The Vertex Deletion Number and Splitting Number of a Triangulation of $C_n \times C_m$

Cândido F. X. de Mendonça N.
Érico F. Xavier *Jorge Stolfi* *Luerbio Faria*
Celina M. H. de Figueiredo

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Cândido F. X. de Mendonça N.[1,2]
xavier@din.uem.br

Érico F. Xavier[1]
exavier@dcc.unicamp.br

Luerbio Faria[3,5]
luerbio@cos.ufrj.br

Jorge Stolfi[1]
stolfi@dcc.unicamp.br

Celina M. H. de Figueiredo[4,5]
celina@cos.ufrj.br

Abstract

The vertex deletion number $\phi(G)$ of a graph G is the minimum number of vertices that must be deleted from G to produce a planar graph. The splitting number $\sigma(G)$ of G is the smallest number of vertex splitting operations that must be applied to G to make it planar. Here we determine these topological invariants for the graph family $\mathcal{T}_{C_n \times C_m}$, a regular triangulation of the torus obtained by adding parallel diagonal edges to the faces of the rectangular toroidal grid $C_n \times C_m$. Specifically, we prove that the obvious upper bound $\phi = \sigma = \min\{n, m\}$ is also a lower bound.
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1 Introduction

A *map* M is a graph G_M plus a permutation φ_M of its edge endpoints. As is well-known, a map defines implicitly a topological embedding of the graph G_M in some 2D manifold. The orbits of φ_M are the faces of the embedding.

A (*two-dimensional*) *toroidal mesh* is a map M whose underlying manifold is a torus, with two authomorphisms τ and ς such that (i) $\varsigma\tau = \tau\varsigma$, (2) $u\tau$ and $u\varsigma$ are neighbors of u, for any u, and (3) the set of vertices $\{u\tau^m\varsigma^n : m, n \in Z\}$ covers the whole graph G.

Because of their symmetry and regularity, toroidal meshes are popular topologies for the connection networks of SIMD parallel machines. The automorphisms τ and ς represent the

[1]Institute of Computing, State University of Campinas (Unicamp), SP, Brazil
[2]Informatics Department, State Univ. of Maringá (UEM), PR, Brazil
[3]Teachers Preparation School, State Univ. of Rio de Janeiro (UERJ), São Gonçalo, RJ, Brazil
[4]Institute of Mathematics, Federal Univ. of Rio de Janeiro (UFRJ), RJ, Brazil
[5]COPPE Computer Systems Dept., Federal Univ. of Rio de Janeiro (UFRJ), RJ, Brazil

basic "parallel data shifting" operations whereby each node passes some datum to a specific neighbor in the network. One important information is the topological invariants such as vertex deletion number, splitting number, skewness and crossing number as a mesure of nonplanarity of a graph G_M. There are several application which make use of this information such as Graph Drawing applications and VLSI design.

One of the most popular of the regular toroidal meshes is the $C_n \times C_m$ graphs for which entire articles were dedicated to proving the minimum number of crossings in optimum drawings [15, 3, 6, 1, 2, 23], and other planarity invariants such as skewness and splitting number [20, 10, 21, 24]. In this work we give a proof that the vertex deletion number and splitting number of $\mathcal{T}_{C_n \times C_m}$ is $\min\{n, m\}$. This graph consists of a regular triangulation of the torus formed by adding the edges $v_{i,j}v_{(i+1) \bmod n, (j+1) \bmod m}$ to each vertex of $C_n \times C_m$.

A *simple drawing* of a graph G is a drawing of G on the plane such that no edge crosses itself, adjacent edges do not cross, crossing edges do so only once, edges do not cross vertices, and no more than two edges cross at a common point. A graph is *planar* when there is a simple drawing for this graph in the plane such that no edges cross. In what follows, all drawings are assumed to be simple.

In our proofs we depend heavily on the following characterization by Kuratowski[18]: a graph is planar if and only if it does not contains a subdivision of K_5 or $K_{3,3}$ (see Figure 1) as a subgraph.

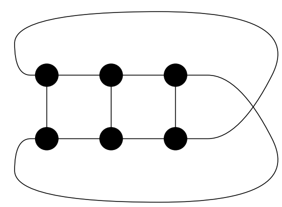

Figure 1: $K_{3,3}$.

A drawing of a graph G is optimum when it has the minimum number of crossings among all drawings of G. This number is called the *crossing number* of G and is denoted by $\nu(G)$.

The *skewness* $\kappa(G)$ is the smallest integer $k \geq 0$ such that the removal of k edges from G yields a planar graph.

The *vertex deletion number* $\phi(G)$ is the smallest integer $k \geq 0$ such that the removal of k vertices from G yields a planar graph.

The *splitting number* $\sigma(G)$ of a graph is the smallest integer $k \geq 0$ such that a planar graph can be obtained from G by k vertex splitting operations. A *vertex splitting operation*, or simply *splitting*, of a vertex $v \in V(G)$ partitions the set of neighbors of v into two nonempty sets P_1 e P_2 and adds to $G \backslash v$ two new and nonadjacent vertices v_1 and v_2, such that P_1 is the set of neighbors of v_1 and P_2 is the set of neighbors of v_2. If a graph H is obtained from G by a sequence of k splittings, we say that H is the *resulting graph* of this set of k splittings in G.

Some aspects of the study of splitting number have been considered by Eades and Mendonça [8, 7]: they successfully used splitting numbers in layout algorithm design.

Very little is known about vertex deletion number, splitting numbers, skewness or crossing numbers for specific classes of graphs. The corresponding decision problems are all NP-complete [12, 11, 13]. For a fixed k, CROSSING NUMBER turns to be polynomial [12], recently Robertson and Seymour [22] have shown VERTEX DELETION NUMBER, SPLITTING NUMBER and SKEWNESS also turn to be polynomial. The difficulty of finding the values of these invariants can justify entire articles in which just one type of graph is considered. For instance, the crossing numbers for the graphs $C_3 \times C_3$, $C_4 \times C_4$, $C_6 \times C_6$ and $C_7 \times C_7$ were recently established [15, 6, 1, 2], the splitting number for the graph Q_4 was established in [10]. The knowledge of the smallest nonplanar element in a class of graphs can help to find the values or bounds for this invariant for every element in the class. For instance, the crossing number of the $C_3 \times C_n$ was established in [3] using the crossing number of the $C_3 \times C_3$. Also the splitting number of the Q_4 which is isomorphic to $C_4 \times C_4$ was used in [20] to determine the lower bound for the graphs $C_n \times C_m$ where $n, m \geq 4$.

The vertex deletion number has been computed for $C_n \times C_m$. This number is (except for a few values of n and m) the same as the vertex splitting number and skewness [21]. The splitting number has been computed for complete graphs [16], for complete bipartite graphs [17] and for $C_n \times C_m$ graphs [20]. The skewness has been computed for Q_n cubes [5] and for $C_n \times C_m$ graphs [20]. The crossing number has been computed for $C_n \times C_m$ graphs [23]. Bound for the crossing number have been computed for complete graphs [14] for the complete bipartite graphs [4] and for n-cubes [9, 19, 25].

Note that the vertex deletion number is trivial for the complete graphs K_n (which is $n - 4$ if $n > 4$) and for the complete bipartite graphs $K_{n,m}$ (which is $min\{n, m\} - 2$ if $min\{n, m\} > 2$).

Research on vertex deletion number can be also justified by the interdependency of the crossing number, skewness, splitting number and vertex deletion number. The following three Lemmas show that for any graph G, $\nu(G) \geq \kappa(G) \geq \sigma(G) \geq \phi(G)$.

Lemma 1.1 *For all graph G, $\nu(G) \geq \kappa(G)$,*

Proof. Consider an optimum drawing of a graph G with $\nu(G)$ crossings, now for each pair of edges that cross remove one of the edges. The removal of this set of edges of size at most $\nu(G)$ produces a planar graph from G which implies that $\nu(G) \geq \kappa(G)$. □

Lemma 1.2 *For all graph G, $\kappa(G) \geq \sigma(G)$.*

Proof. Let H be a subgraph of G obtained by the removal of $k = \kappa(G)$ edges of G. For each edge $e_i = u_i v_i$ $(i = 1, 2, ..., k)$ removed from G to build H, build a splitting operation in u_i such that the new vertices u_i' and u_i'' have neighborhood $N(u_i') = N(u_i) \setminus \{v_i\}$ and $N(u_i'') = \{v_i\}$. □

Lemma 1.3 *For all graph G, $\sigma(G) \geq \phi(G)$.*

Proof. Delete vertices instead of splitting them. □

A *chordless circuit* or simply *circuit* C_k, $k \geq 3$ of a graph G is a set of vertices $C_k = \{v_0, v_1, ..., v_{k-1}\}$ where each vertex v_i has exactly two neighbors $v_{(i-1) \bmod k}$ and $v_{(i+1) \bmod k}$ in C_k. We say that a C circuit is a *k-circuit* if it is a circuit of k vertices.

Let q and r be the maximum common divisor and minimum common multiple of n and m, respectively. A *triangulation of* $C_n \times C_m$, denoted by $\mathcal{T}_{C_n \times C_m}$, is a graph with nm vertices where each vertex $v_{i,j}$ $(i = 0, 1, ..., n-1$ and $j = 0, 1, ..., m-1)$ has exactly six neighbors $v_{(i-1) \bmod n, j}$, $v_{(i+1) \bmod n, j}$, $v_{i,(j-1) \bmod m}$, $v_{i,(j+1) \bmod m}$ $v_{(i-1) \bmod n,(j-1) \bmod m}$ and $v_{(i+1) \bmod n,(j+1) \bmod m}$. Let a *row n-circuit* be the m n-circuits

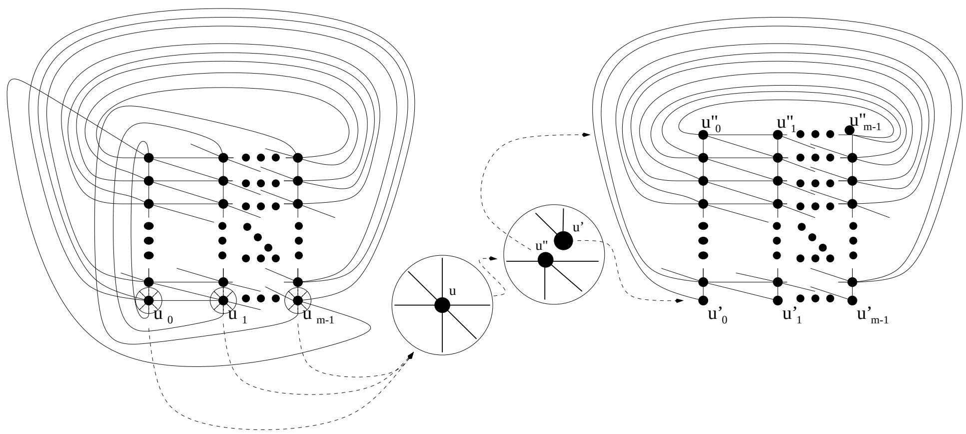

Figure 2: $\sigma(\mathcal{T}_{C_n \times C_m}) \leq \min\{n, m\}$.

$R_n^j = \{v_{0,j}, v_{1,j}, ..., v_{n-1,j}\}$ (for $j = 0, 1, ..., m-1$), a *column m-circuit* be the n m-circuits $C_m^i = \{v_{i,0}, v_{i,1}, ..., v_{i,m-1}\}$ (for $i = 0, 1, ..., n-1$), and a *diagonal r-circiuit* be the q r-circuits $C_r^k = \{v_{k,0}, v_{(k+1)\bmod n,1}, ..., v_{(k+r-1)\bmod n,(r-1)\bmod m}$ (for $k = 0, 1, ..., q-1$. Note that this triangulation does not cover all regular triangulation of the torus.

Two graphs G and H are *isomorphic* if there is a bijection $\psi : VG \to VH$ such that two distinct vertices x and y of G are adjacent if and only if the vertices $\psi(x)$ and $\psi(y)$ are adjacent in H. Such a function is called an *isomorphism* from G to H. It is obvious that $\mathcal{T}_{C_n \times C_m}$ is isomorphic to $\mathcal{T}_{C_m \times C_n}$.

An *automorphism* of a graph G is an isomorphism between G and itself. We observe that $C_n \times C_m$ has $4nm$ automorphisms if $n \neq m$, and $8nm$ if $n = m$.

Given a graph G and a subgraph S of G, we say that G is *S-transitive* if for each pair F, H subgraphs of G, where F and H are isomorphic to S, there is an automorphism α of G such that if $v \in V(F)$, then $\alpha(v) \in V(H)$.

It is an easy exercise to show that the graph $\mathcal{T}_{C_n \times C_m}$ is vertex-transitive. Therefore, a particular vertex may be chosen without loss of generality.

Our strategy in this work is as follows. In section 2 we show that the upper bound of the splitting number of $\mathcal{T}_{C_n \times C_m}$ is at most $\min\{n, m\}$. In section 3 we show that the lower bound of the vertex deletion number of $\mathcal{T}_{C_n \times C_m}$ is at least $\min\{n, m\}$.

2 Upper bounds for $\sigma(\mathcal{T}_{C_n \times C_m})$

Theorem 2.1 *The splitting number of $\mathcal{T}_{C_n \times C_m}$ is at most* $\min\{n, m\}$.

Proof. Without loss of generality we may suppose that $m \leq n$. Figure 2 displays a planar drawing of the graph obtained after $\min\{n, m\} = m$ splitting operations of the $\mathcal{T}_{C_n \times C_m}$. Therefore, $\sigma(\mathcal{T}_{C_n \times C_m}) \leq \min\{n, m\} = m$. □

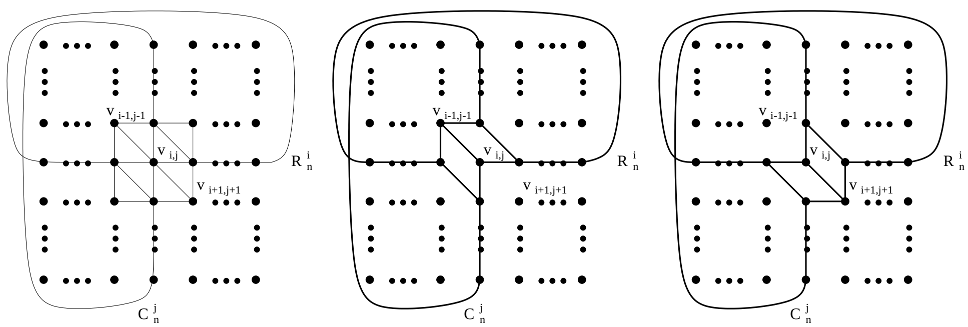

Figure 3: If $v_{(i-1)\bmod n,(j-1)\bmod n}$ or $v_{(i+1)\bmod n,(j+1)\bmod n}$ do not belong to D then H contains a subdivision of $K_{3,3}$

3 Lower bounds for $\phi(\mathcal{T}_{C_n\times C_m})$

Lemma 3.1 *The vertex deletion number of $\mathcal{T}_{C_2\times C_3}$ is at least* 2.

Proof. The graph $\mathcal{T}_{C_2\times C_3}$ contains a subgraph isomorphic to K_6. Therefore, $\phi(\mathcal{T}_{C_2\times C_3}) \geq 2$. □

Lemma 3.2 *The vertex deletion number of $\mathcal{T}_{C_n\times C_n}$ is at least n.*

Proof. Let $G = (V, E)$ be the graph $\mathcal{T}_{C_n\times C_n}$ and D be a subset of the vertices of G such that $|D| = k = n - 1 \geq 2$. Let H be the subgraph of G induced by $V \setminus D$. Let s be number of different rows n-circuits that intersects D. We prove the assertion by induction in s.

Since $k < n$, H contains a column n-circuit C_n^j.

Base: there are two cases.

case 1: n is odd and $s < \frac{n}{2}$. In this case by the pigeon hole principle there are at least two consecutive rows n-circuits, lets say R_n^i and $R_n^{(i+1)\bmod n}$. Therefore, $C_n^j \cup R_n^i \cup \{v_{(i+1)\bmod n,(j+1)\bmod n}\} \subset H$ contains a subdivision of $K_{3,3}$ (see Figure 3).

case 2: n is even and $s = \frac{n}{2}$. In this case, if at least 2 rows n-circuits are consecutive we have a subdivision of $K_{3,3}$ as in the previous case. Otherwise (there are not 2 consecutives rows n-circuits) there is at least one vertex $w_l = v_{(l-1)\bmod n,(j-1)\bmod n}$ or $w_l = v_{(l+1)\bmod n,(j+1)\bmod n}$ (for each row n-circuit R_n^l) that does not belong to D. Therefore, $C_n^j \cup R_n^i \cup \{w\} \subset H$ contains a subdivision of $K_{3,3}$ as shown in Figure 3.

Hippothesis: If $s < k$ then H contains a subdivision of $K_{3,3}$.

Thesis: $s = k$. In this case, h contains at least 1 row n-circuit $R_n^i = \{v_{i,0}, v_{i,1}, ..., v_{i,n-1}\}$ such that $D \cap R_n^i = \emptyset$. If at least one of the vertices $v_{(i-1)\bmod n,(j-1)\bmod n}$ or $v_{(i+1)\bmod n,(j+1)\bmod n}$ does not belong to D then H contains a subdivision of $K_{3,3}$ (see Figure 3). Conversely, if both vertices belong to D consider the automorphism φ of H where $\varphi(v_{t,u}) = v'_{t,u} = v_{(t-u+j)\bmod n,(2j-u)\bmod n}$. Note that $\varphi(H)$ keeps the vertex $v_{i,j}$ in the same position. Furthermore, the row n-circuit R'^i_n contains both vertices $v'_{i,(j-1)\bmod n} = v_{(i+1)\bmod n,(j+1)\bmod n}$ and $v'_{i,(j+1)\bmod n} = v_{(i-1)\bmod n,(j-1)\bmod n}$. Therefore, the number of intersections between D and the n rows n-circuits of $\varphi(H)$ is at most $s-1$ which implies by the induction hippothesis it contains a subdivision of $K_{3,3}$.

The subdivision of $K_{3,3}$ found in H implies that it is not planar. Therefore, $\phi(G) \geq n$. □

Corollary 3.3 *The vertex deletion number of* $\mathcal{T}_{C_n \times C_m}$ *is at least* $\min\{n, m\}$*.*

Proof. Whitout loss of generality suppose that $m \geq n$. Now contract the edges belonging to the comlumn m-circuits between the rows n-circuits R_n^0 and R_n^1 $m-n$ times, if $n, m \geq 3$, otherwise contract only $m-n+1$ times. Next, remove all multiple edges. The remaining graph is a $\mathcal{T}_{C_n \times C_n}$ when $n, m \geq 3$ and a $\mathcal{T}_{C_2 \times C_3}$, otherwise. It is a well kown result that both operations (edges contaction an edge deletion) do not increase the vertex deletion number. Therefore, in this case, it follows from this fact and from Lemma 3.2 and Lemma 3.1 that $\phi(\mathcal{T}_{C_n \times C_m}) \geq n = \min\{n, m\}$. □

Theorem 3.4 *The vertex deletion number and splitting number of* $\mathcal{T}_{C_n \times C_m}$ *is* $\min\{n, m\}$*.*

Proof. The assertion follows from Theorem 2.1 and Corollary 3.3. □

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