covered graphs and bipartite matching covered graphs that are necessary for the proof of the Main Theorem. The first result is well-known and easily proved.

LEMMA 2.1
Every matching covered graph is 2-connected. □

2.1 Tight Cuts

Let G be a matching covered graph. Two cuts $C := \partial(X)$ and $D := \partial(Y)$ of G *cross* if each of the four sets $X \cap Y$, $X \cap \overline{Y}$, $\overline{X} \cap Y$ and $\overline{X} \cap \overline{Y}$ is nonnull. A collection $\mathcal{C}$ of cuts of G is *laminar* if no two of its cuts cross. For any tight cut $C := \partial(X)$ of G, the graphs $G\{X \to x\}$ and $G\{\overline{X} \to \overline{x}\}$ are the *C-contractions* of G. It is easy to prove that both C-contractions of G are matching covered. The following result is a fundamental property of cuts.

LEMMA 2.2 (SUBMODULARITY)
Let G be a matching covered graph, $C := \partial(X)$ and $D := \partial(Y)$ two cuts of G. Let λ denote the set of edges that joins a vertex of $X \cap \overline{Y}$ to a vertex of $\overline{X} \cap Y$. Let $I := \partial(X \cap Y)$, $U := \partial(\overline{X} \cap \overline{Y})$. For any set M of edges of G, the following equality holds: $|M \cap C| + |M \cap D| = |M \cap I| + |M \cap U| + 2|M \cap \lambda|$. □

COROLLARY 2.3 (MODULARITY)
Let G be a matching covered graph, $C := \partial(X)$ and $D := \partial(Y)$ two tight cuts of G. If $|X \cap Y|$ is odd then each of $\partial(X \cap Y)$ and $\partial(\overline{X} \cap \overline{Y})$ is tight and no edge of G joins a vertex of $X \cap \overline{Y}$ to a vertex of $\overline{X} \cap Y$. □

2.1.1 Tight Cut Decompositions

Let G be a matching covered graph. We may apply to G a *tight cut decomposition procedure*, which produces a list of bricks and braces, called a *tight cut decomposition* of G. If G itself is a brick or a brace then the list is the singleton containing graph G. Otherwise, let C be any nontrivial tight cut of G. Then, both C-contractions of G are matching covered. Recursively apply the tight cut decomposition procedure to each C-contraction of G. The resulting lists are then combined to produce the tight cut decomposition of G. We remark that, associated with a tight cut decomposition of G there is a maximal laminar collection $\mathcal{C}$ of nontrivial tight cuts of G. Based on the modularity property, Lovász proved the following remarkable result on tight cut decompositions [2].

THEOREM 2.4
Any two applications of the tight cut decomposition procedure to a matching covered graph produce the same list of bricks and braces, up to multiple edges. □

2.1.2 Tight Cuts in Bipartite Graphs

Let G be a matching covered graph, $\{A, B\}$ its bipartition. Let X be a set of vertices of G such that $|X|$ is odd. Then $|X \cap A|$ and $|X \cap B|$ are distinct: whichever of $X \cap A$ and $X \cap B$ has more vertices than the other is the *majority part* of X and is denoted X_+, the other part of X is its *minority part* and is denoted X_-. The following property is easily proved, by means of a counting argument.

LEMMA 2.5
Let G be a bipartite matching covered graph, $C := \partial(X)$ a cut of G such that $|X|$ is odd. Then, C is tight if and only if (i) $|X_+| = |X_-| + 1$ and (ii) no edge of C is incident with a vertex of X_-. □

COROLLARY 2.6
Every tight cut decomposition of a bipartite matching covered graph consists solely of braces. □

2.2 Bipartite Matching Covered Graphs

The following lemma provides a characterization of bipartite matching covered graphs. It follows immediately from Theorem 4.1.1 in Lovász and Plummer's book [3].

LEMMA 2.7
Let G be a graph with bipartition $\{A, B\}$ and at least four vertices. Assume that G has a perfect matching. Then, the following assertions are equivalent:

(i) *Graph G is matching covered.*

(ii) *For every partition $\{A', A''\}$ of A and every partition $\{B', B''\}$ of B such that $|A'| = |B'|$, graph G has at least one edge that joins A' to B''.*

(iii) *For every vertex a of A and every vertex b of B, graph $G - a - b$ has a perfect matching.* □

For any set X of vertices of a graph G, we denote by $N_G(X)$ the set of vertices of G that are adjacent to some vertex of X. If G is understood we simply write $N(X)$ instead of $N_G(X)$. The next result characterizes braces.

LEMMA 2.8
Let G be a matching covered graph and bipartition $\{A, B\}$. Then, G is a brace if and only if $G - a_1 - a_2 - b_1 - b_2$ has a perfect matching, for any two vertices a_1 and a_2 in A and any two vertices b_1 and b_2 in B.

<u>Proof</u>: By hypothesis, G is matching covered, whence $|A| = |B|$. Assume that G is a brace but $H := G - a_1 - a_2 - b_1 - b_2$ has no perfect matching. By Hall's Theorem, $A - a_1 - a_2$ has a subset X such that $|N_H(X)| \leq |X| - 1$. Then, in G, $|N(X)| \leq |X| + 1$, with equality only if $\{a_1, a_2\} \subseteq N(X)$. As G is a brace, it is matching covered. By Halls's Theorem, $|N(X)| \geq |X|$. By Lemma 2.7, $|N(X)| = |X| + 1$. Moreover, $N(X)$ includes $\{a_1, a_2\}$. By Lemma 2.5, cut $\partial(X \cup N(X))$ is tight in G. As $N(X)$ includes $\{a_1, a_2\}$ and $A - X$ includes $\{b_1, b_2\}$, cut C is nontrivial. This is a contradiction. We deduce that $G - a_1 - a_2 - b_1 - b_2$ has a perfect matching.

Conversely, assume that $G - a_1 - a_2 - b_1 - b_2$ has a perfect matching, for any two vertices a_1 and a_2 of A and any two vertices b_1 and b_2 of B. Assume, to the contrary, that G is not a brace. As G is matching covered, it follows that G has a nontrivial tight cut, $C := \partial(X)$. Then, $|X_+| = |X_-| + 1$ and $|\overline{X}_+| = |\overline{X}_-| + 1$, by Lemma 2.5. As C is nontrivial, each of X_+ and $\overline{X}_+$ contains two or more vertices. Adjust notation so that $X_+ \subset A$. Then, $\overline{X}_+ \subset B$. Let a_1 and a_2 be any two vertices of X_+, b_1 and b_2 any two vertices of $\overline{X}_+$. Then, $H := G - a_1 - a_2 - b_1 - b_2$ has no perfect matching, because $|N_H(X_-)| = |X_-| - 1$. This is a contradiction. We deduce that G is a brace, as asserted. □

Lemma 2.9

Let G be a brace on six vertices or more. Then, G is 3-connected. Moreover, for any set X of three vertices of G that meets both parts of the bipartition of G, the graph $G - X$ is connected.

<u>Proof</u>: Let $\{A, B\}$ denote the bipartition of G. Let X be a set of three vertices of G that meets both A and B. Adjust notation so that $X \cap A$ is a doubleton and $X \cap B$ is a singleton. Let v be any vertex of $B - X$. By Lemma 2.8, $G - X - v$ has a perfect matching. Thus, every connected component of $G - X - v$ has an even number of vertices. If v is not adjacent to any vertex of some connected component K of $G - X - v$, then let v' be any vertex of $B \cap V(K)$: the graph $G - X - v'$ does not have a perfect matching, a contradiction. Thus, v is adjacent to some vertex of every connected component of $G - X - v$. Consequently, $G - X$ is connected, as asserted.

To prove that G is 3-connected, let Y be a set of vertices of G with two or fewer vertices, let us prove that $G - Y$ is connected. As G, a brace, is matching covered, then G is 2-connected, whence either $G - Y$ is connected or Y is a doubleton. We may thus assume that Y is a doubleton. If Y meets both A and B, then for every vertex a in $A - Y$ and every vertex b in B, graph $G - Y - a - b$ has a perfect matching, by Lemma 2.8. By Lemma 2.7, $G - Y$ is matching covered, whence connected. We may thus assume that Y is a subset of one of A and B. Adjust notation so that Y is a subset of A. As G has six or more vertices, $A - Y$ is nonnull, let v be a vertex of $A - Y$. Let w be a vertex of G that is adjacent to v. Then, v and w lie in the same connected component of $G - Y$. By the previous analysis, $G - Y - w$ is connected. But $G - Y - w$ has at least as many connected components as $G - Y$. We deduce that $G - Y$ is connected. As asserted, G is 3-connected. □

3 Proof of the Main Theorem

The proof of the Main Theorem is divided in two parts: in the first part, *containment*, we prove that, starting with C_4 and using only the three operations of brace expansion, only braces are generated. In the second part, *completeness*, we prove that every brace on four or more vertices may be generated from C_4 by repeated applications of the three operations.

3.1 Containment

THEOREM 3.1 (CONTAINMENT)
Let $G_0 = C_4, G_1, \ldots, G_s$, $s \geq 0$, be a sequence of graphs such that $G_0 = C_4$ and, for $i = 1, 2, \ldots, s$, G_i is obtained from G_{i-1} by the application of one of the three operations of brace expansion. Then, G_s is a brace.

Proof: By induction on s. If $s = 0$ then, as C_4 is a brace, the assertion holds immediately. We may thus assume that $s > 0$. Let $G := G_s$, $H := G_{s-1}$. By the induction hypothesis, H is a brace. By definition, G is obtained from H by the application of one of the three operations of brace expansion. Let $\{A, B\}$ denote the bipartition of H. We now analyze separately three cases, depending on the number r of vertex expansions used by the operation that produces G from H.

CASE 1 $r = 0$.

In that case, $G - e = H$, where e joins two vertices x and y of H, where $x \in A$ and $y \in B$. Then, $\{A, B\}$ is a bipartition of G. Moreover, by Lemma 2.7, G is matching covered. To prove that G is a brace, let C be a tight cut of G. Each perfect matching of $G - e$ is a perfect matching of G. Thus, $C - e$ is a tight cut of $G - e$. As $G - e = H$, it follows that $C - e$ is trivial in $G - e$, whence C is trivial in G. This conclusion holds for each tight cut C of G. We deduce that G is a brace, in this case.

CASE 2 $r = 1$.

Adopt the definition of and notation used in the expansion of one vertex, as described in Section 1.3. Then, $H = (G - e)\{X \to x\}$, where $X := \{x_0, x_1, x_2\}$. The ends of e are vertices x_0 and y, and y lies in A, the part of the bipartition of H that contains vertex x. Thus, $\{A', B'\}$ is the bipartition of G, where $A' := (A - x) \cup \{x_1, x_2\}$ and $B' := B \cup \{x_0\}$. Let $C := \partial(X)$. Then, one of the $(C - e)$-contractions of $G - e$ is C_4, up to multiple edges, the other $(C - e)$-contraction is brace H. Thus, $G - e$ is matching covered. Moreover, as G is bipartite, it follows that G is also matching covered, by Lemma 2.7.

We now prove that G is a brace. Consider first the case in which H has precisely four vertices. In that case, C_4 is the underlying simple graph of H and G has precisely six vertices. By definition, each of x_1 and x_2 is adjacent to at least two vertices of $G - x_0$. Thus, each of x_1 and x_2 is adjacent to the three vertices of $B \cup \{x_0\}$. Vertex y is adjacent in H to the two vertices of B. By definition, it is joined to x_0 by edge e, in G. We deduce that each of the three vertices of $(A - x) \cup \{x_1, x_2\}$ is adjacent to the three vertices of $B \cup \{x_0\}$, whence G is $K_{3,3}$, up to muliple edges. Thus, G is a brace in this case.

We may thus assume that H has six or more vertices. Let $\mathcal{B}$ denote the family of braces obtained in any tight cut decomposition of G. Assume, to the contrary, that $\mathcal{B}$ contains two or more braces. We first show that every brace in $\mathcal{B}$ contains six or more vertices. As H is a brace on six or more vertices, each vertex of H is adjacent to three or more vertices. Thus, in G, each vertex of $V(G) - X$ is adjacent to three or more vertices. Vertices x_1 and x_2 are adjacent in $G - x_0$ to two or more vertices, whence they are adjacent to three or more vertices in G. Vertex x_0 is adjacent to x_1, x_2 and y. We deduce that each vertex of G is adjacent to three or more vertices. Consequently, each brace of $\mathcal{B}$ contains six or more vertices.

For each tight cut D of G, cut $D - e$ is tight in $G - e$. Thus, $\mathcal{B} - e := \{B - e : B \in \mathcal{B}\}$ is a (possibly partial) tight cut decomposition of $G - e$. On the other hand, cut $C - e$ is nontrivial and tight in $G - e$. One of the $(C - e)$-contractions of $G - e$ is H, the other is C_4, up to multiple edges. By Theorem 2.4, every tight cut decomposition of $G - e$ produces two braces, C_4 and H, up to multiple edges. Thus, one of the braces of $\mathcal{B}$ contains only four vertices, a contradiction. We deduce that G is a brace, in this case.

Case 3 $r = 2$.

Adopt the definition of and notation used in the expansion of two vertices, as described in Section 1.3. Then, $H = (G - e)\{X \to x\}\{Y \to y\}$, where $X := \{x_0, x_1, x_2\}$ and $Y := \{y_0, y_1, y_2\}$. The ends of e are vertices x_0 and y_0. Thus, $\{A'', B''\}$ is the bipartition of G, where $A'' := (A - x) \cup \{y_0, x_1, x_2\}$ and $B'' := B \cup \{x_0, y_1, y_2\}$. Let $C := \partial(X)$, $D := \partial(Y)$. Then, one of the $(C - e)$-contractions of $G - e$ is C_4, up to multiple edges. The other $(C - e)$-contraction has tight cut $D - e$, and one of its $(D - e)$-contractions is C_4, up to multiple edges, the other is brace H. Thus, $G - e$ is matching covered. As G is bipartite, it follows that G is also matching covered, by Lemma 2.7. Moreover, every tight cut decomposition of $G - e$ produces two copies of C_4 plus brace H.

We now prove that G is a brace. Let $\mathcal{B}$ denote the family of braces obtained in any tight cut decomposition of G. Assume, to the contrary, that $\mathcal{B}$ contains two or more braces. We first show that every brace in $\mathcal{B}$ contains six or more vertices. We do this by proving that every vertex of G is adjacent to three or more vertices. For this, let v be a vertex of G. Vertex x_0 is adjacent to y_0, x_1, and x_2. Likewise, vertex y_0 is adjacent to x_0, y_1 and y_2. By definition of the operation, x_1 is adjacent to at least two vertices in $G - x_0$ and is also adjacent to x_0. Thus, x_1 is adjacent to three or more vertices in G. Likewise, each of x_2, y_1 and y_2 is adjacent to three or more vertices in G. Assume thus that v does not lie in $X \cup Y$. Then, v is a vertex of H. If H has six vertices or more then, in H, v is adjacent to three or more vertices, whence in G it is also adjacent to three or more vertices. We may thus assume that H has precisely four vertices. Adjust notation so that v lies in B. Then, v is the vertex of $B - y$. Vertex x_1 must be adjacent in $G - y_0$ to at least one vertex not in $\{y_1, y_2\}$, by definition of the operation. Thus, x_1 is adjacent to v. Likewise, x_2 is also adjacent to v. In addition, the vertex of $A - x$ is adjacent in H to v. We deduce that v is adjacent to three or more vertices in G. This conclusion holds for each vertex v of G. Thus, every brace in $\mathcal{B}$ contains six or more vertices.

For each tight cut of G, its restriction to $G-e$ is tight in $G-e$. Thus, $\mathcal{B}-e := \{B-e : B \in \mathcal{B}\}$ is a (possibly partial) tight cut decomposition of $G-e$. On the other hand, we have seen that $G-e$ has a tight cut decomposition consisting of two C_4's and H, up to multiple edges. By Theorem 2.4, every tight cut decomposition of $G-e$ produces those three braces, up to multiple edges. As every brace of $\mathcal{B}$ contains six or more vertices, it follows that $\mathcal{B}$ consists of precisely two braces, H and L. Moreover, $L-e$ has a tight cut decomposition in two C_4's, whence it consists of precisely six vertices. Graph $K_{3,3}$ is the only simple brace on six vertices. Thus, L is $K_{3,3}$, up to multiple edges. As $\mathcal{B}$ consists of precisely two braces, each of the two braces has precisely one contraction vertex. As e is an edge of L, at least one of x_0 and y_0 is a vertex of L.

If no end of e in L is a contraction vertex, adjust notation, by interchanging A with B, x with y and x_i with y_i, for $i = 0, 1, 2$, so that the contraction vertex and x_0 lie in distinct parts of the bipartition of L. If one of the ends of e in L is a contraction vertex, adjust notation, by interchanging A with B, x with y and x_i with y_i, for $i = 0, 1, 2$, so that x_0 is a vertex of L. In both alternatives, x_0 is a vertex of L. Moreover, x_0 and the contraction vertex lie in distinct parts of the bipartition of L. At least one of x_1 and x_2 is a vertex of L, because both are adjacent to x_0 in G. Adjust notation so that x_1 is a vertex of L. The contraction vertex and x_1 lie in the same part of the bipartition of L. Thus, y_1 and y_2 are both vertices of L. We deduce that x_1 is adjacent in G only to vertices in a subset of $\{x_0, y_1, y_2\}$. This contradicts the definition of the operation, which requires vertex x_1 to be adjacent in $G - x_0$ to a vertex not in $\{y_1, y_2\}$. Thus, $\mathcal{B}$ consists of only one brace, the graph G.

We have analyzed three cases, each case corresponding to one of the three operations of brace expansion. In each case, we deduced that G is a brace. The proof of the containment part of the Main Theorem is complete. □

3.2 Bipartite Graphs Derived from Braces

Assume that G is a bipartite and matching covered graph. By Corollary 2.6, the tight cut decomposition of G produces only braces. By Theorem 2.4, all tight cut decompositions of G produce the same family of braces, up to multiple edges. We denote by $b(G)$ the number of braces in any tight cut decomposition of G.

In order to prove the completeness part of the Main Theorem, we must study properties of matching covered graphs of the form $G-e$, where G is a brace and e is an edge of G. An edge e of a matching covered graph G is *removable* if the graph $G-e$ is also matching covered.

3.2.1 Tight Cuts in Bipartite Graphs Derived from Braces

LEMMA 3.2
Let G be a brace, e a removable edge of G, $C := \partial(X)$ a cut of G such that $C-e$ is tight in $G-e$. Then, $C-e$ is a nontrivial tight cut of $G-e$ if and only if e is the only edge of C that has one end in X_-.

<u>Proof</u>: By hypothesis, $C - e$ is tight in $G - e$. By Lemma 2.5, $|X_+| = |X_-| + 1$. As G has perfect matchings, we deduce that $\{X_+ \cup \overline{X}_-, X_- \cup \overline{X}_+\}$ is the bipartition of G. If edge e does not lie in C, or if it has one end in X_+, then C is tight in G. In that case, as G is a brace, C is trivial. Assume thus that edge e lies in C and has one end in X_-. Then, the end of e in $\overline{X}$ lies in $\overline{X}_-$. We deduce that both X and $\overline{X}$ have nonnull minority parts, whence C is nontrivial. □

Lemma 3.3

Let G be a brace, e a removable edge of G, v an end of e, $\mathcal{C}$ a laminar collection of nontrivial cuts of G such that, for each cut $C \in \mathcal{C}$, the cut $C - e$ is tight in $G - e$. Then, set inclusion is a total order in the collection of the shores of the cuts in $\mathcal{C}$ that contain vertex v.

<u>Proof</u>: Let w denote the end of e distinct from v. Let $C := \partial(X)$ be a cut in $\mathcal{C}$. Adjust notation, by interchanging X with $\overline{X}$ if necessary, so that v lies in X. By definition, $C - e$ is a nontrivial tight cut of $G - e$. As G is a brace, e is the only edge of C that is incident with a vertex in X_-. Thus, v lies in X_-. Likewise, w lies in $\overline{X}_-$. These conclusions hold for each cut C in $\mathcal{C}$.

Let $D := \partial(Y)$ denote a cut in $\mathcal{C} - C$. Adjust notation so that v lies in Y. Then, v lies in $X \cap Y$ and w lies in $\overline{X} \cap \overline{Y}$. As $\mathcal{C}$ is laminar, cuts C and D do not cross. Thus, one of $X \cap \overline{Y}$ and $Y \cap \overline{X}$ is empty. We deduce that either $X \subset Y$ or $Y \subset X$, where the inclusion is proper, because C and D are distinct. □

Lemma 3.4 (The Three Case Lemma)

Let G be a brace, e a removable edge of G. Let H be a brace obtained by a tight cut decomposition of $G - e$. Then, H has at most two contraction vertices.

<u>Proof</u>: Let $\{x_1, x_2, \cdots, x_s\}$ denote the set of contraction vertices of H. Then, there exists a disjoint collection $\mathcal{X} := \{X_1, X_2, \cdots, X_s\}$ of shores of nontrivial tight cuts of $G - e$ such that

$$H = (G - e)\{X_1 \to x_1\}\{X_2 \to x_2\} \cdots \{X_s \to x_s\}.$$

Each X_i in $\mathcal{X}$ contains precisely one of the ends of edge e. Moreover, $\mathcal{X}$ is disjoint. Therefore, $\mathcal{X}$ contains at most two elements. We conclude that H contains at most two contraction vertices. □

A simple counting argument establishes the following result as an immediate consequence of Lemma 3.2.

Corollary 3.5

Let G be a brace, e a removable edge of G, C a cut of G. Then, cut $C - e$ is a nontrivial tight cut of $G - e$ if and only $|M \cap C| = 1 + 2|M \cap \{e\}|$, for each perfect matching M of G. □

Lemma 3.6
Let G be a brace, e a removable edge of G, v an end of e, $C := \partial(X)$ and $D := \partial(Y)$ two cuts of G. Assume that cuts C and D cross, v lies in $X \cap Y$ and the cuts $C - e$ and $D - e$ are both tight in $G - e$. Then, the cuts $\partial(X \cap Y) - e$ and $\partial(\overline{X} \cap \overline{Y}) - e$ are both nontrivial and tight in $G - e$.

Proof: Let Z denote the shore of D such that $|X \cap Z|$ is odd. Let $I := \partial(X \cap Z)$ and $U := \partial(\overline{X} \cap \overline{Z})$. By Corollary 2.3, the cuts $I - e$ and $U - e$ are both tight in $G - e$. Moreover, no edge of $G - e$ joins a vertex of $X \cap \overline{Z}$ to a vertex of $\overline{X} \cap Z$.

We assert that at least one of I and U is nontrivial. For this, assume the contrary. Let w denote the vertex of $X \cap Z$ and x the vertex of $\overline{X} \cap \overline{Z}$. Graph $G - e - w - x$ is not connected. Therefore, for each end y of e, graph $G - w - x - y$ is not connected. This is a contradiction to Lemma 2.9. As asserted, one of I and U is nontrivial. Adjust notation so that I is nontrivial.

By Lemma 2.5, edge e lies in each of C, D and I. We deduce that e joins a vertex of $X \cap Z$ to a vertex of $\overline{X} \cap \overline{Z}$. By hypothesis, the end v of e lies in $X \cap Y$. Thus, $Z = Y$. To complete the proof we must now show that U is also nontrivial.

No edge of G joins a vertex of $X \cap \overline{Y}$ to a vertex of $\overline{X} \cap Y$. Let M be any perfect matching of G. By Lemma 2.2,

$$|M \cap C| + |M \cap D| = |M \cap I| + |M \cap U|. \tag{1}$$

Cuts $C - e$, $D - e$ and $I - e$ are all nontrivial and tight in $G - e$. By Corollary 3.5, each of $|M \cap C|$, $|M \cap D|$ and $|M \cap I|$ is equal to $1 + 2|M \cap \{e\}|$. From this and (1), we deduce that $|M \cap U|$ is also equal to $1 + 2|M \cap \{e\}|$. By Corollary 3.5, cut $U - e$ is also nontrivial and tight in $G - e$. □

Corollary 3.7
Let G be a brace, e a removable edge of G. If $b(G - e) \leq 3$ then the set of nontrivial tight cuts of $G - e$ is laminar.

Proof: Let C and D be two nontrivial cuts of G such that $C - e$ and $D - e$ are both tight in $G - e$. Assume, to the contrary, that C and D cross. By Lemma 3.6, there are two (distinct) nontrivial cuts I and U of G, such that $\{I, U, C\}$ is a laminar collection of three cuts that contain edge e. Moreover, $\{I - e, U - e, C - e\}$ is a laminar collection of three nontrivial tight cuts of $G - e$. By Theorem 2.4, $b(G - e) \geq 4$. This is a contradiction to the hypothesis that $b(G - e) \leq 3$. □

3.2.2 Removable Edges in Bipartite Graphs

The next lemma, proved by Lovász and Vempala [4], plays a fundamental role in the proof of the Main Theorem.

Lemma 3.8
Let G be a matching covered graph with bipartition $\{A, B\}$, and $a \in A$ a vertex with degree $d \geq 3$. Let $ab_1, \ldots ab_d$ be the edges of G incident with a. Assume that the edges $ab_1, \ldots, ab_r$ ($r \leq d$) are not removable. Assume also that $r > 0$. Then there are partitions $\{A_0, A_1, \ldots, A_r\}$ of A and $\{B_0, B_1, \ldots, B_r\}$ of B, such that $a \in A_0$, and, for $i = 1, \ldots, r$, $|A_i| = |B_i|$, $b_i \in B_i$ and ab_i is the only edge of G that joins a vertex of B_i to a vertex not in A_i. (see Figure 5).

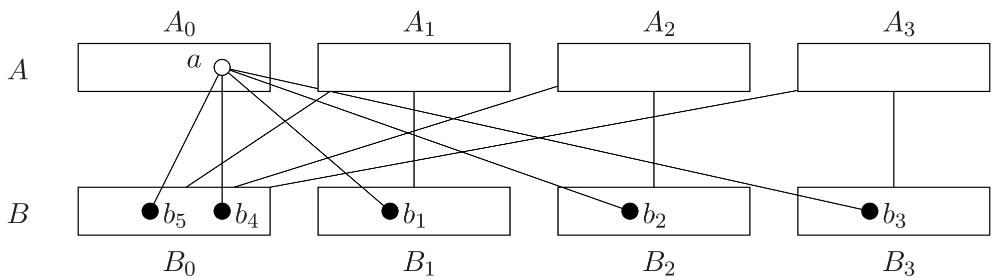

Figure 5: Illustration of Lemma 3.8, where $d = 5$ and $r = 3$.

Proof: Let i be an index in $\{1, \ldots, r\}$. As G is matching covered and $d > 1$, graph $G - ab_i$ has a perfect matching. Edge ab_i is not removable. By Lemma 2.7, there are nonempty proper subsets A_i of A and B_i of B such that $|A_i| = |B_i|$ and ab_i is the only edge with one end in $A - A_i$ and one end in B_i. Let $A_0 := A - \cup_{1 \leq i \leq r} A_i$ and $B_0 := B - \cup_{1 \leq i \leq r} B_i$. Then, a lies in A_0.

Let us first show that $B_1, B_2, \ldots, B_r$ are disjoint. Consider, say, B_1 and B_2. Let $S := B_1 \cap B_2$ and suppose that $S \neq \emptyset$. Edge ab_1 is the only edge of G that joins a vertex of B_1 to a vertex not in A_1. Thus, vertex b_2 does not lie in B_1. Likewise, vertex b_1 is not in B_2. Thus, neither b_1 nor b_2 is in S, and so $a \notin N(S)$. Then, $N(S) \subseteq A_1 \cap A_2$. As $d \geq 3$, there is a perfect matching M of G that contains neither ab_1 nor ab_2. Then, vertices of A_1 are all matched by M with vertices of B_1. Similarly, vertices of A_2 are all matched with vertices of B_2. As G is matching covered and S is a nonnull proper subset of B, we have that $|N(S)| \geq |S| + 1$, by Lemma 2.7. Then, a vertex v of $N(S)$ must be matched with a vertex w not in S. As $N(S) \subseteq A_1 \cap A_2$, v lies in $A_1 \cap A_2$. Since M matches each vertex of A_i with a vertex of B_i for $i = 1, 2$, we deduce that $w \in B_1 \cap B_2$. Thus, $w \in S$, a contradiction. We conclude that $B_1 \cap B_2 = \emptyset$. Likewise, we deduce that sets $B_1, B_2, \ldots, B_r$ are disjoint.

Let us now show that $A_1, A_2, \ldots, A_r$ are disjoint. Consider, say, A_1 and A_2. Note that $N(B_1 \cup B_2) = A_1 \cup A_2 \cup \{a\}$, and $|A_i| = |B_i|$. Thus, $|N(B_1 \cup B_2)| \leq |B_1| + |B_2| + 1 = |B_1 \cup B_2| + 1$, with equality if, and only if, A_1 and A_2 are disjoint. On the other hand, since $d \geq 3$, vertex $b_3 \notin (B_1 \cup B_2)$ and so, $B_1 \cup B_2$ is a proper subset of B. As G is matching covered, $|N(B_1 \cup B_2)| \geq |B_1 \cup B_2| + 1$, by Lemma 2.7. Thus, $|N(B_1 \cup B_2)| = |B_1 \cup B_2| + 1$. We conclude that $A_1 \cap A_2 = \emptyset$. Likewise, we deduce that sets $A_1, A_2, \ldots, A_r$ are disjoint.

Consequently, $|A_0| = |B_0|$. □

COROLLARY 3.9
In a brace on six or more vertices, every edge is removable.

Proof: Let G be a brace on six or more vertices, $\{A, B\}$ the bipartition of G, $e = ab$ any edge of G, where $a \in A$ and $b \in B$. Assume, to the contrary, that e is not removable in G. Then, A has a partition $\{A_0, A_1\}$ and B a partition $\{B_0, B_1\}$ such that $|A_0| = |B_0|$ and e is the only edge of G that joins a vertex of A_0 to a vertex of B_1.

Let $X := A_1 \cup B_1 \cup \{a\}$, $C := \partial(X)$. Then, C is a tight cut of G. As G is a brace, C is trivial, whence one of X and $\overline{X}$ is a singleton. Clearly, X contains at least three vertices. We deduce that $\overline{X}$ is a singleton. Thus, each of A_0 and B_0 is a singleton. Likewise, let $Y := A_0 \cup B_0 \cup \{b\}$. Then, cut $D := \partial(Y)$ is tight and $\overline{Y}$ is a singleton, whence each of A_1 and B_1 is a singleton. We deduce that G has only four vertices, a contradiction. □

COROLLARY 3.10
Let G be a brace, e a removable edge of G, a a vertex of G. If the degree of a in $G - e$ is three or more then at most one edge of $G - e$ incident with a is not removable in $G - e$.

Proof: Let d denote the degree of a in $G - e$, assume that $d \geq 3$. Let $\{A, B\}$ denote the bipartition of G. Adjust notation so that a lies in A. Let $ab_1, \dots ab_d$ be the edges of $G - e$ incident with a. Assume that the edges $ab_1, \dots, ab_r$ $(r \leq d)$ are not removable. If $r = 0$ then the assertion holds immediately. We may thus assume that $r > 0$. By Lemma 3.8, there are partitions $\{A_0, A_1, \dots, A_r\}$ of A and $\{B_0, B_1, \dots, B_r\}$ of B, such that $a \in A_0$, and, for $i = 1, \dots, r$, $|A_i| = |B_i|$, $b_i \in B_i$ and the edge ab_i is the only edge of $G - e$ that joins a vertex in B_i to a vertex not in A_i.

Assume, to the contrary, that $r \geq 2$. Then, at least one of B_1 and B_2, say B_1, does not contain the end of e in B. Consequently $N(B_1) = A_1 \cup \{a\}$. Let $X := B_1 \cup N(B_1)$, $C := \partial(X)$. As $|B_1| = |A_1|$, it follows that cut C is tight in G. Clearly, X is nontrivial. As $r \geq 2$ and $|B_2| = |A_2|$, the set $\overline{X}$, which includes $B_2 \cup A_2$, is not a singleton either. Consequently, C is a nontrivial tight cut of G, a contradiction to the hypothesis that G is a brace. The assertion holds. □

3.3 Thin Edges

Let G be a brace, e a removable edge of G such that $G - e$ is not a brace. Let $\mathcal{B}$ denote a tight cut decomposition of $G - e$. By Lemma 3.4, precisely two braces in $\mathcal{B}$ have only one contraction vertex, the remaining braces, if any, have two contraction vertices. The two braces in $\mathcal{B}$ having only one contraction vertex are said to be *external in* $\mathcal{B}$. Assume further that $b(G - e) \leq 3$. Then, by Corollary 3.7, the tight cut decomposition of $G - e$ is unique. In that case, we may omit $\mathcal{B}$ and simply say that a brace is *external in* $G - e$. With these observations in mind, we may now define the notion of thin edge.

Let G be a brace. An edge e of G is *thin* if it is removable in G, $b(G - e) \leq 3$ and the following additional properties hold:

(i) If $b(G - e) = 2$ then at least one of the two braces of $G - e$ has only four vertices.

(ii) If $b(G - e) = 3$ then both external braces of $G - e$ have only four vertices.

In the brace shown in Figure 6, the edge e is thin but the edge f is not. Figure 7 depicts the three types of thin edges of a brace G: for $s = 1, 2, 3$, the tight cut decomposition of $G - e_s$ contains precisely s braces.

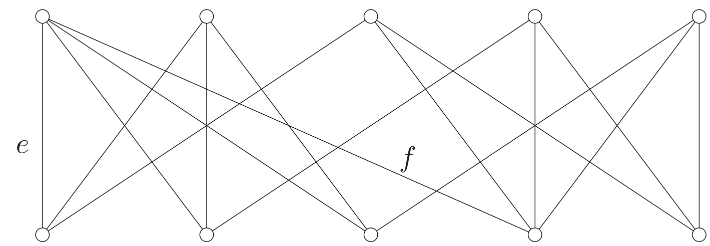

Figure 6: Example of an edge e that is thin and of an edge f that is not thin.

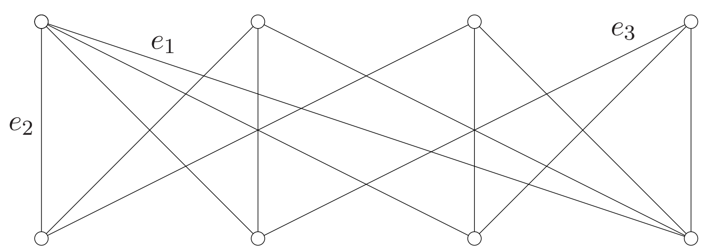

Figure 7: The three types of thin edges.

3.4 Reducing Completeness to the Existence of Thin Edges

We now reduce the proof of the completeness part of the Main Theorem to the existence of thin edges in braces.

Lemma 3.11
Assume that every simple brace on 8 or more vertices has a thin edge. Then, every brace G on four or more vertices may be generated from C_4 by repeated applications of the three operations of brace expansion.

Proof: By induction on $|E(G)|$. Consider first the case in which G has a removable edge e such that $G-e$ is a brace. In that case, by induction hypothesis, $G-e$ may be generated from C_4. Moreover, G may be generated from $G-e$ by the operation of edge addition (expansion of zero vertices). The assertion holds in this case.

We may thus assume that for every removable edge e of G, $b(G-e) > 1$. In particular, this implies that G is free of multiple edges. If G has only four vertices then it is C_4, the only simple brace on four vertices. In that case, the assertion holds immediately. If G has only six vertices then it is $K_{3,3}$, the only simple brace on six vertices. The brace $K_{3,3}$ may be generated from C_4, as indicated in Figure 3.

We may thus assume that G is free of multiple edges and has 8 vertices, or more. In that case, by hypothesis, G has a thin edge, e. We have assumed that $b(G-f) > 1$ for every removable edge f. By definition of thin edge, $b(G-e)$ is either two or three. We analyze those two cases separately.

Case 1 $b(G-e) = 2$.

Let $C := \partial(X)$ denote a nontrivial cut of G such that $C-e$ is tight in $G-e$ and the two braces of $G-e$ are the $(C-e)$-contractions of $G-e$. By definition of thin edge, one of the braces has only four vertices. Adjust notation, by interchanging X with $\overline{X}$ if necessary, so that $(G-e)\{\overline{X} \to \overline{x}\}$ has only four vertices. Let $H := (G-e)\{X \to x\}$ denote the other $(C-e)$-contraction of $G-e$. Let $\{A, B\}$ denote the bipartition of brace H.

As G is free of multiple edges, the subgraph of G spanned by X is $K_{2,1}$. Let x_0 denote the vertex of X_-, x_1 and x_2 the two vertices of X_+. Then, x_0 is an end of edge e in G. Let y denote the other end of e. Then, x and y both lie in A (see Figure 2). As G is a brace, x_1 and x_2 both have degree three or more in G, whence x has degree four or more in H. We deduce that G is obtained from brace H by the expansion of vertex x. As G has 8 vertices or more, brace H has six vertices or more. By induction hypothesis, H may be generated from C_4, whence so too may G. The assertion holds in this case.

Case 2 $b(G-e) = 3$.

Let $C := \partial(X)$ and $D := \partial(Y)$ denote two noncrossing nontrivial cuts of G such that $C-e$ and $D-e$ are both tight in $G-e$. By definition, both extremal braces of $G-e$ have only four vertices. Adjust notation, by interchanging X with $\overline{X}$ if necessary, so that $|X| = 3$. Likewise, adjust notation so that $|Y| = 3$. As G is free of multiple edges, the subgraphs $G[X]$ and $G[Y]$ spanned by X and Y, respectively, are equal to $K_{2,1}$. Let x_0 and y_0 denote the vertex of X_- and of Y_-, respectively. Let x_1 and x_2 denote the two vertices of X_+, let y_1 and y_2 denote the two vertices of Y_+. As G has more than four vertices and is a brace, we deduce that e joins x_0 and y_0 (see Figure 2). Let $H := (G-e)\{X \to x\}\{Y \to y\}$ be the brace of $G-e$ with two contraction vertices. Let $\{A, B\}$ denote the bipartition of H. Adjust notation so that x lies in A, whereupon y lies in B. As G is a brace on more than four vertices, each of x_1 and x_2 has degree three or more in G, whence x has degree four or more in H. Likewise, y has degree four or more in H. Assume, to the contrary, that $N_G(x_1) = \{x_0, y_1, y_2\}$. Let $Z := (X - x_2) \cup Y$. Then, $Z_- = \{x_1, y_0\}$ and $Z_+ = \{x_0, y_1, y_2\}$. Moreover, $N_G(Z_-) = Z_+$, whence $F := \partial(Z)$ is a tight cut of G. As Z has five vertices and

G has 8 or more, we deduce that F is a nontrivial tight cut of G. This is a contradiction. We conclude that x_1 is adjacent to some vertex that does not lie in $X \cup Y$. A similar conclusion holds for x_2, y_1 and y_2. Consequently, G is obtained from brace H by the expansion of vertices x and y. As G has 8 vertices or more, H has four vertices, or more. By induction hypothesis, H may be generated from C_4. Thus, G may also be generated from C_4. The assertion holds also in this case. The proof of the Lemma reduces the completeness of the three operations of brace expansion to the existence of thin edges in simple braces on 8 or more vertices. □

3.5 The Existence of Thin Edges

We now complete the proof of the Main Theorem, by proving that every simple brace on 8 vertices or more has a thin edge. In fact, we prove a slightly more general result.

Theorem 3.12 (The Thin Edge Theorem)
Every brace distinct from K_2 and C_4 has a thin edge.

Proof: Let G be a brace distinct from K_2 and C_4. If G has a removable edge e such that $G-e$ is a brace then the assertion holds immediately. We may thus assume that $b(G-e) > 1$, for each removable edge e of G. In particular, this implies that G is free of multiple edges.

If G has only two vertices then it is K_2, the only simple brace on two vertices. If G has only four vertices then it is C_4, the only simple brace on four vertices. If G has only six vertices then it is $K_{3,3}$, the only simple brace on six vertices: in that case, every edge e of G is thin, with $b(G-e) = 2$ and each brace of $G-e$ with only four vertices.

We may thus assume that G has 8 vertices, or more. We may also assume that G is simple. Let e be any edge of G. By Corollary 3.9, edge e is removable. We have assumed that $b(G-e) > 1$. Consider a tight cut decomposition $\mathcal{B}$ of $G-e$. Let $\mathcal{C}$ denote the corresponding (nonnull) maximal laminar collection of nontrivial cuts of G such that $C-e$ is tight in $G-e$, for each cut C in $\mathcal{C}$. Let v be one of the ends of e. Let $\mathcal{X}$ denote the set of shores of the cuts of $\mathcal{C}$ that contain vertex v. By Lemma 3.3, set inclusion is a total order in $\mathcal{X}$. Thus, we may enumerate the shores in $\mathcal{X}$ in a sequence

$$\emptyset \subset X_1 \subset X_2 \subset \ldots \subset X_{b(G-e)-1} \subset V(G). \tag{2}$$

We remark that

$$\begin{aligned} G_1 &:= (G-e)\{\overline{X_1} \to \overline{x_1}\}, \\ G_i &:= (G-e)\{X_{i-1} \to x_{i-1}\}\{\overline{X_i} \to \overline{x_i}\}, \quad i = 2, 3, \ldots, b(G-e)-1 \text{ and} \\ G_{b(G-e)} &:= (G-e)\{X_{b(G-e)} \to x_{b(G-e)}\} \end{aligned}$$

are the braces of $\mathcal{B}$. We define the *rank* of e, denoted $r(e)$, to be the maximum number of vertices of the braces of $G-e$. More precisely, $r(e) := \max\{|V(H)| : H \in \mathcal{B}\}$. The value $r(e)$ is well-defined, by Theorem 2.4. We define now the following invariant of G.

$$r^\star := \max\{r(e) : e \in E(G)\}.$$

Lemma 3.13
Assume that G is simple and has 8 vertices or more. If $r^\star < 6$ then G is the cube and each edge of G is thin.

Proof: As G has more than two vertices, $r^\star \geq 4$. By hypothesis, $r^\star < 6$. Thus, $r^\star = 4$. Let e be any edge of G, v one of its ends. Consider a tight cut decomposition $\mathcal{B}$ of $G - e$, let $\mathcal{C}$ denote the corresponding laminar collection of nontrivial cuts of G, let $\mathcal{X}$ denote the set of shores of the cuts of $\mathcal{C}$ that contain vertex v, as in (2). Note that as $r^\star = 4$, it follows that both X_1 and the complement of $X_{b(G-e)-1}$ have precisely three vertices each. Moreover, $|X_i - X_{i-1}| = 2$, for $i = 2, 3, \ldots, b(G-e) - 1$. Consequently,

$$b(G-e) = |V(G)|/2 - 1. \tag{3}$$

Note that $G_1 := (G-e)\{\overline{X_1} \to \overline{x_1}\}$ is an external brace of $\mathcal{B}$, $\overline{x_1}$ its only contraction vertex. Let $C_1 := \partial_G(X_1)$. Then, $C_1 - e$ is a nontrivial tight cut of $G - e$. By definition, v lies in X_1. By Lemma 3.2, v lies in $(X_1)_-$ and each vertex of G adjacent to v in $G - e$ lies in $(X_1)_+$. As G is simple, the brace G_1 consists of at least $d+1$ vertices, where d is the degree of v in G. By hypothesis, $d + 1 \leq |V(G_1)| \leq r^\star = 4$, whence $d = 3$. That is, v has degree three in G. This conclusion holds for each end v of each edge e of G. We deduce that G is cubic.

As G has 8 vertices or more, it follows from (3) that $b(G-e) \geq 3$. We have seen that $|X_i - X_{i-1}| = 2$, for $i = 2, 3, \ldots, b(G-e)-1$. In particular, $X_2 - X_1$ consists of precisely two vertices, one in each part of the bipartition of G. Let v' denote the vertex of $X_2 - X_1$ that lies in $(X_2)_-$. By Lemma 3.2, all the neighbors of v' in G lie in X_2. The three neighbors of v' are then the three vertices of $(X_2)_+$. We deduce that the edges of $\partial(v) - e$ lie in a quadrilateral. This conclusion holds for each edge e incident with v. In other words, any two vertices of G that lie in $N(v)$ have two common adjacent vertices, one of which is v.

Let $W := N(v)$. If the three vertices of W have a common adjacent vertex other than v, say v'', then the cut $C := \partial(W \cup \{v, v''\})$ is tight in G, because G is cubic; moreover, C is nontrivial, because G has 8 vertices or more. This is a contradiction. We deduce that v is the only vertex of G that is adjacent to each vertex of W. As G is cubic, and each pair of vertices of W have two adjacent vertices in common, it follows that $|N(W)| = 4$. Consequently, cut $D := \partial(W \cup N(W)$ is tight in G. If G has 10 vertices, or more, then D is nontrivial, a contradiction. We deduce that G has fewer than 10 vertices. By hypothesis, G has 8 vertices, or more. We deduce that G has exactly 8 vertices. We conclude that G is the cube, as asserted. In that case, from (3), we deduce that $b(G-e) = 3$. As $r^\star = 4$, each brace in $\mathcal{B}$ has only four vertices. Consequently, e is thin. This conclusion holds for each edge e of G. □

We may assume that $r^\star \geq 6$. We may also assume that G has 8 vertices or more, and is simple. Let $R := \{e \in E(G) : r(e) = r^\star\}$. Let e be an edge of R, $\mathcal{B}$ a tight cut decomposition of $G - e$, $H^\star$ a brace in $\mathcal{B}$ on $r^\star$ vertices. If possible, choose e, $\mathcal{B}$ and $H^\star$ so that $H^\star$ has two contraction vertices. If $H^\star$ has two contraction vertices, say x and y, let $C := \partial_G(X)$ and $D := \partial_G(Y)$ be two cuts in $\mathcal{C}$ such that $H^\star = (G-e)\{X \to x\}\{Y \to y\}$: adjust notation so that $|X| \geq |Y|$. If $H^\star$ has only one contraction vertex, say x, let $C := \partial_G(X)$ be a cut in $\mathcal{C}$

such that $H^\star = (G-e)\{X \to x\}$. In both alternatives,

$$r^\star = r(e) \le 1 + |\overline{X}|, \tag{4}$$

with equality only if $H^\star$ has precisely one contraction vertex.

It now suffices to show that $|X| < 5$. To see this, note first that as C is nontrivial, it follows that $|X| = 3$. If H has only one contraction vertex then $b(G-e) = 2$, and one of the braces of $\mathcal{B}$ has only four vertices. Consequently, edge e is thin. If H has two contraction vertices then, as D is nontrivial and $|Y| \le |X|$, it follows that $|Y| = 3$. Consequently, $b(G-e) = 3$ and both external braces of $G-e$ have only four vertices, whence e is thin. In both alternatives, e is thin.

To prove that $|X| < 5$, assume the contrary. In the remaining of the proof we deduce a contradiction to the definition of edge e. For this, note that edge e has one end in the minority parts of each of X and $\overline{X}$. Let v denote the end of e in X_-, w its other end. Let $\{A, B\}$ denote the bipartition of G. Adjust notation so that v lies in A. As $|X| \ge 5$, we have that X_- contains two or more vertices. Let v' be a vertex of $X_- - v$. As G has more than four vertices, vertex v' has degree three or more in G. By Lemma 3.8, at least two edges incident with v' are removable in $G-e$. Let e' and e'' denote any two such edges. Let w' and w'' denote the ends of e' and e'' in B. The vertices of G adjacent to v' in G lie all in X_+. Thus, w' and w'' lie both in X_+.

We now show that one of e' and e'' contradicts the choice of e. We do this by proving properties of edge e'. Similar properties hold for edge e''. We have already observed that, as G has more than four vertices, edges e' and e'' are both removable in G.

Let $\mathcal{B}'$ be a tight cut decomposition of $G-e'$, $\mathcal{C}'$ the corresponding collection of nontrivial cuts of G such that $C' - e$ is tight in $G-e'$, for each cut C' in $\mathcal{C}'$.

Lemma 3.14
Cut C crosses at least one cut of $\mathcal{C}'$.

Proof: Assume, to the contrary, that no cut of $\mathcal{C}'$ crosses cut C. Let C' be a cut in $\mathcal{C}'$. Edge e' has both ends in X and lies in C'. Thus, X meets both shores of C'. By hypothesis, cuts C and C' do not cross. Thus, $\overline{X}$ is a subset of one of the shores of C'. This conclusion holds for each cut C' in $\mathcal{C}'$. Let X' denote the minimal shore among the shores of cuts of $\mathcal{C}'$ that include $\overline{X}$. Then, $H' := (G-e')\{\overline{X'} \to \overline{x'}\}$ is a brace of $\mathcal{B}'$. Edge e lies in C and is incident with vertex v, in turn a vertex of X_-. By Lemma 2.5, cut $C - e'$ is not tight in $G-e'$. We deduce that $\overline{X}$ is a proper subset of X'. Thus, $1 + |\overline{X}| < |V(H')| \le r(e')$. From this and (4), we deduce that $r^\star < r(e')$, a contradiction. The assertion of the Lemma holds. □

Lemma 3.15
Let C' be a cut of $\mathcal{C}'$ that crosses cut C, let X' denote the shore of C' that contains the end v' of e' in X_-. Let $I := \partial(X \cap X')$, $U := \partial(\overline{X} \cap \overline{X}')$. Then, $|X \cap X'|$ is odd, cut $I - e - e'$ is nontrivial and tight in $G - e - e'$, and cut $U - e$ is tight in $G - e$.

Proof: Let Z denote the shore of C' such that $|X \cap Z|$ is odd. Cut $C - e$ is tight in $G - e$, whence it is tight in $G - e - e'$. Cut $C' - e'$ is tight in $G - e'$, whence it is also tight in $G - e - e'$. By Corollary 2.3, each of $I - e - e'$ and $U - e - e'$ is tight in $G - e - e'$. Moreover, no edge of $E(G) - e - e'$ joins a vertex of $X \cap \overline{Z}$ to a vertex of $\overline{X} \cap Z$. Edge e' has both ends in X. Thus, no edge of $E(G) - e$ joins a vertex of $X \cap \overline{Z}$ to a vertex of $\overline{X} \cap Z$. Moreover, by Lemma 2.5, cut $U - e$ is tight in $G - e$. Let M be any perfect matching of $G - e$. By Lemma 2.2,

$$1 + |M \cap C'| = |M \cap C| + |M \cap C'| = |M \cap I| + |M \cap U| = |M \cap I| + 1,$$

whence $|M \cap I| = |M \cap C'$. This conclusion holds for each perfect matching M of $G - e$. As cut $C' - e'$ is nontrivial and tight in $G - e'$, it follows, by Corollary 3.5, that $I - e - e'$ is nontrivial and tight in $G - e - e'$.

Each $(C' - e - e')$-contraction of $G - e - e'$ is matching covered and thus 2-connected. Therefore, the subgraph $(G - e - e')[Z]$ of $G - e - e'$ spanned by the shore Z of C' is connected. Consequently, $G - e - e'$ has an edge, f, that joins a vertex of $X \cap Z$ to a vertex in $\overline{X} \cap Z$. Edge f lies in $C - e$, whence its end in X lies in X_+, in turn a subset of B. Edge f also lies in $I - e - e'$, in turn a nontrivial tight cut of $G - e - e'$. Consequently, B includes the majority part of $X \cap Z$.

Likewise, each $(C - e - e')$-contraction of $G - e - e'$ is matching covered and thus 2-connected. Therefore, the subgraph $(G - e - e')[X]$ of $G - e - e'$ spanned by the shore X of C is connected. Let f' be an edge that joins a vertex of $X \cap Z$ to a vertex in $X \cap \overline{Z}$. Edge f' lies in $I - e - e'$, and the majority part of $X \cap Z$ is a subset of B, whence the end of f' in $X \cap Z$ lies in B. Edge f' lies in $C' - e'$, in turn a nontrivial tight cut of $G - e'$, whence its end in Z lies in the majority part of Z. But that end of f' lies in B. Thus, B includes the majority part of Z. As v', the end of e' in X_-, lies in A, it follows that the the majority part of X' is a subset of B. We deduce that $Z = X'$. As asserted, $|X \cap X'|$ is odd, cut $I - e - e'$ is nontrivial and tight in $G - e - e'$ and cut $U - e$ is tight in $G - e$. □

Consider the subset $\mathcal{C}'_0$ of $\mathcal{C}'$ consisting of those cuts that cross cut C. Consider the set $\mathcal{X}'_0$ of the shores of the cuts of $\mathcal{C}'_0$ that contain the end v' of e'. By Lemma 3.3, $\mathcal{X}'_0$ is totally ordered by set inclusion. Let X' denote the minimal element of $\mathcal{X}'_0$. By Lemma 3.15, $|X \cap X'|$ is odd. Adopt the definition of I and U as in the statement of Lemma 3.15. Let $H' := (G - e')\{\overline{X'} \to \overline{x'}\}$. As C' lies in $\mathcal{C}'$, the tight cut decomposition $\mathcal{B}'$ of $G - e'$ has a brace $H'^{\star}$ of H' that has vertex $\overline{x'}$ as a contraction vertex. If $H'^{\star}$ has two contraction vertices then let y' denote the other contraction vertex of $H'^{\star}$, let Y' be the shore of a cut D' in $\mathcal{C}'$ such that $H'^{\star} = H'\{Y' \to y'\}$.

Lemma 3.16

$$r(e') \geq 2 + |\overline{X} \cap X'|, \tag{5}$$

with equality only if $H'^{\star}$ has two contraction vertices.

Proof: By Lemma 3.15, cut I is nontrivial, whence $|V(H')| > 2 + |\overline{X} \cap X'|$. If H' is a brace then $r(e') \geq |V(H')|$, whence inequality (5) holds strictly. Assume thus that H' is not a brace. Then $H'^{\star}$ has two contraction vertices. Moreover, the end v' of e' in X_- lies in Y'. By the minimality of X', cuts C and D' do not cross. Thus, Y' is a subset of $X \cap X'$. Consequently, $r(e') \geq |V(H'^{\star})| \geq 2 + |\overline{X} \cap X'|$, with equality only if $D' = I$. In both alternatives, we establish the validity of inequality (5), with equality only if $H'^{\star}$ has two contraction vertices and $Y' = X \cap X'$. □

LEMMA 3.17
Cut $U - e$ is nontrivial and tight in $G - e$. Moreover, $Y = \overline{X} \cap \overline{X'}$. Consequently, $H^{\star} = (G-e)\{X \to x\}\{(\overline{X} \cap \overline{X'}) \to y\}$.

<u>Proof</u>: By definition of X, the graph $H^{\star}$ is a brace of $G_X := (G-e)\{X \to x\}$ that has vertex x as one of its contraction vertices. By Lemma 3.15, cut $U - e$ is tight in $G - e$. Assume, to the contrary, that $U - e$ is trivial. Then, $\overline{X} \cap \overline{X'}$ is a singleton, whence

$$r^{\star} = |V(H^{\star})| \leq |V(G_X)| \leq 2 + |\overline{X} \cap X'|, \tag{6}$$

with equality only if $H^{\star} = G_X$. As $r^{\star} \geq r(e')$, we deduce from (5) that equality holds in (5) and in (6). Thus, $H'^{\star}$ is a brace on $r^{\star}$ vertices and two contraction vertices, whereas $H^{\star} = G_X$ has only one contraction vertex. This is a contradiction to the definition of edge e. As asserted, cut $U - e$ is nontrivial and tight in $G - e$.

Consequently, G_X is not a brace. Thus, $H^{\star}$ has two contraction vertices. By definition of D and Y, $H^{\star} = G_X\{Y \to y\}$, where $D = \partial_G(Y)$ is a cut in $\mathcal{C}$. Thus, $D - e$ is nontrivial and tight in $G - e$. Moreover, edge e must lie in D. As Y and X are disjoint, it follows that Y contains the end w of e in $\overline{X}$, whereas the end v of e in X lies in $\overline{Y}$. As $U - e$ is nontrivial and tight in $G - e$, w lies in $\overline{X'}$, whereas v does not lie in $\overline{X} \cap \overline{X'}$. Thus, $\overline{X} \cap \overline{X'}$ and Y have a nonnull intersection. So too have their complements. We assert that those two sets are equal. For this, assume the contrary. Consider first the case in which cuts U and D cross. By Lemma 3.6, $F := \partial_G(Y \cup (\overline{X} \cap \overline{X'}))$ is nontrivial and $F - e$ is tight in $G - e$: in that case, $F - e$ is a nontrivial tight cut of $H^{\star}$, a contradiction. We deduce that cuts U and D do not cross. As $\overline{X} \cap \overline{X}'$ and Y have a nonnull intersection and so too have their complements, it follows that one of $\overline{X} \cap \overline{X'}$ and Y is a subset of the other. If Y is a a proper subset of $\overline{X} \cap \overline{X'}$ then $U - e$ is a nontrivial tight cut of $H^{\star}$, again a contradiction. If $\overline{X} \cap \overline{X'}$ is a proper subset of Y then $|V(H^{\star}| < 2 + |\overline{X} \cap X'|$: in that case, by Lemma 3.16, we deduce that $r(e') > r^{\star}$, a contradiction. The only possibility left is the equality of Y and $\overline{X} \cap \overline{X'}$, as asserted. □

A conclusion similar to that of Lemma 3.17 holds for edge e'', the edge incident incident with v' and distinct from e' that is also removable in $G - e$. More precisely, graph G has a nontrivial cut $C'' := \partial(X'')$ that crosses cut C, such that $C'' - e''$ is tight in $G - e''$ and $Y = \overline{X} \cap \overline{X''}$.

Cuts $C - e$ and $U - e$ are both nontrivial and tight in $G - e$. The end v of e in X lies in A, whereas the end w of e in $\overline{X}$ lies in B. Thus, A includes the majority parts of both $\overline{X}$ and Y. We deduce that $\overline{X} - Y$ has the same number of vertices in A and in B. As $2 + |\overline{X} - Y| = r^{\star} \geq 6$, we deduce that $\overline{X} - Y$ has two or more vertices in B. Let b_1 and b_2

be two distinct vertices of B in $\overline{X} - Y$. As cut $U - e$ is nontrivial, and since A includes the majority part of Y, the set Y contains two or more vertices in A. Let a_1 and a_2 denote two vertices of $A \cap Y$.

As G is a brace, the graph $G - a_1 - a_2 - b_1 - b_2$ has a perfect matching. As b_1 and b_2 are two vertices that lie in the majority part of X', and since neither a_1 nor a_2 lies in X', we deduce that edge e' lies in M. (In fact, e' is the only edge of M in C'.) As $\overline{X} \cap \overline{X'} = Y = \overline{X} \cap \overline{X''}$, we deduce that b_1 and b_2 lie both in X'', whereas a_1 and a_2 lie both in $\overline{X''}$. Consequently, edge e'' also lies in M. Thus, M contains both edges e' and e''. This is a contradiction, because those two edges are incident with vertex v'. As asserted, e is a thin edge of G. The proof of the Thin Edge Theorem completes the proof of the Main Theorem. □

References

[1] M. H. de Carvalho, C. L. Lucchesi, and U. S. R. Murty. How to build a brick. *Discrete Math.*, 2003. To appear. See also `www.ic.unicamp.br/~lucchesi`.

[2] L. Lovász. Matching structure and the matching lattice. *J. Combin. Theory Ser. B*, 43:187–222, 1987.

[3] L. Lovász and M. D. Plummer. *Matching Theory*. Number 29 in Annals of Discrete Mathematics. Elsevier Science, 1986.

[4] L. Lovász and S. Vempala. On removable edges in matching covered graphs. Unpublished manuscript.

[5] W. McCuaig. Brace generation. *J. Graph Theory*, 38:124–169, 2001.

List of Assertions

(Main Theorem) A graph G on four vertices or more is a brace if and only if it may be obtained from C_4 by repeated applications of the three operations of brace expansion.

Every matching covered graph is 2-connected. □

(Submodularity) Let G be a matching covered graph, $C := \partial(X)$ and $D := \partial(Y)$ two cuts of G. Let λ denote the set of edges that joins a vertex of $X \cap \overline{Y}$ to a vertex of $\overline{X} \cap Y$. Let $I := \partial(X \cap Y)$, $U := \partial(\overline{X} \cap \overline{Y})$. For any set M of edges of G, the following equality holds: $|M \cap C| + |M \cap D| = |M \cap I| + |M \cap U| + 2|M \cap \lambda|$. □

(Modularity) Let G be a matching covered graph, $C := \partial(X)$ and $D := \partial(Y)$ two tight cuts of G. If $|X \cap Y|$ is odd then each of $\partial(X \cap Y)$ and $\partial(\overline{X} \cap \overline{Y})$ is tight and no edge of G joins a vertex of $X \cap \overline{Y}$ to a vertex of $\overline{X} \cap Y$. □

Any two applications of the tight cut decomposition procedure to a matching covered graph produce the same list of bricks and braces, up to multiple edges. □

Let G be a bipartite matching covered graph, $C := \partial(X)$ a cut of G such that $|X|$ is odd. Then, C is tight if and only if (i) $|X_+| = |X_-| + 1$ and (ii) no edge of C is incident with a vertex of X_-. □

Every tight cut decomposition of a bipartite matching covered graph consists solely of braces. □

Let G be a graph with bipartition $\{A, B\}$ and at least four vertices. Assume that G has a perfect matching. Then, the following assertions are equivalent:

(i) Graph G is matching covered.

(ii) For every partition $\{A', A''\}$ of A and every partition $\{B', B''\}$ of B such that $|A'| = |B'|$, graph G has at least one edge that joins A' to B''.

(iii) For every vertex a of A and every vertex b of B, graph $G - a - b$ has a perfect matching. □

Lemma 2.8 {`lem:brace-char`} 7

Let G be a matching covered graph and bipartition $\{A, B\}$. Then, G is a brace if and only if $G - a_1 - a_2 - b_1 - b_2$ has a perfect matching, for any two vertices a_1 and a_2 in A and any two vertices b_1 and b_2 in B.

Lemma 2.9 {`lem:3-connected`} 7

Let G be a brace on six vertices or more. Then, G is 3-connected. Moreover, for any set X of three vertices of G that meets both parts of the bipartition of G, the graph $G - X$ is connected.

Theorem 3.1 8

(Containment) Let $G_0 = C_4, G_1, \ldots, G_s,\ s \geq 0$, be a sequence of graphs such that $G_0 = C_4$ and, for $i = 1, 2, \ldots, s$, G_i is obtained from G_{i-1} by the application of one of the three operations of brace expansion. Then, G_s is a brace.

Lemma 3.2 {`lem:tight:G-e`} 11

Let G be a brace, e a removable edge of G, $C := \partial(X)$ a cut of G such that $C - e$ is tight in $G - e$. Then, $C - e$ is a nontrivial tight cut of $G - e$ if and only if e is the only edge of C that has one end in X_-.

Lemma 3.3 {`lem:nested`} 11

Let G be a brace, e a removable edge of G, v an end of e, $\mathcal{C}$ a laminar collection of nontrivial cuts of G such that, for each cut $C \in \mathcal{C}$, the cut $C - e$ is tight in $G - e$. Then, set inclusion is a total order in the collection of the shores of the cuts in $\mathcal{C}$ that contain vertex v.

Lemma 3.4 {`lem:threeCase`} 11

(The Three Case Lemma) Let G be a brace, e a removable edge of G. Let H be a brace obtained by a tight cut decomposition of $G-e$. Then, H has at most two contraction vertices.

Corollary 3.5 {`cor:counting-3`} 11

Let G be a brace, e a removable edge of G, C a cut of G. Then, cut $C - e$ is a nontrivial tight cut of $G - e$ if and only $|M \cap C| = 1 + 2|M \cap \{e\}|$, for each perfect matching M of G. □

Lemma 3.6 {`lem:crossing-tight-derived`} 12

Let G be a brace, e a removable edge of G, v an end of e, $C := \partial(X)$ and $D := \partial(Y)$ two cuts of G. Assume that cuts C and D cross, v lies in $X \cap Y$ and the cuts $C - e$ and $D - e$ are both tight in $G - e$. Then, the cuts $\partial(X \cap Y) - e$ and $\partial(\overline{X} \cap \overline{Y}) - e$ are both nontrivial and tight in $G - e$.

Corollary 3.7 {`cor:smallBeautiful`} 12

Let G be a brace, e a removable edge of G. If $b(G - e) \leq 3$ then the set of nontrivial tight cuts of $G - e$ is laminar.

Lemma 3.8 {`lem:non-rem`} 13

Let G be a matching covered graph with bipartition $\{A, B\}$, and $a \in A$ a vertex with degree $d \geq 3$. Let $ab_1, \ldots ab_d$ be the edges of G incident with a. Assume that the edges

$ab_1, \ldots, ab_r$ ($r \le d$) are not removable. Assume also that $r > 0$. Then there are partitions $\{A_0, A_1, \ldots, A_r\}$ of A and $\{B_0, B_1, \ldots, B_r\}$ of B, such that $a \in A_0$, and, for $i = 1, \ldots, r$, $|A_i| = |B_i|$, $b_i \in B_i$ and ab_i is the only edge of G that joins a vertex of B_i to a vertex not in A_i. (see Figure 5).

In a brace on six or more vertices, every edge is removable.

Let G be a brace, e a removable edge of G, a a vertex of G. If the degree of a in $G - e$ is three or more then at most one edge of $G - e$ incident with a is not removable in $G - e$.

Assume that every simple brace on 8 or more vertices has a thin edge. Then, every brace G on four or more vertices may be generated from C_4 by repeated applications of the three operations of brace expansion.

(The Thin Edge Theorem) Every brace distinct from K_2 and C_4 has a thin edge.

Assume that G is simple and has 8 vertices or more. If $r^\star < 6$ then G is the cube and each edge of G is thin.

Cut C crosses at least one cut of $\mathcal{C}'$.

Let C' be a cut of $\mathcal{C}'$ that crosses cut C, let X' denote the shore of C' that contains the end v' of e' in X_-. Let $I := \partial(X \cap X')$, $U := \partial(\overline{X} \cap \overline{X}')$. Then, $|X \cap X'|$ is odd, cut $I - e - e'$ is nontrivial and tight in $G - e - e'$, and cut $U - e$ is tight in $G - e$.

$$r(e') \ge 2 + |\overline{X} \cap X'|, \tag{5}$$

with equality only if $H'^\star$ has two contraction vertices.

Cut $U - e$ is nontrivial and tight in $G - e$. Moreover, $Y = \overline{X} \cap \overline{X'}$. Consequently, $H^\star = (G - e)\{X \to x\}\{(\overline{X} \cap \overline{X'}) \to y\}$.