INSTITUTO DE COMPUTAÇÃO
UNIVERSIDADE ESTADUAL DE CAMPINAS

The Perfect Matching Polytope and Solid Bricks

Marcelo H. de Carvalho   *Cláudio L. Lucchesi*
U. S. R. Murty

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The Perfect Matching Polytope and Solid Bricks *

Marcelo H. de Carvalho
UFMS, Campo Grande, Brasil and
University of Waterloo, Waterloo, Canada

Cláudio L. Lucchesi
UNICAMP, Campinas, Brasil

U. S. R. Murty
University of Waterloo, Waterloo, Canada

Abstract

The *perfect matching polytope* of a graph G is the convex hull of the set of incidence vectors of perfect matchings of G. Edmonds (1965) showed that a vector x in $\mathbf{R}^E$ belongs to the perfect matching polytope of G if and only if it satisfies the inequalities: (i) $x \geq 0$ (*non-negativity*), (ii) $x(\partial(v)) = 1$, for all $v \in V$ (*degree constraints*) and (iii) $x(\partial(S)) \geq 1$, for all odd subsets S of V (*odd set constraints*). We are interested in the problem of characterizing graphs whose perfect matching polytopes are determined by non-negativity and the degree constraints. (It is well-known that bipartite graphs have this property.) The appropriate context for studying this problem is the theory of matching covered graphs.

An edge of a graph is *admissible* if there is some perfect matching of the graph containing that edge. A graph is *matching covered* if it is connected, has at least two vertices and each of its edges is admissible. A cut C of a matching covered graph G is *tight* if $|M \cap C| = 1$ for every perfect matching M of G, and is *separating* if each of the two graphs obtained by shrinking a shore of C to a single vertex is also matching covered. Every tight cut is a separating cut, but the converse is not true. A non-bipartite matching covered graph is a *brick* if it has no nontrivial tight cuts and is a *solid brick* if it has no nontrivial separating cuts. We show that the above-mentioned problem may be reduced to one of recognizing solid bricks. (The complexity status of this problem is unknown.) We include a brief account of how we were led to solid bricks, present some examples and a proof of a recent theorem of Reed and Wakabayashi.

1 The Perfect Matching Polytope

For a graph G, as usual, $\mathbf{R}^E$ denotes the set of all functions from the edge set E of G into the field of reals. It is convenient to think of the elements of $\mathbf{R}^E$ as vectors whose coordinates are indexed by the edges of G.

We denote the set of all perfect matchings of a graph G by $\mathcal{M}$ and, for any $M \in \mathcal{M}$, the incidence vector of M by χ^M. The *perfect matching polytope* of G, denoted by $\mathcal{P}oly(G)$,

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is the convex hull of $\{\chi^M : M \in \mathcal{M}\}$. (For brevity, we shall refer to the perfect matching polytope simply as the *matching polytope*).

In a landmark paper, Edmonds (1965) gave a linear inequality description of $\mathcal{P}oly(G)$. To present Edmonds' description of $\mathcal{P}oly(G)$, we need the notion of a coboundary. For a subset S of the vertex set V, the *coboundary* of S is the set of all edges of G with precisely one end in S. We denote the coboundary of S by $\partial(S)$. (When S is a singleton $\{v\}$, we simply write $\partial(v)$ instead of $\partial(\{v\})$.) We refer to a subset of E that is the coboundary of a subset of V as a *cut* of G. For any element x of $\mathbf{R}^E$ and any cut C of G, we denote $\sum_{e \in C} x(e)$ by $x(C)$.

Let M be any perfect matching of G. For any vertex v of G, there is precisely one edge of M incident with V. Thus, for any x in $\mathcal{P}oly(G)$, $x(\partial(v)) = 1$. Also, for any odd subset S of V, $|M \cap \partial(S)|$ must be odd. Thus, $x(\partial(S)) \geq 1$.

Theorem 1.1 (Edmonds, 1965)
A vector x in $\mathbf{R}^E$ belongs to the matching polytope $\mathcal{P}oly(G)$ of a graph G if and only if it satisfies the following system of linear inequalities:

$$\begin{array}{rcll} x & \geq & 0 & & \textit{(non-negativity)} \\ x(\partial(v)) & = & 1 & \textit{for all } v \in V & \textit{(degree constraints)} \\ x(\partial(S) & \geq & 1 & \textit{for all odd } S \subset V & \textit{(odd set constraints)} \end{array}$$

A natural question that arises is whether there are simpler descriptions of $\mathcal{P}oly(G)$ than the one given above. This is indeed the case for bipartite graphs. A vector x in $\mathbf{R}^E$ is *r-regular* if $\partial(v) = r$, for all $v \in V$.

Theorem 1.2
For a bipartite graph G, a vector x in $\mathbf{R}^E$ belongs to $\mathcal{P}oly(G)$ if and only if it is non-negative and 1-regular.

There are non-bipartite graphs whose matching polytopes consist precisely of non-negative 1-regular vectors. However, in general, it is not possible to describe the matching polytope of a graph just by the non-negativity and the degree constraints. The vector x shown in Figure 1 is non-negative and 1-regular, but it is not in the perfect matching polytope of the graph because $x(C) < 1$ for the indicated cut C.

The above observations suggest the following problem:

Problem 1.3 *Characterize graphs G for which a vector in $\mathbf{R}^E$ is in $\mathcal{P}oly(G)$ if and only if it is non-negative and 1-regular.*

The appropriate context for studying this problem is the theory of matching covered graphs.

2 Matching Covered Graphs

The following theorem due to Tutte provides a characterization of graphs that have a perfect matching.

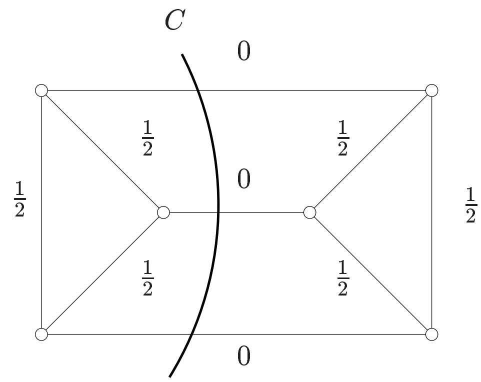

Figure 1: A non-negative 1-regular vector that is not in the matching polytope

THEOREM 2.1 (TUTTE, 1947)
A graph G has a perfect matching if and only if

$$\mathcal{O}(G - S) \leq |S|, \quad \text{for all } S \subset V,$$

where $\mathcal{O}(G - S)$ denotes the number of odd components of $G - S$.

In a graph G with a perfect matching, we shall refer to a subset B of V with $\mathcal{O}(G-B) = |B|$ as a *barrier* of G. See Figure 2.

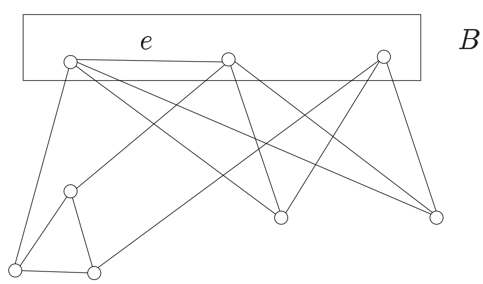

Figure 2: A barrier

An edge of a graph is *admissible* if there is a perfect matching of the graph that contains it. (In the graph shown in Figure 2, edge e is inadmissible, all other edges are admissible.) The following characterization of admissible edges may be deduced easily from Tutte's theorem.

THEOREM 2.2
Let G be a graph with a perfect matching. An edge e of G is admissible if and only if there is no barrier of G that contains both ends of e.

If an edge e of a graph G is not admissible, then $x_e = 0$ for every vector x in $\mathcal{P}oly(G)$. Thus, in studying the properties of matching polytopes of graphs, we may restrict our attention to graphs that do not contain inadmissible edges. Furthermore, if a graph G is disconnected and has G_1, G_2,...,G_k as its components, $\mathcal{P}oly(G)$ may be expressed as the sum of polytopes that are isomorphic to $\mathcal{P}oly(G_1)$, $\mathcal{P}oly(G_2)$,..., $\mathcal{P}oly(G_k)$. Thus, we may restrict our attention to graphs that are connected.

A *matching covered graph* is a connected graph in which every edge is admissible. There is an extensive theory of matching covered graphs (see [10], [9] and [1]). We shall review below some of the definitions and theorems of this subject that are essential for understanding this note.

2.1 Tight Cuts

Every cut C in a matching covered graph G is the coboundary of two complementary subsets of V called the *shores* of C. For a cut $C := \partial(X)$ of G, we denote the graph obtained from G by shrinking the shore $\overline{X}$ to a single vertex $\overline{x}$ by $G\{X;\overline{x}\}$, or simply by $G\{X\}$ if there is no need to refer to the vertex resulting from shrinking $\overline{X}$. Similarly, we denote the graph obtained from G by shrinking the shore X to a single vertex x by $G\{\overline{X};x\}$, or simply by $G\{\overline{X}\}$. We refer to $G\{X\}$ and $G\{\overline{X}\}$ as the *C-contractions of G*.

Let G be a matching covered graph. A cut C of a G is a *separating cut* of G if both C-contractions of G are also matching covered and is *tight cut* of G if every perfect matching of G has exactly one edge in C. Figure 3 shows several examples of tight cuts.

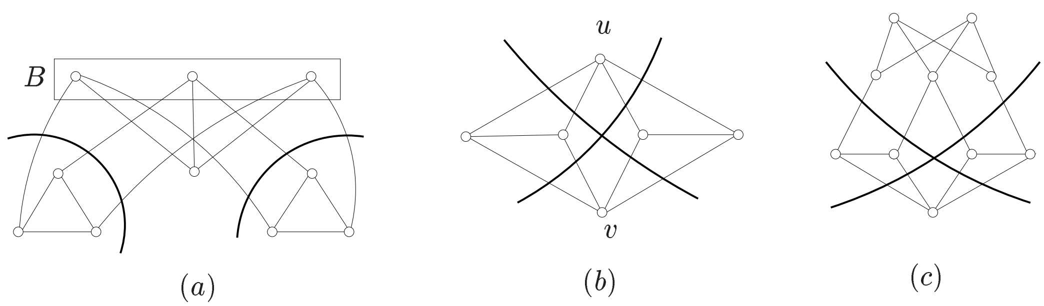

Figure 3: Tight Cuts

The reason for our interest in tight cuts is due to the following property they have:

PROPOSITION 2.3
For any tight cut of a matching covered graph G, both the C-contractions of G are also matching covered.

By the above proposition, every tight cut of a matching covered graph is also a separating cut. However, the converse is not true. For example, the cut C in Figure 1 is a separating cut but it is not tight.

There are two important types of tight cuts associated with nontrivial barriers and 2-separations that are not barriers. If B is a barrier of G and K is a nontrivial odd component of $G - B$, then $\partial(K)$ is a tight cut; such a tight cut is called *barrier cut*. If $\{u, v\}$ is a 2-separation of G and L is an even component of $G - \{u, v\}$, then $\partial(L + u)$ and $\partial(L + v)$ are tight cuts; such tight cuts are called 2-*separation cuts*. Figure 3(a) shows an example of barrier cuts and Figure 3(b) shows an example of 2-separation cuts.

A matching covered graph may have a tight cut that is neither a barrier cut nor a 2-separation cut (see Figure 3(c)). However, graphs with nontrivial tight cuts have the following surprising property.

Theorem 2.4 (Edmonds, Lovász and Pulleyblank, 1982; Szigeti, 2002)
Any matching covered graph that has a nontrivial tight cut has either a barrier cut or a 2-separation cut.

A matching covered graph that is bipartite and has no nontrivial tight cuts is a *brace*, and one that is non-bipartite and has no nontrivial tight cuts is a *brick*. One may establish the following characterizations of braces and bricks that lead to polynomial-time algorithms for their recognition. (The characterization of bricks requires Theorem 2.4.)

Theorem 2.5
A bipartite matching covered graph G with bipartition (X, Y) is a brace if and only if, for any two distinct vertices $x_1, x_2 \in X$ and any two distinct vertices $y_1, y_2 \in Y$, the graph $G - \{x_1, x_2, y_1, y_2\}$ has a perfect matching.

A graph G is *bicritical* if $G - \{u, v\}$ has a perfect matching for any $u, v \in V(G)$. (Using Tutte's Theorem, it can be shown that a bicritical graph cannot have any nontrivial barriers.)

Theorem 2.6
A graph G is a brick if and only if it is 3-connected and bicritical.

By Proposition 2.3, for any nontrivial tight cut C of a matching covered graph G may be decomposed into two smaller matching covered graphs. Thus, by repeated decomposing along nontrivial tight cuts, any matching covered graph may be decomposed into matching covered graphs free of nontrivial tight cuts, that is, into bricks and braces. This procedure is known as a *tight cut decomposition* of G. (Figure 4 illustrates a tight cut decomposition along the two indicated tight cuts. The list of graphs resulting from this decomposition are the 5-wheel W_5, K_4 and $K_{3,3}$.)

The tight cut decomposition procedure has the following striking property.

Theorem 2.7 (Lovász, 1987)
Any two tight cut decompositions of a matching covered graph yield the same list of bricks and braces (except possibly for multiplicities of edges).

In particular, any two tight cut decompositions of a matching covered graph G yield the same number of bricks; we denote this number by $b(G)$. (The number of bricks of the graph in Figure 4 is two.) A *near-brick* is a matching covered graph with just one brick.

The properties of a matching covered graph can often be analyzed by analyzing its bricks and braces separately. For example:

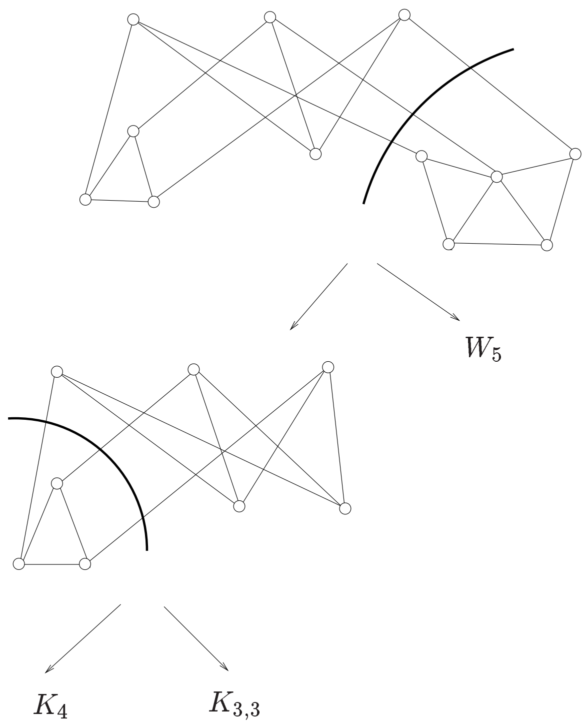

Figure 4: A tight cut decomposition

THEOREM 2.8
Let G be a graph, C a tight cut of G and let G_1 and G_2 be the two C-contractions of G. A vector x in $\mathbf{R}^E$ is in $\mathcal{P}oly(G)$ if and only if the restrictions of x to $E(G_1)$ and $E(G_2)$ are in $\mathcal{P}oly(G_1)$ and $\mathcal{P}oly(G_2)$, respectively.

Thus, to check if a vector x is in $\mathcal{P}oly(G)$, it suffices to check if the restrictions of x to the edge sets of the bricks and braces of G are in their matching polytopes.

To further explore the connections between the perfect matching polytope of a matching covered graph G and its tight cut decompositions, we require the following notion. Let $\mathcal{C}$ be a family of cuts of G. A vector x in $\mathbf{R}^E$ is said to be *r-regular over $\mathcal{C}$* if $x(C) = r$ for every $C \in \mathcal{C}$. (Thus an r-regular vector is one that is regular over the family of all trivial cuts.) We denote the family of all tight cuts (including all trivial cuts) of G by $\mathcal{T}(G)$. Since, for any tight cut C and any every perfect matching M of G, $|C \cap M| = 1$, we have:

PROPOSITION 2.9
Every vector x in $\mathcal{P}oly(G)$ is 1-regular over $\mathcal{T}(G)$.

When $b(G) \leq 1$ (that is, when G is either bipartite or is a near-brick), it can be shown that regularity over the family of all trivial cuts implies regularity over $\mathcal{T}(G)$. This is not true when $b(G) > 1$. Figure 5 shows an example of a 12-regular vector that is not regular over $\mathcal{T}(G)$ (the tight cut C_1 has weight 6).

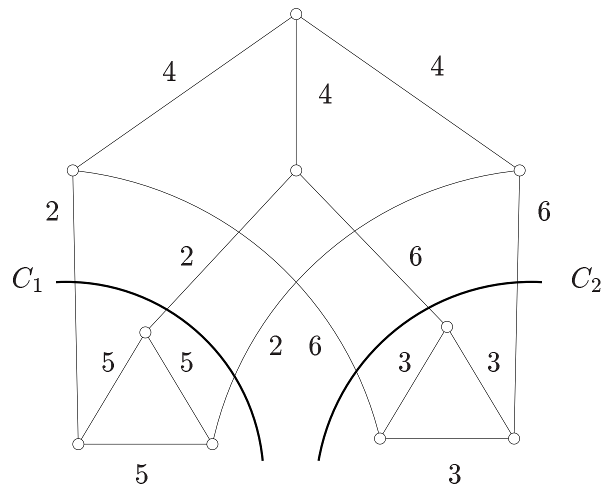

Figure 5: A regular vector that is not regular over $\mathcal{T}$

For the graph G in the above Figure, let x denote the vector obtained by dividing all the coefficients in the above example by 12. This vector x is non-negative and 1-regular. However, since $x(C_1) = \frac{1}{2} < 1$, x is not in $\mathcal{P}oly(G)$. Thus, the perfect matching polytope of this graph cannot be described just by non-negativity and the degree constraints.

We have already noted that if G is bipartite, a vector x in $\mathbf{R}^E$ belongs to $\mathcal{P}oly(G)$ if and only if it is non-negative and 1-regular. Thus, in seeking an answer to Problem 1.3, we may restrict our attention to near-bricks. In fact, in view of Theorem 2.8, we may restrict our attention to bricks.

The appropriate generalization of Problem 1.3 to all matching covered graphs is the following:

Problem 2.10 *Characterize matching covered graphs G for which a vector in $\mathbf{R}^E$ is in $\mathcal{P}oly(G)$ if and only if it is non-negative and 1-regular over $\mathcal{T}(G)$.*

2.2 The dimension of the perfect matching polytope

The *matching space* of G, denoted by $\mathcal{L}in(G)$, is the set of all rational linear combinations of vectors in $\{\chi^M : M \in \mathcal{M}(G)\}$. Using Edmonds' Theorem one may establish the following characterization of the matching space.

Theorem 2.11
A vector x in $\mathbf{R}^E$ belongs to $\mathcal{L}in(G)$ if and only if it is regular over $\mathcal{T}(G)$.

Thus $\mathcal{L}in(G)$ is the set of solutions to the following system of linear equations. (The "dummy variable" r, denoting the degree of regularity, can take any value.)

$$x(C) - r = 0, \quad C \in \mathcal{T}(G) \tag{1}$$

We shall denote the dimension of $\mathcal{L}in(G)$ by $dim(G)$. Edmonds, Lovász and Pulleyblank [8] determined the $dim(G)$ by computing the rank of the coefficient matrix of the above system of equations. The idea they used for computing this rank was also contained in Naddef's earlier work [12].

When G is bipartite, every vector that is regular is also regular over $\mathcal{T}$. Thus, the rank of the coefficient matrix of (1) is the same as the rank of the incidence matrix of G which is $n-1$. Therefore, when G is bipartite $dim(G) = m+1-(n-1) = m-n+2$.

When G is a near-brick, it is still true that every vector that is regular is also regular over $\mathcal{T}$. But the rank of the incidence matrix of a non-bipartite graph is n. Thus, if $b=1$, $dim(G) = m+1-(n) = m-n+1$.

In general, given any non-bipartite matching covered graph G, using a tight cut decomposition of G, it is possible to find $b-1$ nontrivial tight cuts C_1, C_2,...,C_{b-1} such that any vector that is regular over $\{\nabla(v), v \in V\} \cup \{C_1, C_2, ..., C_{b-1}\}$ is also regular over $\mathcal{T}(G)$. (For example, for the graph in Figure 5, it can be shown that any vector that is regular over $\{\nabla(v), v \in V\} \cup \{C_1\}$ is also regular over $\mathcal{T}(G)$.) Thus the rank of the coefficient matrix of (1) is at most $n+b-1$. It can be shown, by induction on b, that the rank is in fact equal to $n+b-1$. We therefore have:

Theorem 2.12 (Edmonds, Lovász, Pulleyblank, 1982)
$dim(G) = m-n+2-b$.

2.3 Removable Edges

An edge e of a matching covered graph G is *removable* if $G-e$ is matching covered. A removable edge e is b-invariant if $b(G-e) = b(G)$. If an edge e of a matching covered graph G is b-invariant, it follows from Theorem 2.12 that $dim(G) = dim(G-e)+1$. In this case, if $\mathcal{B}(G-e)$ is any basis of $\mathcal{L}in(G-e)$ and M_e is any perfect matching of G containing the edge e, then $\mathcal{B}(G-e) \cup \{\chi^{M_e}\}$ is a basis of $\mathcal{L}in(G)$.

Of the three bricks shown in Figure 6, K_4 and $\overline{C_6}$ do not have any removable edges. Although every edge of the Petersen graph P is removable, none is b-invariant. (Since P is a brick, $b(P) = 1$. However, it can be checked that, for any e of P, $b(P-e) = 2$.)

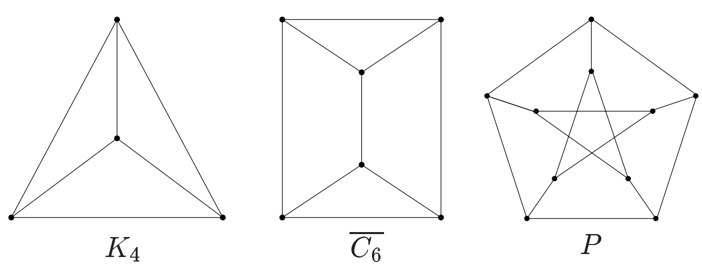

Figure 6: The three basic bricks

Confirming a conjecture of Lovász, we proved the following theorem concerning the existence of b-invariant edges in bricks.

THEOREM 2.13 (CARVALHO, LUCCHESI, MURTY, 2002)
Every brick distinct from the three basic bricks shown in Figure 6 has a b-invariant edge.

2.4 Characteristic of a Separating Cut

The notion of the characteristic of a matching covered graph, defined below, played a crucial role in our proof of Theorem 2.13. Here we shall recall the pertinent properties of this parameter.

For an odd cut C of a matching covered graph G and each for an odd integer i, we define $\mathcal{M}_i(C)$ as follows:

$$\mathcal{M}_i(C) = \{M : M \in \mathcal{M}, |\, M \cap C\,| = i\}. \tag{2}$$

For each separating cut C of G, the *characteristic* $\lambda(C)$ *of* C is defined as follows:

$$\lambda(C) := \begin{cases} \min\{i > 1 : \mathcal{M}_i(C) \neq \emptyset\}, & \text{if } C \text{ is not tight} \\ \infty, & \text{otherwise.} \end{cases}$$

The *characteristic* $\lambda(G)$ *of* a matching covered graph G is defined as follows:

$$\lambda(G) := min\{\lambda(C) : \ C \text{ is separating}\}.$$

This parameter satisfies the following interesting properties.

THEOREM 2.14
Let G be a matching covered graph, and let e be a removable edge of G. Then $\lambda(G - e) \geq \lambda(G)$.

THEOREM 2.15
The characteristic of a matching covered graph G is the minimum of the characteristics of its bricks and braces.

A matching covered graph G is *solid* if $\lambda(G) = \infty$. In other words, G is solid if, and only if, each of its separating cuts is tight.

It is easy to show that all bipartite matching covered graphs are solid. In particular, all braces are solid. However, not all bricks are solid. For example, both $\overline{C_6}$ and P have characteristics three and five, respectively, and hence, they are not solid. It follows from Theorem 2.15 that a matching covered graph G is solid if and only if each of its bricks is solid.

To prove Theorem 2.13 it was necessary for us to establish a property of the Petersen graph that distinguishes it from all other bricks. The following theorem, proved in [2], turned out to be what we needed.

THEOREM 2.16
Every nonsolid brick has characteristic three or five. The only simple brick of characteristic five is the Petersen graph.

2.5 Robust Cuts

Our attempts to prove Theorem 2.13 by induction led us to consider separating cuts (which were used earlier by Lovász in [9]). Our strategy to prove that theorem was to first show that every solid brick distinct from K_4 has a b-invariant edge and then to try to prove the assertion for nonsolid bricks by decomposing them along separating cuts and applying induction. If C is a separating cut of a brick G, the two C-contractions of G are matching covered, by definition, but they need not be bricks or even near-bricks. This difficulty had to be overcome before implementing our inductive strategy for proving Theorem 2.13. This led us to the notion of a robust cut. A cut C of a brick G is a *robust cut* of G if both the C-contractions of G are near-bricks. We proved the following theorem in [2].

Theorem 2.17
Every nonsolid brick has a robust cut.

Our proof of Theorem 2.17 required the use of a certain partial order defined on the set of all nontrivial separating cuts of a brick. Let C and D be two nontrivial separating cuts of a brick G. Cut D *precedes* C (written as $D \preceq C$) if $|M \cap D| \leq |M \cap C|$ for each perfect matching M of G. We were able to show that any nontrivial separating cut of a brick that is minimal with respect to this precedence relation is a robust cut. Underlying this proof was an algorithm which accepts a separating cut of G as input and yields a robust cut of G as output.

3 The Matching Polytope and Solid Bricks

In this section we shall provide an answer to Problem 1.3. In view of Theorem 2.8, we may restrict ourselves to bricks.

Theorem 3.1
For a brick G, $\mathcal{P}oly(G)$ consists of all non-negative 1-regular vectors if and only if G is solid.

Proof: Firstly suppose that G is not solid. We wish to show that there is some non-negative 1-regular vector in $\mathbf{R}^E$ that does not belong to $\mathcal{P}oly(G)$. Since G is nonsolid, it has a nontrivial separating cut C. Let M_0 be a perfect matching of G such that $|M_0 \cap C| > 1$. (Such a perfect matching must exit; otherwise C would be tight.) Also, since C is separating, for every edge e of G, there is a perfect matching M_e of G such that $M_e \cap C = \{e\}$. Now let

$$x := \frac{1}{|M_0| - 1} \left(\left(\sum_{e \in M_0} \chi^{M_e} \right) - \chi^{M_0} \right)$$

Clearly the vector x is non-negative, 1-regular with $x(C) < 1$.

Conversely, suppose that G is solid. We wish to prove that every non-negative 1-regular vector in $\mathbf{R}^E$ belongs to $\mathcal{P}oly(G)$. Assume to the contrary that there is a non-negative 1-regular vector x that does not belong to $\mathcal{P}oly(G)$. Then, by Theorem 1.1 there must exist odd cuts C with $x(C) < 1$. Let $\mathcal{C}$ denote the set of all cuts C for which $x(C) < 1$ and let

$D := \partial(Y)$ be a cut in $\mathcal{C}$ that is minimal with respect to the precedence relation $\preceq$ defined in the previous section. We shall obtain a contradiction by showing that D is a separating cut.

Consider the D-contraction $G_1 := G\{Y\}$. We wish to show that G_1 is matching covered. If it is not, then either there is a subset S of $V(G_1)$ such that either (i) $\mathcal{O}(G_1 - S) > |S|$ or (ii) $\mathcal{O}(G_1 - S) = |S|$, but there is an edge e of G_1 with both its end in S (see Figure 7). In either case, there must be an odd component K of $G_1 - S$ for which $x(\partial(K)) < 1$. Such a component is clearly nontrivial. One may verify that $K \preceq D$, contradicting the choice of D. Therefore G_1 is matching covered. Similarly, $G_2 := G\{\overline{S}\}$ is also matching covered and D is a separating cut. A contradiction. □

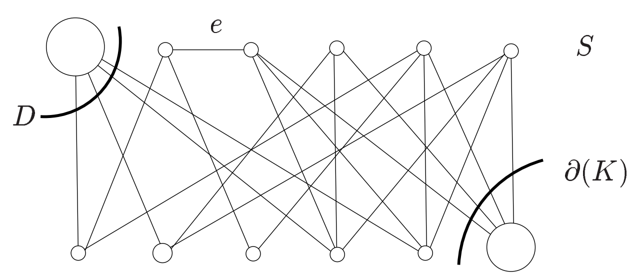

Figure 7: Finding a separating cut using precedence

The above theorem may also be proved using the well-known characterization of the facets of the matching polytope. From this theorem, we may now deduce the following answer to Problem 2.10.

Theorem 3.2
The perfect matching polytope $\mathcal{P}oly(G)$ of a matching covered graph G consists of non-negative and 1-regular over $\mathcal{T}(G)$ if and only if G is solid.

4 A Characterization of Solid Bricks

If a brick G is nonsolid, a separating cut of G serves as a succinct certificate for demonstrating that G is nonsolid. Thus, the problem of deciding whether or not an input brick G is solid is in $co-\mathcal{NP}$. We do not know if this decision is problem is in $\mathcal{NP}$. In this section we shall present the following attractive characterization of nonsolid bricks that may be helpful in constructing examples of solid.

Theorem 4.1 (Reed and Wakabayashi, 2003)
A brick G has a nontrivial separating cut if and only if it has two disjoint odd circuits C_1 and C_2 such that $G - (V(C_1) \cup V(C_2))$ has a perfect matching.

The proof of the above theorem that we shall present here is based on the following refinement of Theorem 2.17 proved in [2]. (See the Acknowledgment for an account of the genesis of this theorem.)

Theorem 4.2
Let G be a brick that has a nontrivial separating cut. Then there exist subsets X and Y of V such that:

1. *$Y \subseteq \overline{X}$,*
2. *the graphs $G_1 := G\{X\}$ and $G_2 := G\{Y\}$ are bricks, and*
3. *the graph G' obtained from G by shrinking X to a vertex x and Y to a vertex y is a bipartite matching covered graph containing x and y in different parts of its bipartition. (See Figure 8.)*

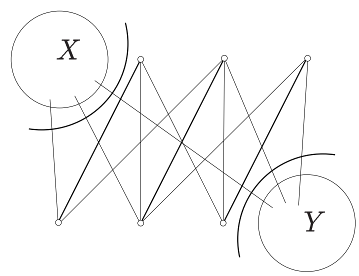

Figure 8: Robust cuts in nonsolid bricks

A graph G is *critical* if $G - v$ has a perfect matching for each $v \in V$. Clearly, any odd circuit is critical. In fact, any critical graph G has an ear-decomposition $(G_1, G_2, ..., G_r)$ such that G_1 is an odd circuit and, for $2 \leq i \leq r$, G_i is obtained from G_{i-1} by adding an odd ear. We therefore have:

Lemma 4.3 (Lovász)
Every nontrivial critical graph G contains an odd circuit C such that $G - V(C)$ has a perfect matching.

In addition to the above lemma, we require the following two lemmas. We shall refer to a subset X of V such that $G[X]$ is critical as a *critical set*.

Lemma 4.4
Let G be a matching covered graph and let X be a critical subset of V and let $C := \partial(X)$. Then the C-contraction $G_1 := G\{X; \overline{x}\}$ is matching covered.

Proof: Clearly G_1 is connected. Since $G[X]$ is critical, for any vertex v in X, the graph $G_1 - \{u, \overline{x}\}$ has a perfect matching. It follows that G_1 has a perfect matching and that there is no nontrivial barrier of G_1 containing $\overline{x}$. Suppose that there is an edge $e := uv$ of

G_1 that is not admissible. Then there is a barrier B of G_1 containing u and v. By the above observation, $\overline{x}$ is not a vertex of B. Thus $\overline{x}$ is in one of the components of $G_1 - B$. But this implies that B is also a barrier of G and that e is inadmissible in G. This is impossible because, by hypothesis, G is matching covered. □

LEMMA 4.5 (SEE [10])

If we delete one vertex from each part of any bipartite matching covered graph. the resulting graph has a perfect matching.

THEOREM 4.6

A brick G is nonsolid if and only if there exists a critical subset X of V such that there is another critical subset Y of V disjoint from X with the property that $G - (X \cup Y)$ has a perfect matching.

Proof: Firstly, suppose that G has a separating cut. Then, by Theorem 4.2, there exist two subsets X and Y of V satisfying the three conditions in the statement of that Theorem. Since $G\{X\}$ and $G\{Y\}$ are bicritical, $G[X]$ and $G[Y]$ are critical. And, by Lemma 4.5, $G - (X \cup Y)$ has a perfect matching.

Conversely, suppose that G is a brick and it has subsets X satisfying the hypothesis. Of all the critical subsets that satisfy those conditions, choose X for which $\partial(X)$ is minimal with respect to precedence. It follows from Lemma 4.4 that $G_1 := G\{X; \overline{x}\}$ is matching covered. To complete the proof, it suffices to show that $G_2 := G\{\overline{X}; x\}$ is matching covered. Clearly G_2 is connected. Let S be any maximal subset of $V(G_2)$ such that $\mathcal{O}(G_2 - S) \geq |S|$. Since G itself is a brick, x must be a vertex of S. By the maximality of S, each component of $G_2 - S$ is odd and critical. Furthermore, by hypothesis $G - (X \cup Y)$ has a perfect matching and, since $G[Y]$ is critical, for any $y \in Y$, $G_2 - \{x, y\}$ has a perfect matching. From this we can conclude that $\mathcal{O}(G_2 - S) = |S|$ and that Y is a subset of the vertex set of one of the components of $G_2 - S$. It follows that $\mathcal{O}(G - S)$ cannot be greater than $|S|$ and hence that G_2 has a perfect matching and that S is a barrier of G_2. (See Figure 9.)

Suppose now that G_2 has an inadmissible edge e. Then there exists a maximal barrier S of G_2 that contains both ends of e. Arguing as above we may conclude that $x \in S$, all components of $G_2 - S$ are odd and critical and that the vertex set of a single component of $G_2 - S$ contains Y. Let Z be the vertex set of the component of $G_2 - S$ that contains S. From the fact that $G_2 - \{x, y\}$ has a perfect matching for any $y \in Y$, it follows that $G - (Z \cup X)$ has a perfect matching. Clearly $|M \cap \partial(Z)| \leq |M \cap \partial(X)|$, for any perfect matching M of G. On the other hand, if M_e is any perfect matching of G containing e, $|M_e \cap \partial(Z)| < |M_e \cap \partial(X)|$. Thus $\partial(Z)$ strictly precedes $\partial(X)$. As Z and X are disjoint critical sets and $G - (Z \cup X)$ has a perfect matching, the choice of X is contradicted. We conclude that G_2 is matching covered and that $\partial(X)$ is a separating cut of G. □

We note that underlying the above proof there is an algorithm. Given any pair of disjoint critical subsets X and Y of V such that $G - (X \cup Y)$ has a perfect matching, this algorithm can be used to find a separating cut of G.

In view of Lemma 4.3, Theorem 4.1 is a corollary of Theorem 4.6.

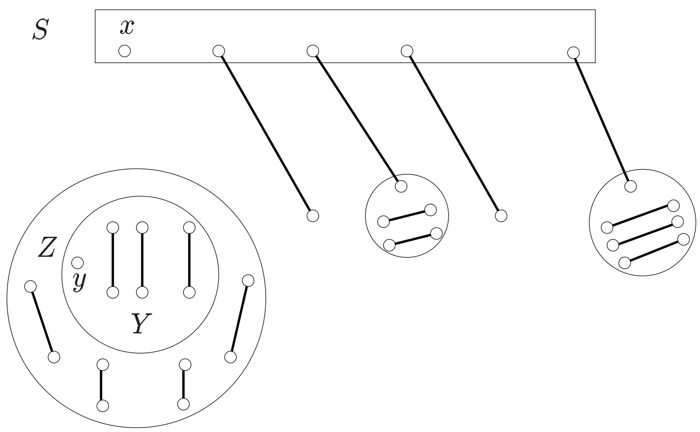

Figure 9: A perfect matching of $G_2 - \{x, y\}$

5 Examples of Solid Bricks

Theorem 4.1 can be used show that certain classes of bricks are solid. A graph is *odd-intercyclic* if any two of its odd circuits have at least one vertex in common. It follows from Theorem 4.1 that every odd-intercyclic brick is solid. (This fact can be established by simple arguments, see [1].) Odd wheels and Möbius ladders described below are examples of odd-intercyclic solid bricks.

Odd Wheels: The *wheel of order* $n \geq 3$, denoted by W_n, is obtained by adjoining a vertex h to a circuit R of length n and joining it to each vertex of R; h and R are referred to as the *hub* and the *rim* of W_n, respectively. (The wheel W_3 of order three is isomorphic to K_4; any one of its four vertices may be regarded as its hub.) A wheel is *odd* or *even* according to the parity of its order. It is easy to show that every odd wheel is an odd-intercyclic brick.

Möbius ladder: The ladder L_{2n}, $n \geq 2$, is obtained from two disjoint paths $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ by adding the edges $x_i y_i$, $1 \leq i \leq n$. The Möbius ladder M_{2n}, $n \geq 2$, is obtained from L_{2n} by joining x_1 to y_n and y_1 to x_n. This graph is Hamiltonian and is isomorphic to the cubic graph obtained from the circuit $(0, 1, ..., 2n - 1)$ by joining each vertex i to the vertex $i + n$ $(mod\ 2n)$. It can be shown that, for any odd integer $n \geq 3$, M_{2n} is a brace, and for any even integer $n \geq 2$, M_{2n} is an odd-intercyclic brick.

A complete characterization of odd-intercyclic graphs was discovered more than ten years ago by Gerards et al [7]. An infinite class of odd-intercyclic graphs may be obtained as follows. Let H be a 2-connected planar bipartite graph and let $(v_1, v_2, ..., v_{2k})$ be a facial circuit of H. Obtain G from H by joining, for $1 \leq i \leq k$, the vertices v_i and v_{i+k} by a new edge. Such a graph G has an embedding on the projective plane so that all faces are even and it is not too difficult to see that it is is odd-intercyclic. See Figure 10.

We do not know the answer to the following problem.

Problem 5.1 *Characterize odd-intercyclic bricks.*

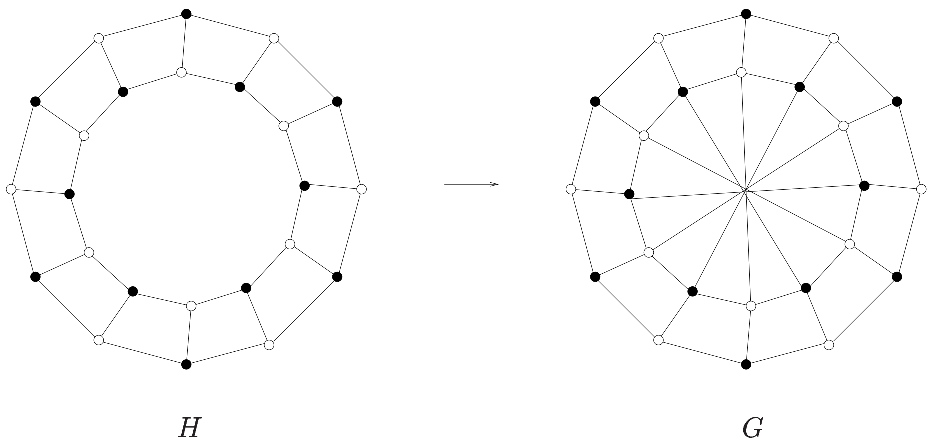

Figure 10: A construction of odd-intercyclic graphs

All the examples of solid bricks that we know that are not odd-intercyclic seem to be ‘close’ to being odd-intercyclic. One such family is described below. (Our proof that the bricks in this family are solid was based on Theorem 4.2. It would be an interesting exercise to prove this result using Theorem 4.1.)

An infinite family of solid bricks that are not odd-intercyclic: Let $n \geq 3$ be an odd integer. Consider the brace M_{2n}. Obtain the cubic graph H from M_{2n} by deleting the vertex n, adding three new vertices u, v and w, and joining u to $n-1$, v to 0, w to $n+1$ and u, v and w to each other. (Thus H is obtained by splicing M_{2n} and K_4.) Now obtain the graph S_{2n+2} from H by joining 1 and $2n-1$. See Figure 11. It can be shown that S_{2n+2} is a solid brick for every odd integer $n \geq 3$.

The first general result concerning solid bricks we proved is contained in [4]. A brick G is *extremal* if the total number of perfect matchings of G is equal to $dim(G)$. We showed that every simple extremal solid brick is an odd wheel. It follows from this that every extremal brick can be obtained by splicing together graphs whose underlying simple graphs are odd wheels.

Using a method of generating bricks that we established in [5], we began a systematic investigation of solid bricks. One general result that we were able to establish is the following.

THEOREM 5.2 (SEE [5])
Every simple planar solid brick is an odd wheel.

As mentioned earlier, we do not know a polynomial-time algorithm for deciding whether or not a given brick is solid. We do not even know if the related decision problem is in $\mathcal{NP}$!

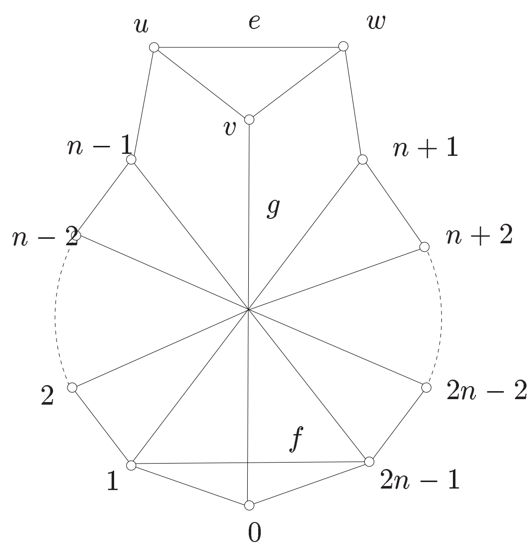

Figure 11: The solid brick $G := S_{2n+2}$

6 Separating Cut Decompositions

By considering cut-contractions with respect to nontrivial separating cuts, one may decompose a matching covered graph G into graphs that are free of nontrivial separating cuts. (In particular, one may decompose a brick into solid bricks.) We shall refer to such a decomposition as a *separating cut decomposition* of G. Unfortunately, separating cut decompositions do not have many of the attractive and useful properties, such as uniqueness, enjoyed by tight cut decompositions. For example, a brick may have two separating cut decompositions giving rise to different numbers of solid bricks, see [3]. However, in spite of this, we believe that separating cut decompositions would be useful in understanding the structure of matching covered graphs (as they were in our work [1] and [2]).

7 Acknowledgment

The notion of a solid brick was introduced by us in [1]. We made crucial use of the properties of solid bricks in our proof of a conjecture of Lovász concerning bricks (see [1] and [2]). But we were unable to determine the complexity status of the problem of deciding whether or not a given brick is solid. To the best of our knowledge, this problem is still open.

In January 2003, Cláudio Lucchesi was invited to present a series of lectures at the University of São Paulo (USP) on the above-mentioned unsolved problem. During the course of these lectures, Cristina Fernandes raised the following perspicacious question: Is it true that a brick G is solid if and only if its perfect matching polytope is determined by non-negativity and the degree constraints. The answer to this question happens to be 'yes', as demonstrated by Theorem 3.1. This was contained in the work that Lucchesi did in 2001 ([11]) on the perfect matching cone of a brick.

Towards the end of Lucchesi's course at USP, Bruce Reed turned up in São Paulo. With his characteristic vigour, he participated in discussions with the USP group and suggested the insightful characterization of nonsolid bricks described in Theorem 4.1. Our attempts to devise an algorithm for the solid brick recognition problem based on this theorem led us to the formulation of Theorem 4.6 and its proof presented here.

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