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The Vertex Deletion Number of $C_n \times C_m$

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The Vertex Deletion Number of $C_n \times C_m$*

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Abstract

The vertex deletion number of a graph G is the smallest integer $k \geq 0$ such that there is an planar induced subgraph of G obtained by the removal of k vertices of G. The $C_n \times C_m$ graphs has distinguished place in Computer Science. Several authors have devoted articles to proving the minimum number of crossings in optimum drawings [15, 3, 6, 1, 2, 22], and other planarity invariants such as skewness and splitting number [20, 10, 23]. In this work we give a proof that the vertex deletion number of the $C_n \times C_m$ is $\min\{n, m\} - \xi_{5,9}(n, m)$, where $\xi_{i,j}(k_1, k_2)$ is the number of true conditions among the following: (i) $k_1 = k_2 \leq i$ and (ii) $k_1 + k_2 \leq j$.

Keywords: vertex deletion number, vertex splitting number, skewness, planarity invariants.

1 Introduction

Graph Drawing applications for visualization or VLSI projects require layout techniques of nonplanar graphs. However, the wealth of layout algorithms are limited to a special class of graphs, particularly to planar graphs. These algorithms are useless for nonplanar graphs. One possible approache to handling nonplanarity in graph drawing algorithms is to consider topological invariants of the graph such as the vertex deletion number which are used as measure of nonplanarity. Research on topological properties of the $C_n \times C_m$ graphs is important for applications such as parallel processing. In this article we prove that the vertex deletion of $C_n \times C_m$ is $\min\{n, m\} - \xi_{5,9}(n, m)$, where $\xi_{i,j}(k_1, k_2)$ is the number of true conditions among the following: (i) $k_1 = k_2 \leq i$ and (ii) $k_1 + k_2 \leq j$ (see Figure 2).

A *simple drawing* of a graph G is a drawing of G on the plane such that no edge crosses itself, adjacent edges do not cross, crossing edges do so only once, edges do not cross vertices, and no more than two edges cross at a common point. A graph is *planar* when there is a simple drawing for this graph in the plane such that no edges cross. In what follows, all drawings

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are assumed to be simple. In our proofs we depend heavily on the following characterization by Kuratowski[18]: a graph is planar if and only if it does not contains a subdivision of K_5 or $K_{3,3}$ as a subgraph. In fact we only use the nonplanarity of the subdivision of $K_{3,3}$ (see Figure 1).

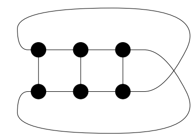

Figure 1: $K_{3,3}$.

A drawing of a graph G is optimum when it has the minimum number of crossings among all drawings of G. This number is called the *crossing number* of G and is denoted by $\nu(G)$.

The *skewness* $\kappa(G)$ is the smallest integer $k \geq 0$ such that the removal of k edges from G yields a planar graph.

The *vertex deletion number* $\phi(G)$ is the smallest integer $k \geq 0$ such that the removal of k vertices from G yields a planar graph.

The *splitting number* $\sigma(G)$ of a graph is the smallest integer $k \geq 0$ such that a planar graph can be obtained from G by k vertex splitting operations. A *vertex splitting operation*, or simply *splitting*, of a vertex $v \in V(G)$ partitions the set of neighbors of v into two nonempty sets P_1 e P_2 and adds to $G \backslash v$ two new and nonadjacent vertices v_1 and v_2, such that P_1 is the set of neighbors of v_1 and P_2 is the set of neighbors of v_2. If a graph H is obtained from G by a sequence of k splittings, we say that H is the *resulting graph* of this set of k splittings in G.

Some aspects of the study of splitting number have been considered by Eades and Mendonça [8, 7]: they successfully used splitting numbers in layout algorithm design.

Very little is known about vertex deletion number, splitting numbers, skewness or crossing numbers for specific classes of graphs. The corresponding decision problems for general graphs are all NP-complete [12, 11, 13]. For a fixed k, CROSSING NUMBER turns to be polynomial [12], recently Robertson and Seymour [21] have shown VERTEX DELETION NUMBER, SPLITTING NUMBER and SKEWNESS also turn to be polynomial. The difficulty of finding the values of these invariants can justify entire articles in which just one type of graph is considered. For instance, the crossing numbers for the graphs $C_3 \times C_3$, $C_4 \times C_4$, $C_6 \times C_6$ and $C_7 \times C_7$ were recently established [15, 6, 1, 2], the splitting number for the graph Q_4 was established in [10]. The knowledge of the smallest nonplanar element in a class of graphs can help to find the values or bounds for this invariant for every element in the class. For instance, the crossing number of the $C_3 \times C_n$ was established in [3] using the crossing number of the $C_3 \times C_3$. Also the splitting number of the Q_4 which is isomorphic to $C_4 \times C_4$ was used in [20] to determine the lowerbound for the graphs $C_n \times C_m$ where $n, m \geq 4$.

The splitting number has been computed for complete graphs [16], for complete bipartite graphs [17] and for the $C_n \times C_m$ graphs [20]. The skewness has been computed for the n-cube graphs Q_n [5] and for the $C_n \times C_m$ graphs [20]. The crossing number has been computed for $C_n \times C_m$ graphs [22]. Bound for the crossing number have been computed for complete graphs [14] for the complete bipartite graphs [4] and for n-cubes [9, 19, 24].

Note that the vertex deletion number is trivial for the complete graphs K_n (which is $n-4$ if $n > 4$) and for the complete bipartite graphs $K_{n,m}$ (which is $\min\{n,m\} - 2$ if $\min\{n,m\} > 2$). However, we show in this work that for the $C_n \times C_m$ this number is not trivial, and except for a few values of n and m it is the same as the vertex splitting number and skewness [20].

Research on vertex deletion number can be also justified by the interdependency of the crossing number, skewness, splitting number and vertex deletion number. The following three Lemmas show that for any graph G, $\nu(G) \geq \kappa(G) \geq \sigma(G) \geq \phi(G)$.
F.or all graph G, $\nu(G) \geq \kappa(G)$,

Proof. Consider an optimum drawing of a graph G with $\nu(G)$ crossings, now for each pair of edges that cross remove one of the edges. The removal of this set of edges of size at most $\nu(G)$ produces a planar graph from G which implies that $\nu(G) \geq \kappa(G)$. □

F.or all graph G, $\kappa(G) \geq \sigma(G)$.

Proof. Let H be a subgraph of G obtained by the removal of $k = \kappa(G)$ edges of G. For each edge $e_i = u_i v_i$ $(i = 1, 2, ..., k)$ removed from G to build H, build a splitting operation in u_i such that the new vertices u_i' and u_i'' have neighborhood $N(u_i') = N(u_i) \setminus \{v_i\}$ and $N(u_i'') = \{v_i\}$. □

F.or all graph G, $\sigma(G) \geq \phi(G)$.

Proof. Delete vertices instead of splitting them. □

A *chordless circuit* or simply *circuit* C_k, $k \geq 3$ of a graph G is a set of vertices $C_k = \{v_0, v_1, ..., v_{k-1}\}$ where each vertex v_i has exactly two neighbors $v_{(i-1)\bmod k}$ and $v_{(i+1)\bmod k}$ in C_k.

A $C_n \times C_m$ *graph* is a graph with nm vertices where each vertex $v_{i,j}$ $(i = 0, 1, ..., n-1$ and $j = 0, 1, ..., m-1)$ has exactly four neighbors $v_{(i-1)\bmod n, j}$, $v_{(i+1)\bmod n, j}$, $v_{i,(j-1)\bmod m}$ and $v_{i,(j+1)\bmod m}$. It is an easy exercise to show that $C_n^j = \{v_{0,j}, v_{1,j}, ..., v_{n-1,j}\}$ is a circuit C_n in $C_n \times C_m$ for $j = 0, 1, ..., m-1$ and that $C_m^i = \{v_{i,0}, v_{i,1}, ..., v_{i,m-1}\}$ is a circuit C_m in $C_n \times C_m$ for $i = 0, 1, ..., n-1$.

Two graphs G and H are *isomorphic* if there is a bijection $\psi : VG \to VH$ such that two distinct vertices x and y of G are adjacent if and only if the vertices $\psi(x)$ and $\psi(y)$ are adjacent in H. Such a function is called an *isomorphism* from G to H. It is obvious that $C_n \times C_m$ is isomorphic to $C_m \times C_n$.

An *automorphism* of a graph G is an isomorphism between G and itself. We observe that $C_n \times C_m$ has $4nm$ automorphisms if $n \neq m$, and $8nm$ if $n = m$.

We define the *i,j-small-values-detection function* $\xi_{i,j} : N^2 \to \{0, 1, 2\}$ so that $\xi_{i,j}(k_1, k_2)$ is the number of true conditions among the following:

(i) $k_1 = k_2 \leq i$, and

(ii) $k_1 + k_2 \leq j$.

$\xi_{5,9}$	3	4	5	6	7
3	2	1	1	1	0
4	1	2	1	0	0
5	1	1	1	0	0
6	1	0	0	0	0
7	0	0	0	0	0

Figure 2: Values of $\xi_{5,9}(n, m)$.

Given a graph G and a subgraph S of G, we say that G is *S-transitive* if for each pair F, H subgraphs of G, where F and H are isomorphic to S, there is an automorphism α of G such that if $v \in V(F)$, then $\alpha(v) \in V(H)$.

It is an easy exercise to show that the graph $C_n \times C_m$ is vertex-transitive. Therefore, a particular vertex may be chosen without loss of generality.

Our strategy in this work is as follows. In section 2 we show that the upperbound of the vertex deletion number of $C_n \times C_m$ is at most $\min\{n,m\} - \xi_{5,9}(n,m)$. In section 3 we show that the lowerbound of the vertex deletion number of $C_n \times C_m$ is at least $\min\{n,m\} - \xi_{5,9}(n,m)$.

2 Upperbounds for $\phi(C_n \times C_m)$

The vertex deletion number of the $C_n \times C_m$ graphs is at most $\min\{n,m\} - \xi_{5,9}(n,m)$.

Proof. Figure 3 (a) displays the $C_3 \times C_3$ graph and a planar drawing of the induced graph after the removal of one vertex indicated by ✱. Dotted lines indicate isomorphism. Thus, $\phi(C_3 \times C_3) \leq 1$. Figure 3 (b), (c), (d) and (e) display the graphs $C_3 \times C_4$, $C_3 \times C_5$, $C_3 \times C_6$ and $C_4 \times C_4$, respectively, and planar drawings of the induced subgraphs after the removal of two vertices. Thus, $\phi(C_3 \times C_4) \leq \phi(C_3 \times C_5) \leq \phi(C_3 \times C_6) \leq \phi(C_4 \times C_4) \leq 2$. Figure 3 (f) displays the $C_4 \times C_5$ and a planar drawing after the removal of three vertices. Thus, $\phi(C_4 \times C_5) \leq 3$. Figure 3 (g) displays the $C_5 \times C_5$ and a planar drawing of the induced subgraph after the removal of four vertices. Thus, $\phi(C_5 \times C_5) \leq 4$ Finally, Figure 3 (h) displays the $C_n \times C_m$ graph and a planar drawing of the induced subgraph after the removal of $\min\{n,m\}$. Thus, $\phi(C_n \times C_m) \leq \min\{n,m\}$ vertices. All these results can be summarized by the inequation $\phi(C_n \times C_m) \leq \min\{n,m\} - \xi_{5,9}(n,m)$. □

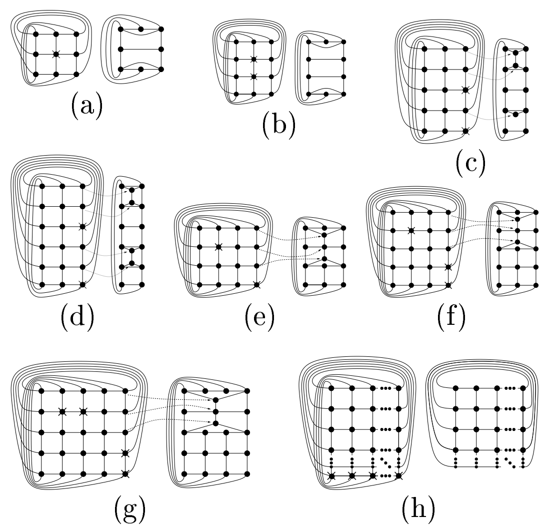

Figure 3: $\phi(C_n \times C_m) \leq \min\{n,m\} - \xi_{5,9}(n,m)$.

3 Lowerbounds for $\phi(C_n \times C_m)$

To prove our claim we must show that some induced subgraphs of the $C_n \times C_m$ contains a subdivision of the $K_{3,3}$. These graphs have in some cases a surprisingly enormous number of analogous cases. Therefore, we will represent such subdivisions of the $K_{3,3}$ and induced subgraphs as follows:

- the vertices $v_{i,j}$ are represented by the integer points $\{(i,j), 0 \leq i < n, 0 \leq j < m\}$,
- deleted vertices are drawn with the symbol $\times$,

- edges used by the subdivision of the $K_{3,3}$ are drawn solid and vertices of the $K_{3,3}$ are drawn as large circles with degree 3,
- each horizontal half-edge drawn along the left edge side of the grid connects to the half-edge at the right side, on the same row; and analogously for vertical half-edges,
- all other vertices are drawn as small dots, and all other edges are omitted.

To reduce the amount of work we wrote two simple combinatorics programs: **Find-Analogous** and **Find-$K_{3,3}$**. The former generates all the non-analogous subgraphs of $C_n \times C_m$ that result from the deletion of k vertices, for given n, m and k. We say that two subgraphs of $C_n \times C_m$ are *analogous* if they are isomorphic by an automorphism of $C_n \times C_m$. Note that two non-analogous subgraphs may be isomorphic. Due to the automorphisms of $C_n \times C_m$ we need to generate only subgraphs with the top left corner vertex deleted. This reduces the number of subgraphs that needs to be considered from $\binom{nm}{k}$ to $\binom{nm-1}{k-1}$. Furthermore when deciding whether a subgraph is analogous to a previously generated one, we need to consider only $4k$ or $8k$ automorphisms of $C_n \times C_m$, instead of $4nm$ or $8nm$.

The second program **Find-$K_{3,3}$** checks whether each subgraph of $C_n \times C_m$ generated by **Find-Analogous** contains a subdivision of $K_{3,3}$. The subdivisions of the $K_{3,3}$ are added to a list by the user. If **Find-$K_{3,3}$** fails to find a subdivision of $K_{3,3}$ it stops printing the subgraph. If **Find-$K_{3,3}$** finds a subdivision of $K_{3,3}$ for each subgraph of $C_n \times C_m$ generated by **Find-Analogous** it prints all solutions.

T.he vertex deletion number of $C_3 \times C_3$ is at least 1.

Proof. The graph $C_3 \times C_3$ contains a subdivision of $K_{3,3}$ as shown in Figure 4. Therefore, it is not planar which implies that $\phi(C_3 \times C_3) \geq 1$. □

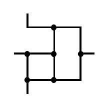

Figure 4: $\phi(C_3 \times C_3) \geq 1$.

T.he vertex deletion number of $C_3 \times C_4$ is at least 2.

Proof. Let G be the subgraph induced by all vertices of the $C_3 \times C_4$ minus one vertex. Without loss of generality we suppose that the deleted vertex is at the top left corner as shown in Figure 5. This graph contains a subdivision of $K_{3,3}$. Therefore, it is not planar which implies that $\phi(C_3 \times C_4) \geq 2$.

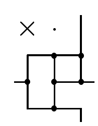

Figure 5: $\phi(C_3 \times C_4) \geq 2$. □

T.he vertex deletion number of $C_3 \times C_5$, $C_3 \times C_6$ and $C_4 \times C_4$ are at least 2.

Proof. All of these graphs contain a subdivision of $C_3 \times C_4$. □

T.he vertex deletion number of $C_3 \times C_7$ is at least 3.

Proof. Figure 6 displays the 7 non-analogous possible ways to delete 2 vertices from $C_3 \times C_7$ generating different induced subgraphs. Figure 6 also displays a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $\phi(C_3 \times C_7) \geq 3$. □

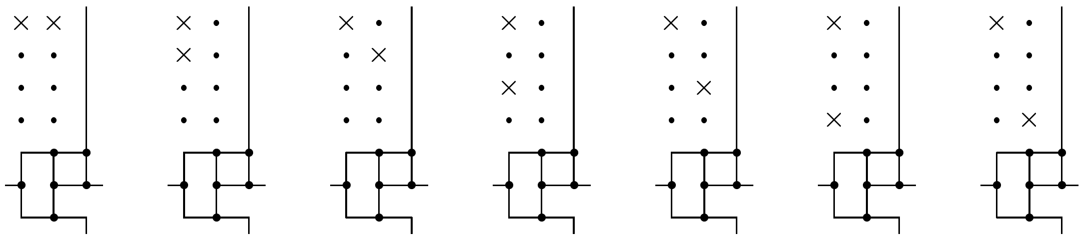

Figure 6: $\phi(C_3 \times C_7) \geq 3$.

T.he vertex deletion number of $C_3 \times C_m$, where $m \geq 7$ is at least 3.

Proof. The graph $C_3 \times C_m$ contains a subdivision of $C_3 \times C_7$ which has vertex deletion number at least 3. □

T.he vertex deletion number of $C_4 \times C_5$ is at least 3.

Proof. Figure 7 displays the 8 non-analogous subgraphs obtained by deleting 2 vertices from $C_4 \times C_5$. Figure 7 also displays a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $\phi(C_4 \times C_5) \geq 3$. □

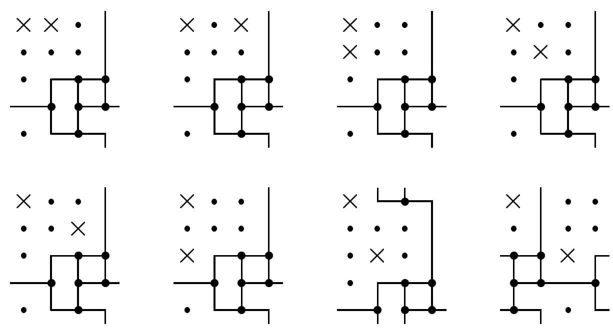

Figure 7: $\phi(C_4 \times C_5) \geq 3$.

T.he vertex deletion number of $C_4 \times C_6$ is at least 4.

Proof. Figure 8 displays the 34 non-analogous subgraphs obtained by deleting 3 vertices from $C_4 \times C_6$. Figure 8 also displays a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $\phi(C_4 \times C_6) \geq 4$. □

T.he vertex deletion number of $C_4 \times C_m$, where $m \geq 6$ is at least 4.

Proof. The graph $C_4 \times C_m$ contains a subdivision of $C_4 \times C_6$ which has vertex deletion number at least 4. □

T.he vertex deletion number of $C_5 \times C_5$ is at least 4.

Proof. Figure 9 displays the 19 non-analogous ways to delete 3 vertices from $C_5 \times C_5$. Figure 9 also displays a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $\phi(C_5 \times C_5) \geq 4$. □

T.he vertex deletion number of $C_5 \times C_6$ is at least 5.

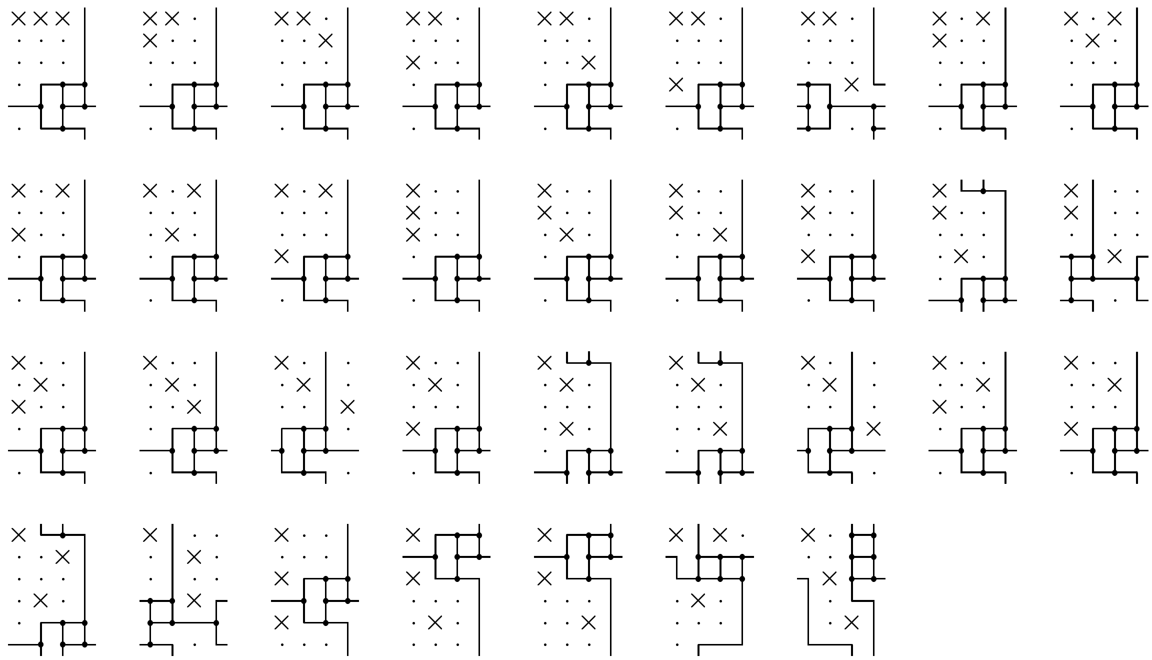

Figure 8: $\phi(C_4 \times C_6) \geq 4$.

Proof. Figures 10 to 14 display the 291 non-analogous subgraphs obtained by deleting 4 vertices from $C_5 \times C_6$. Figure 10 to 14 also display a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $\phi(C_5 \times C_6) \geq 5$. □

T.he vertex deletion number of $C_5 \times C_m$, where $m \geq 6$ is at least 5.

Proof. The graph $C_5 \times C_m$ contains a subdivision of $C_5 \times C_6$ which has vertex deletion number at least 5. □

T.he vertex deletion number of $C_6 \times C_6$ is at least 6.

Proof. Figures 10 to 38 display the 1455 non-analogous subgraphs obtained by deleting 5 vertices from $C_6 \times C_6$. Figures 15 to 38 also display a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $\phi(C_5 \times C_6) \geq 5$. □

T.he vertex deletion number of $C_k \times C_k$ for an integer $k \geq 6$ is at least k.

Proof. We will prove this assertion by induction in k. The induction basis is the graph $C_6 \times C_6$. The induction hypothesis is that for all graphs $C_l \times C_l$ the vertex deletion number is at least l, where $6 \leq l < k$. Without loss of generality we may suppose that the vertex s at the top left corner is deleted. The remaining graph has a subdivision of $C_{k-1} \times C_{k-1}$. It follows from the induction hypothesis the $C_k \times C_k \backslash s$ has vertex deletion number at least $k-1$ and therefore, the graph $C_k \times C_k$ has vertex deletion number at least k. □

T.he vertex deletion number of $C_n \times C_m$, where $n, m \geq 6$ is at least $\min\{n, m\}$.

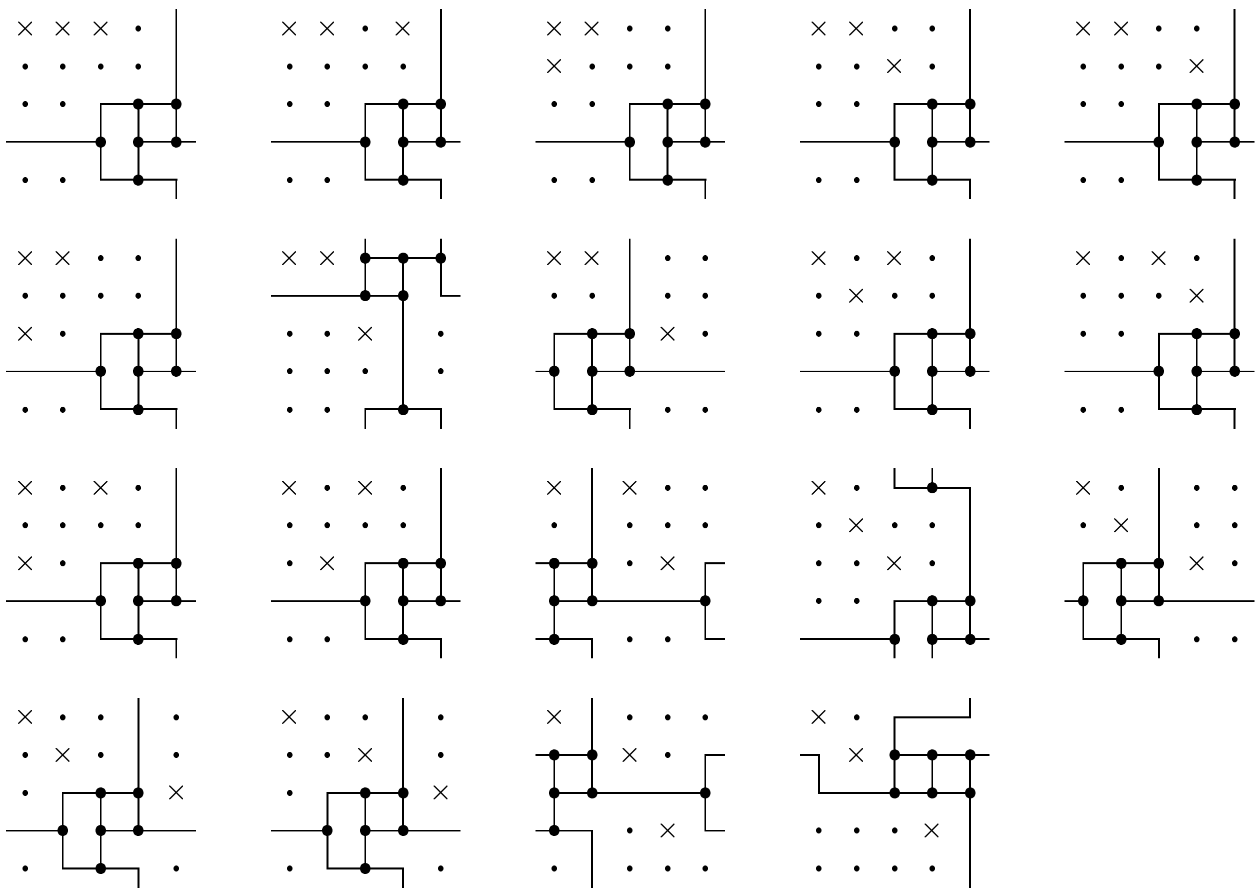

Figure 9: $\phi(C_5 \times C_5) \geq 4$.

Proof. The graph $C_n \times C_m$ contains a subdivision of $C_n \times C_n$ which has vertex deletion number at least n. □

The vertex deletion number of $C_n \times C_m$ is at least $\min\{n, m\} - \xi_{5,9}(n, m)$.

Proof. The assertion follows from Lemma 3, Lemma 3, Corollary 3, Lemma 3, Corollary 3, Lemma 3, Lemma 3, Corollary 3, Lemma 3, Lemma 3, Corollary 3, Lemma 3, Lemma 3 and Corollary 3. □

The vertex deletion number of $C_n \times C_m$ is $\min\{n, m\} - \xi_{5,9}(n, m)$.

Proof. The assertion follows from Theorem 2 and Theorem 3. □

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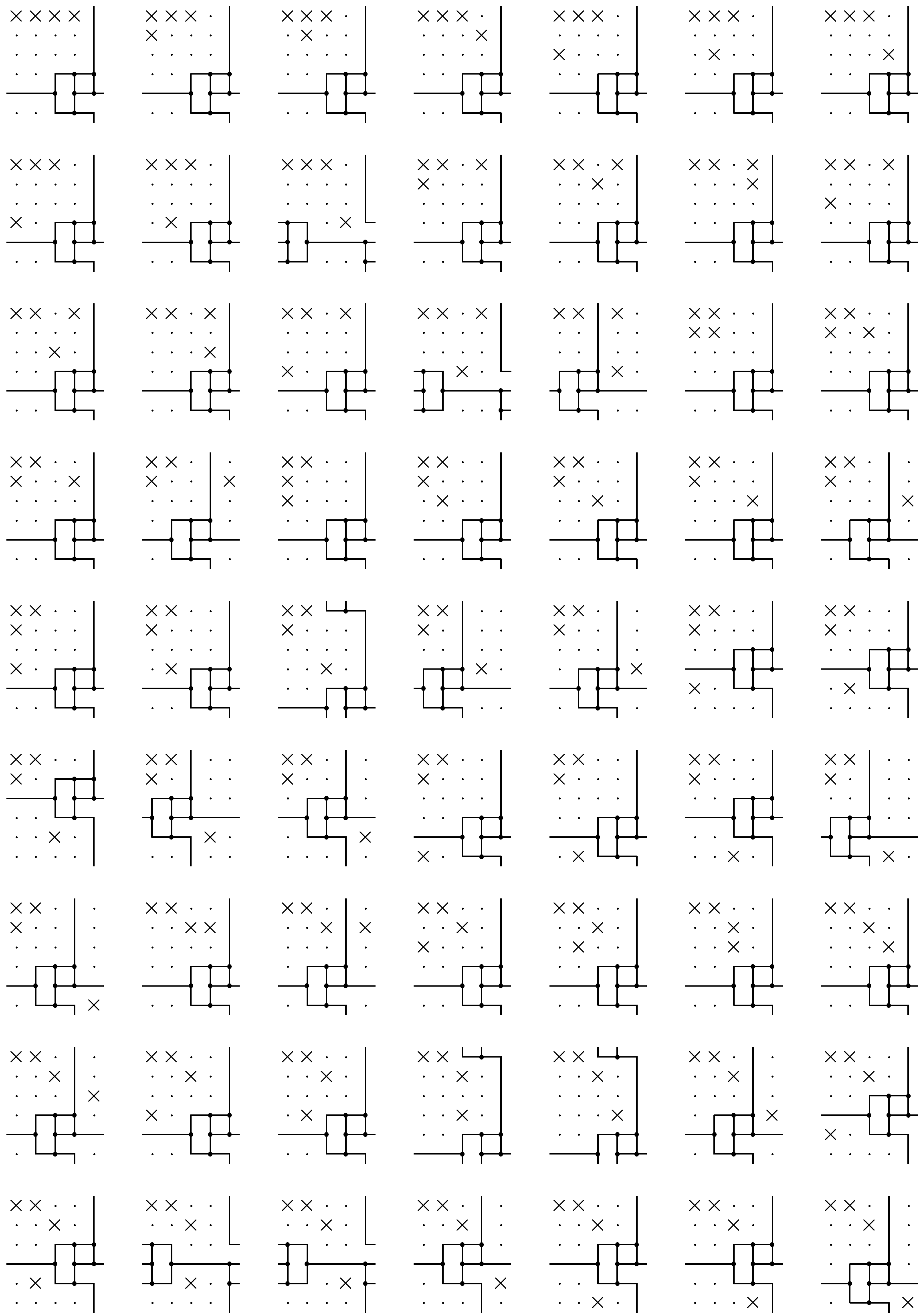

Figure 10: $\phi(C_5 \times C_6) \geq 5$.

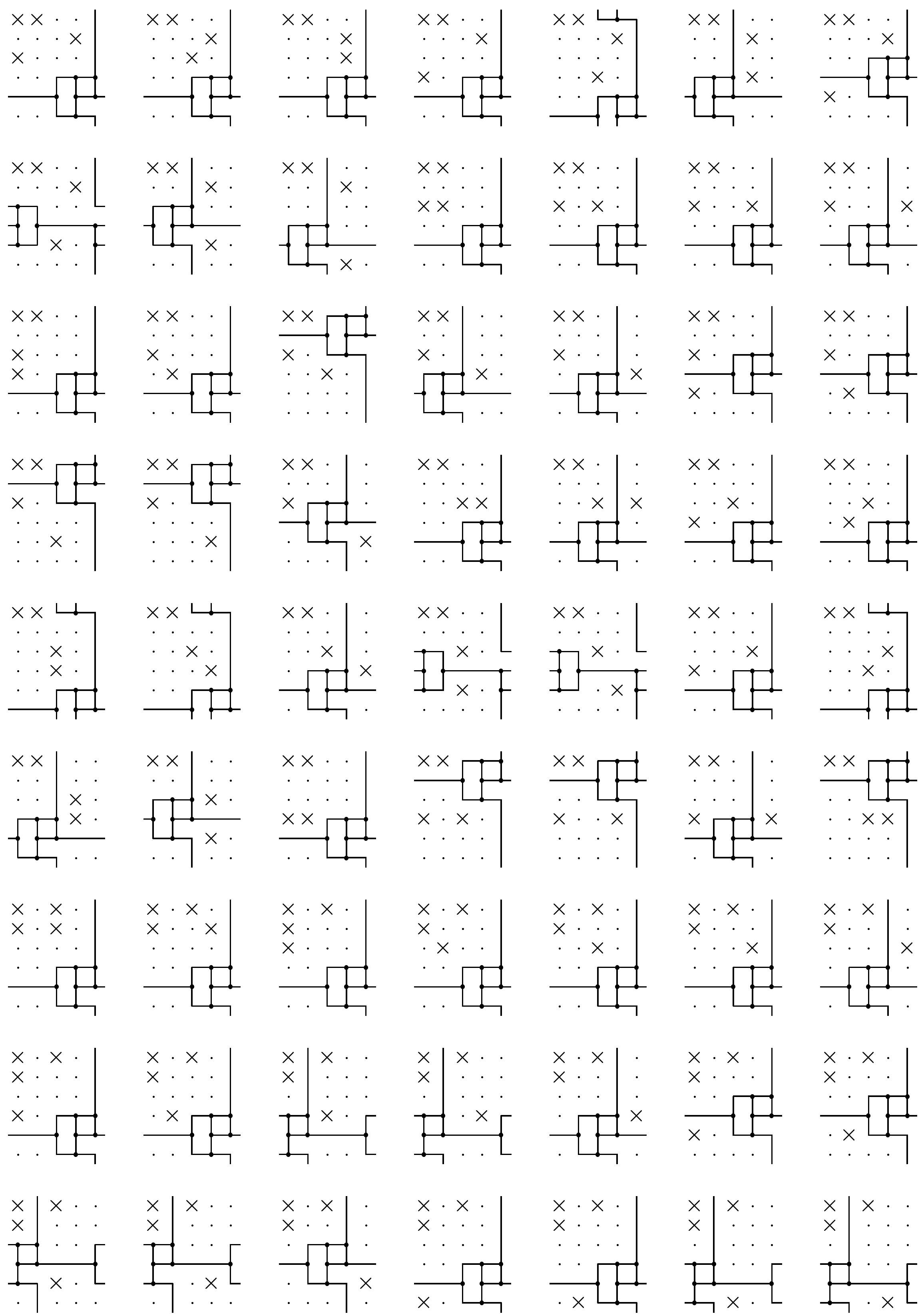

Figure 11: $\phi(C_5 \times C_6) \geq 5$.

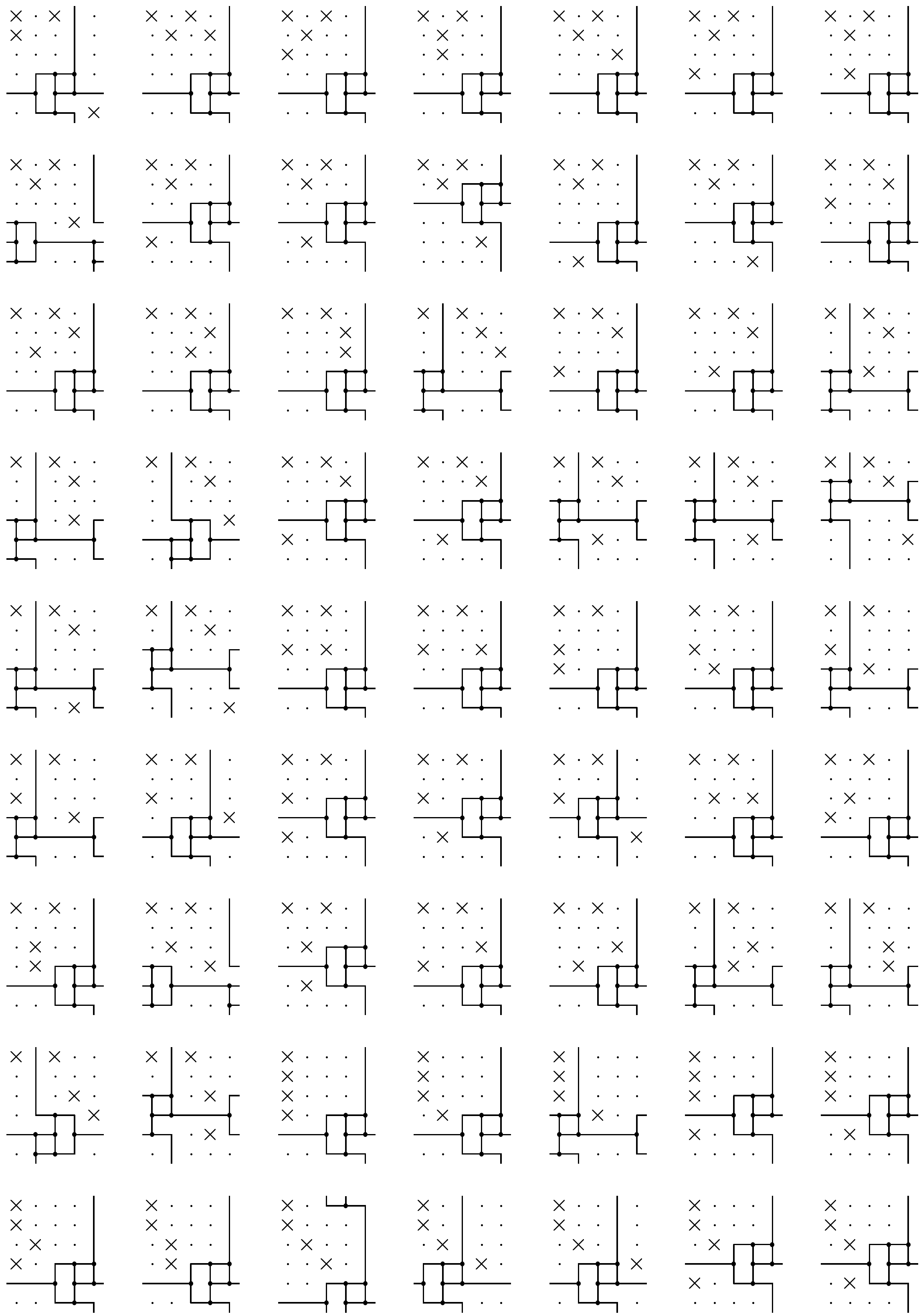

Figure 12: $\phi(C_5 \times C_6) \geq 5$.

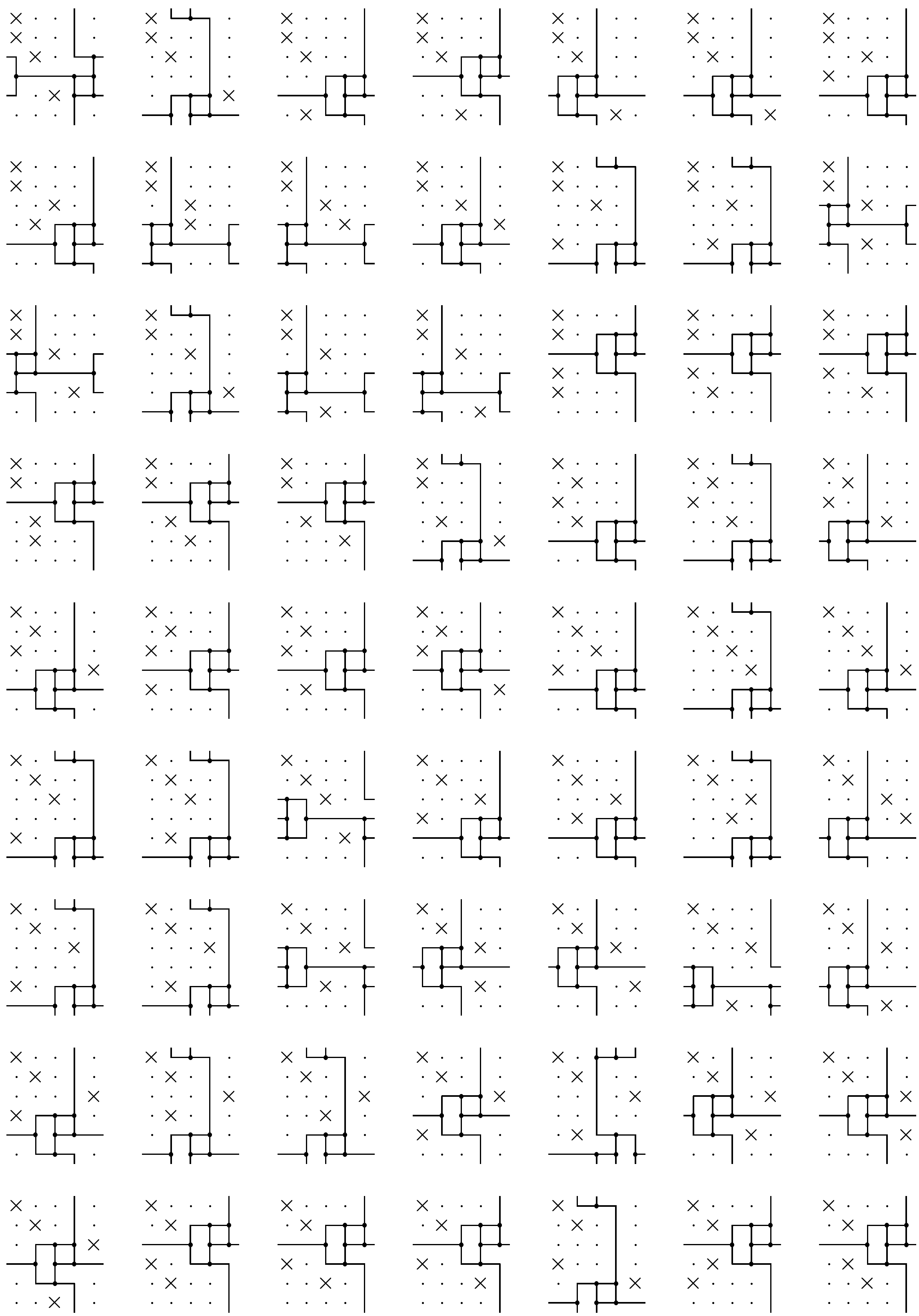

Figure 13: $\phi(C_5 \times C_6) \geq 5$.

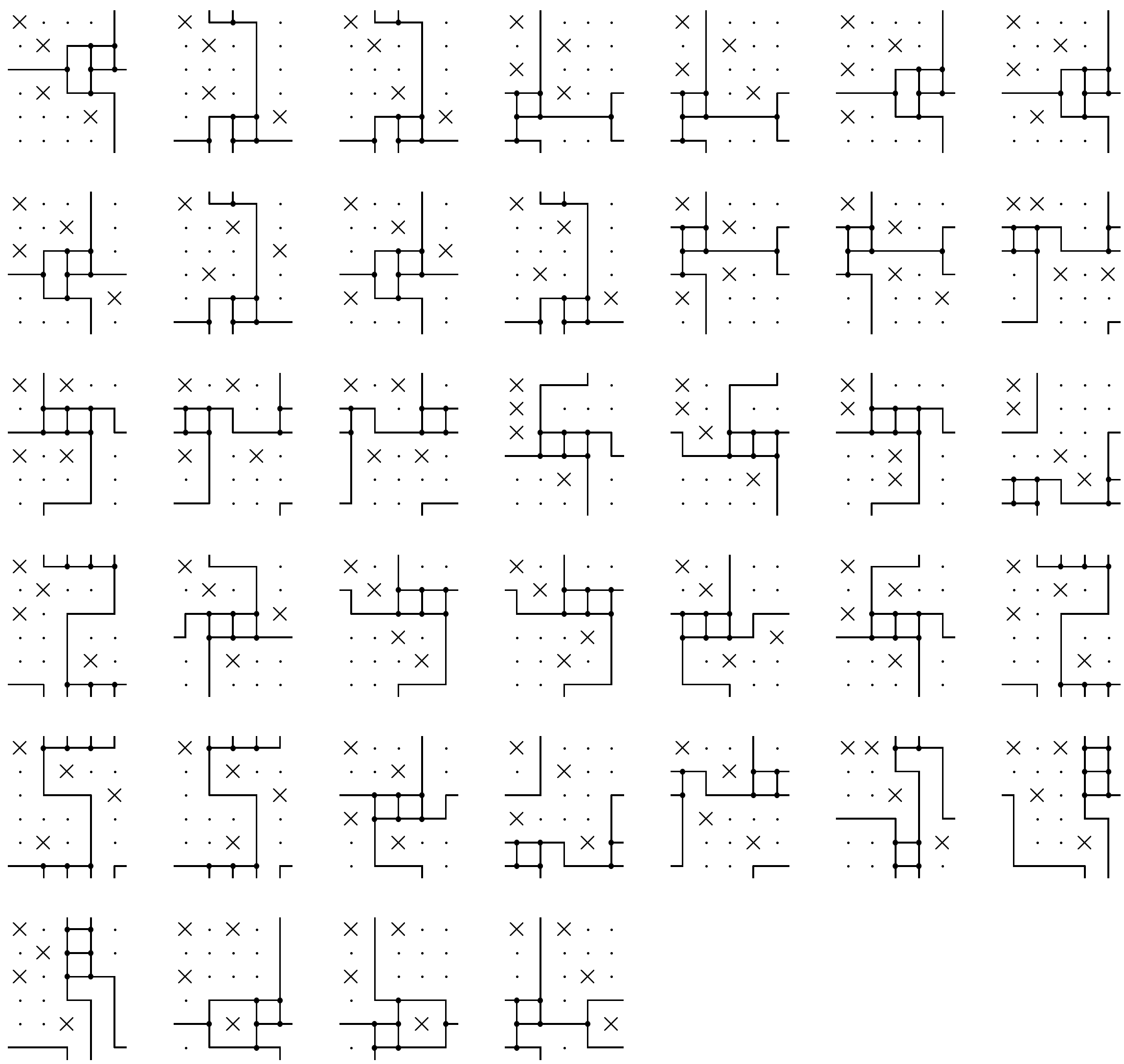

Figure 14: $\phi(C_5 \times C_6) \geq 5$.

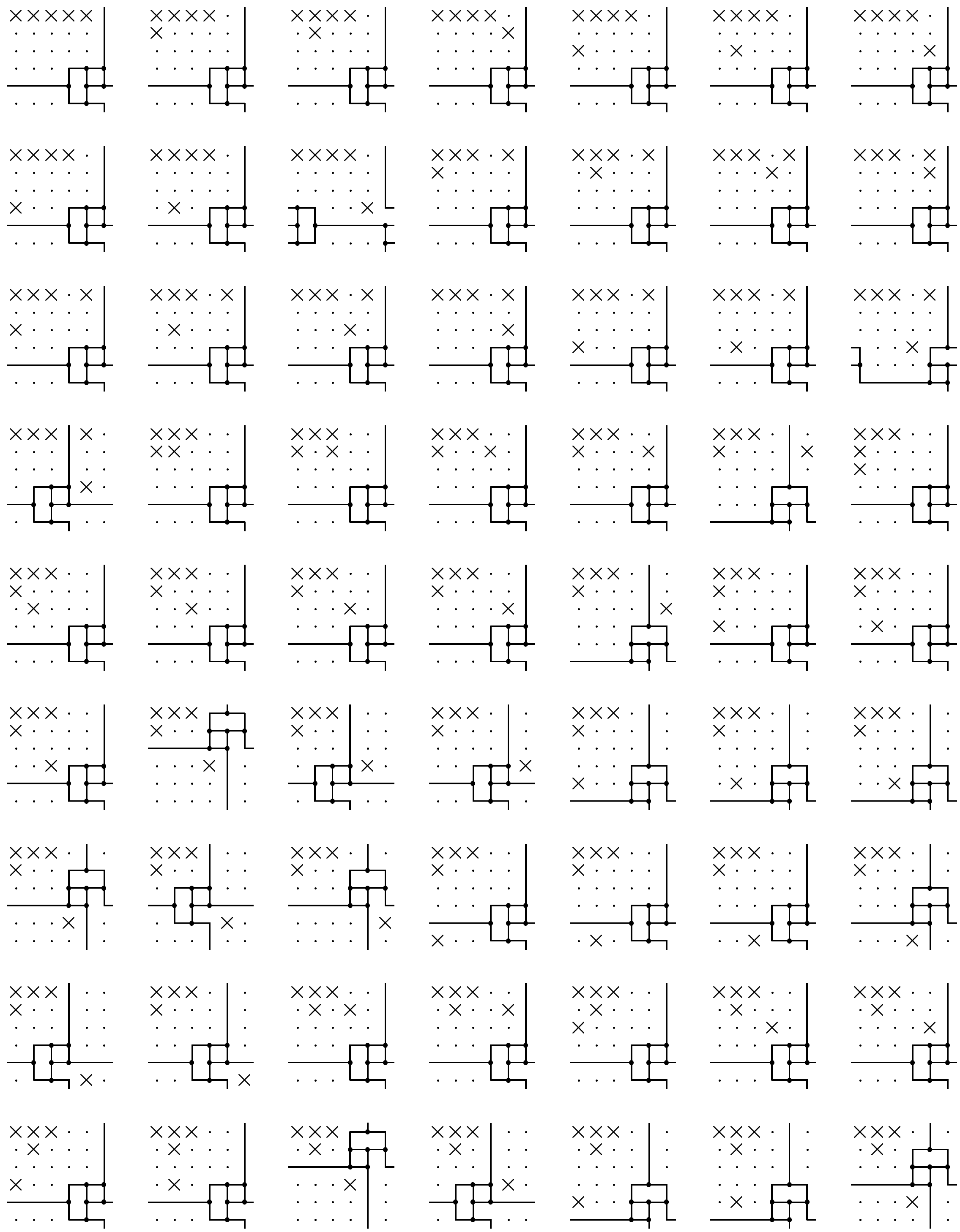

Figure 15: $\phi(C_6 \times C_6) \geq 6$.

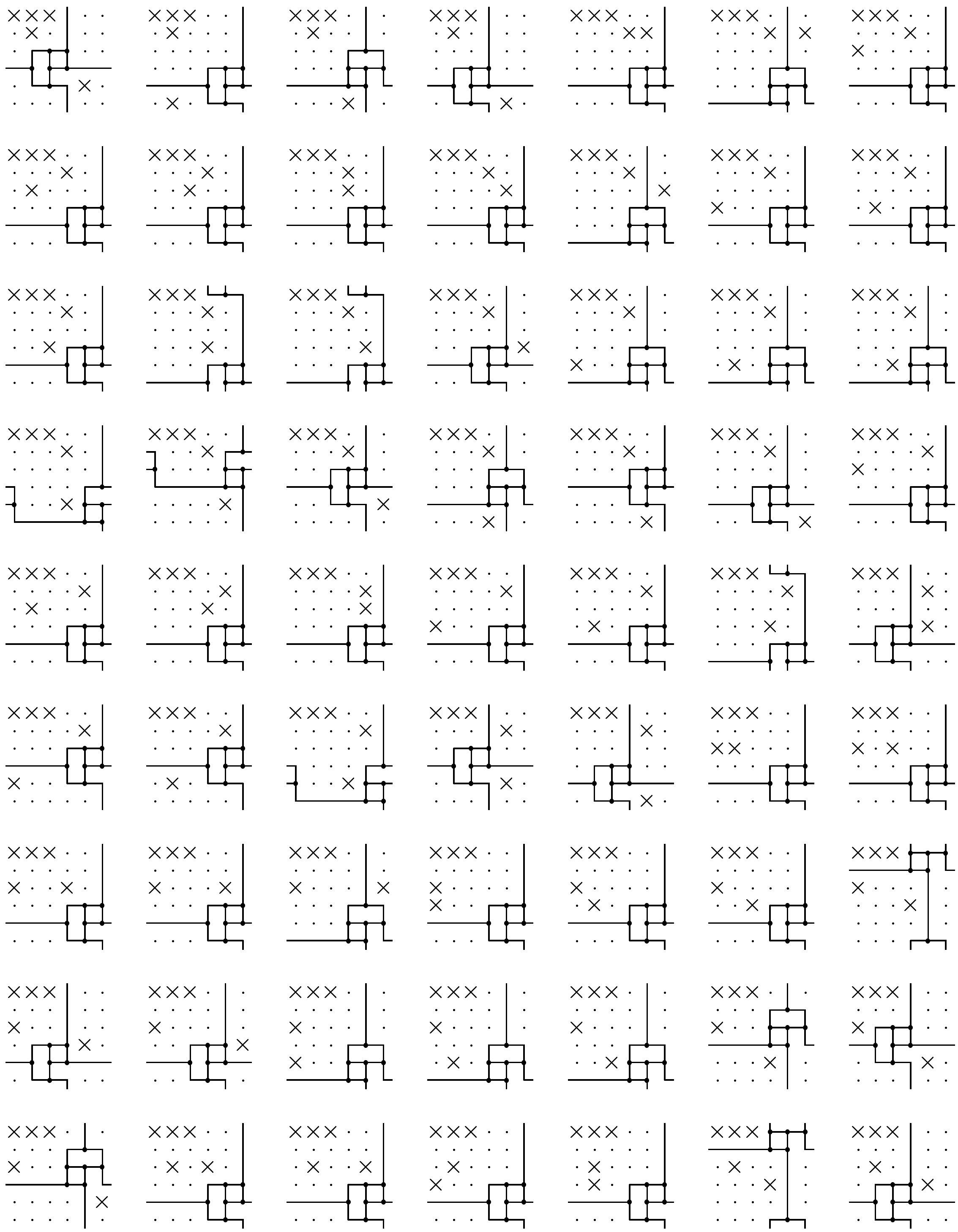

Figure 16: $\phi(C_6 \times C_6) \geq 6$.

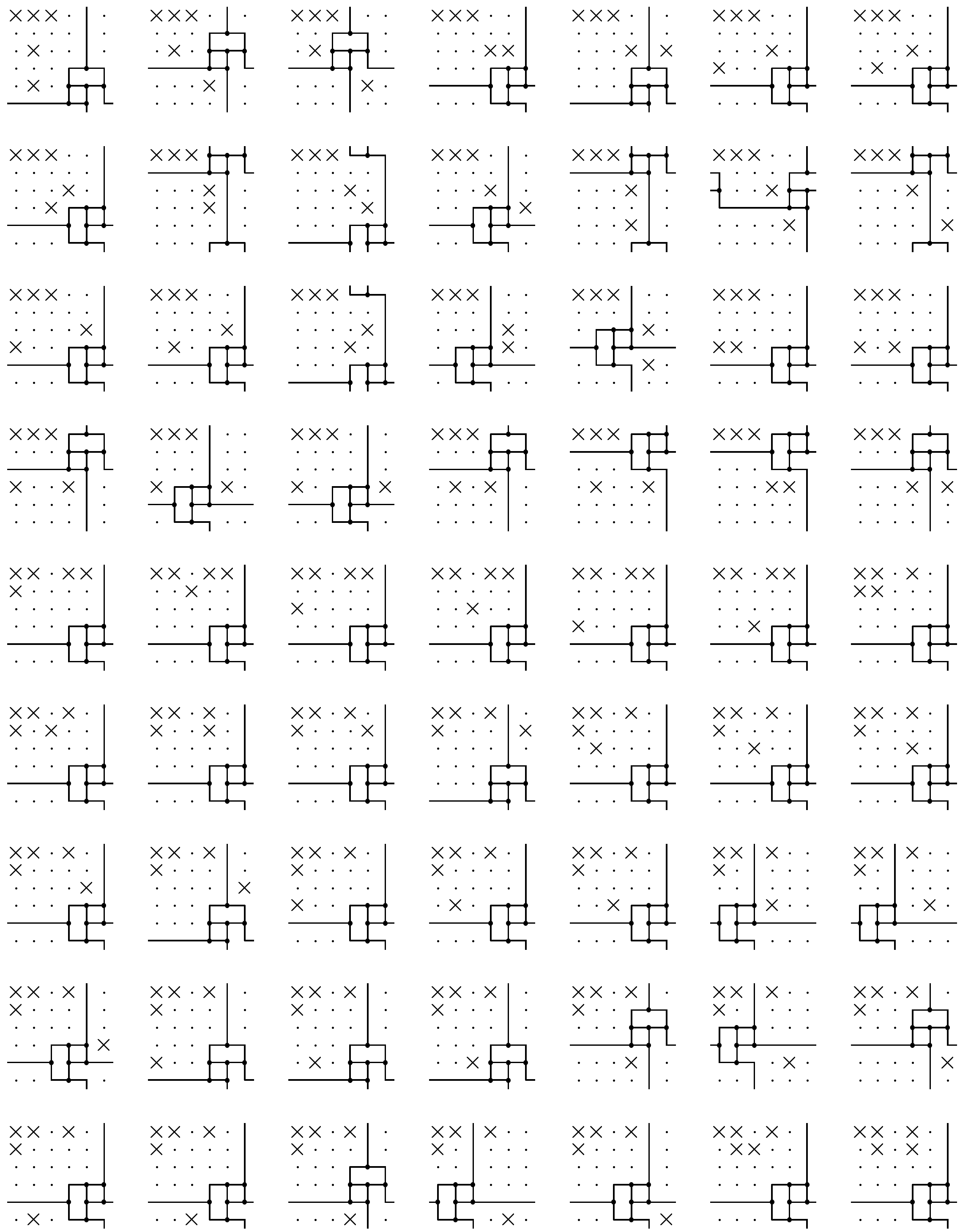

Figure 17: $\phi(C_6 \times C_6) \geq 6$.

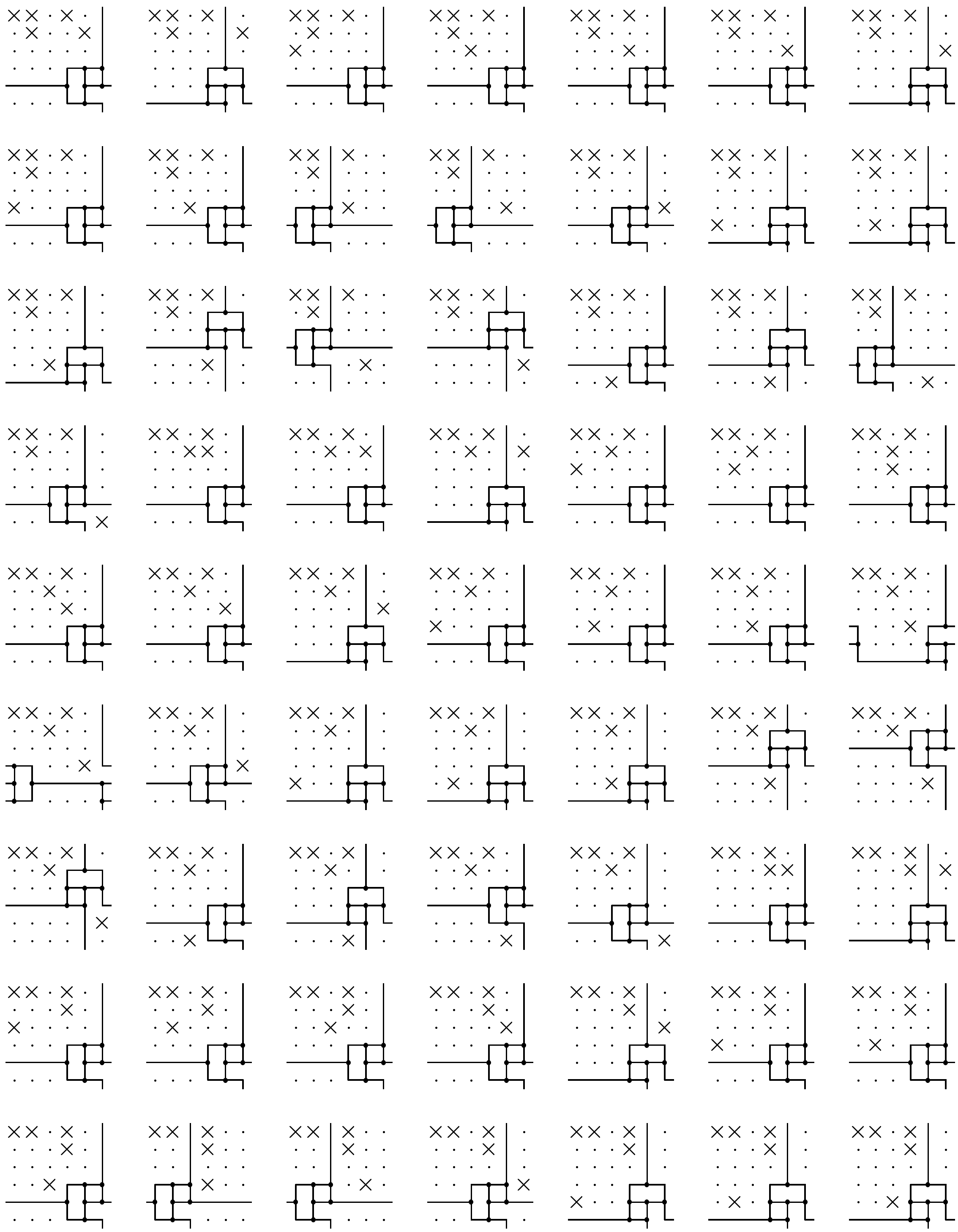

Figure 18: $\phi(C_6 \times C_6) \geq 6$.

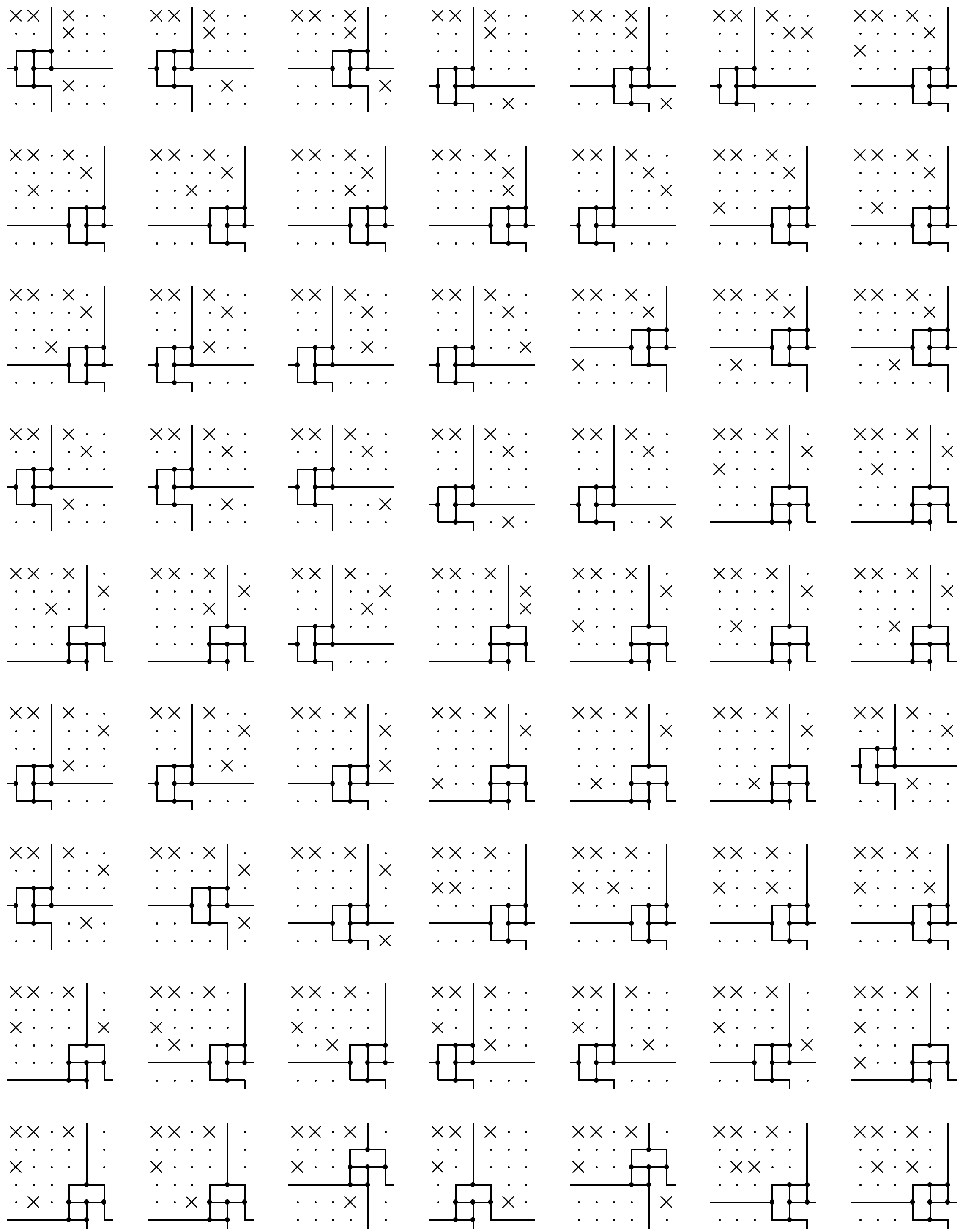

Figure 19: $\phi(C_6 \times C_6) \geq 6$.

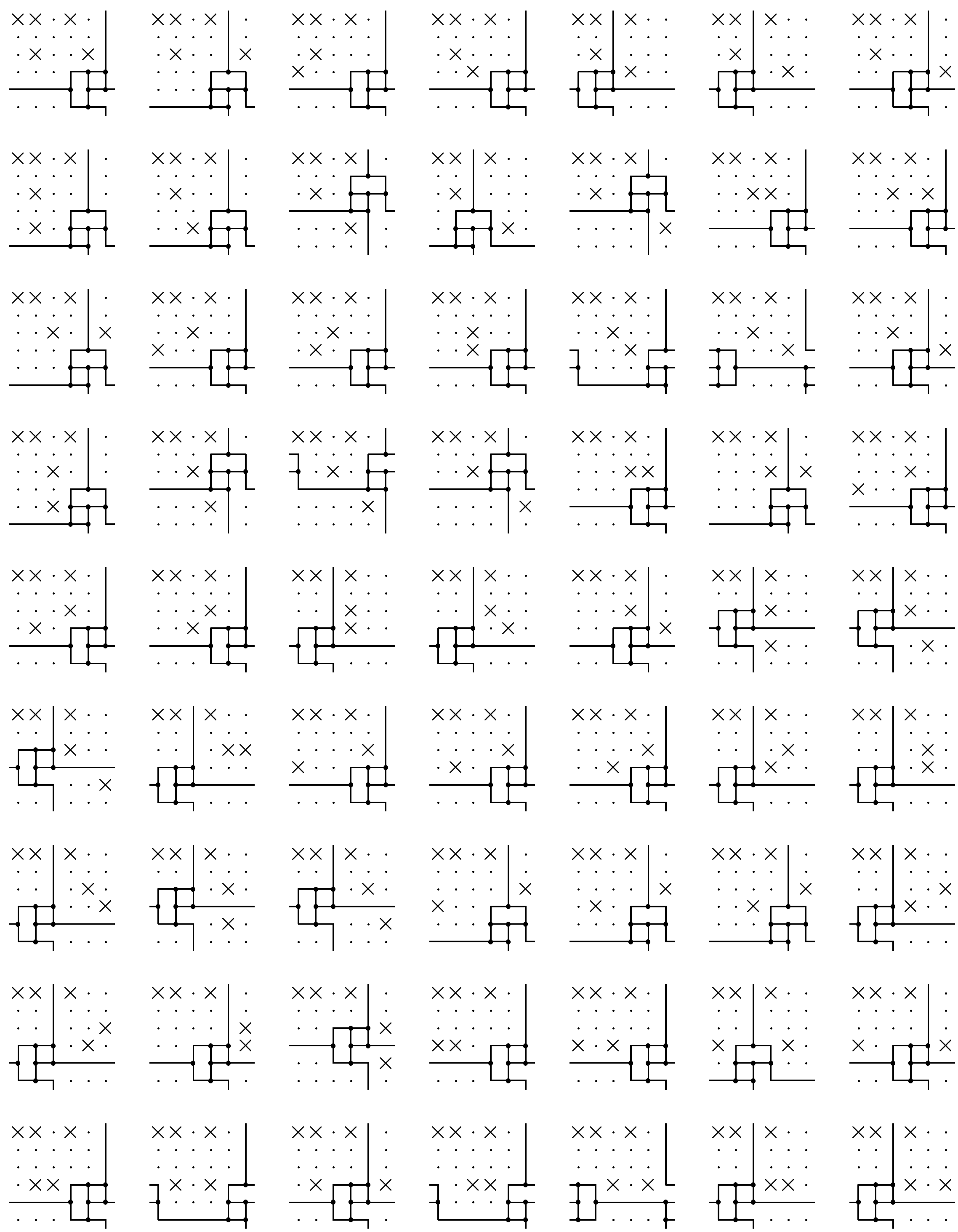

Figure 20: $\phi(C_6 \times C_6) \geq 6$.

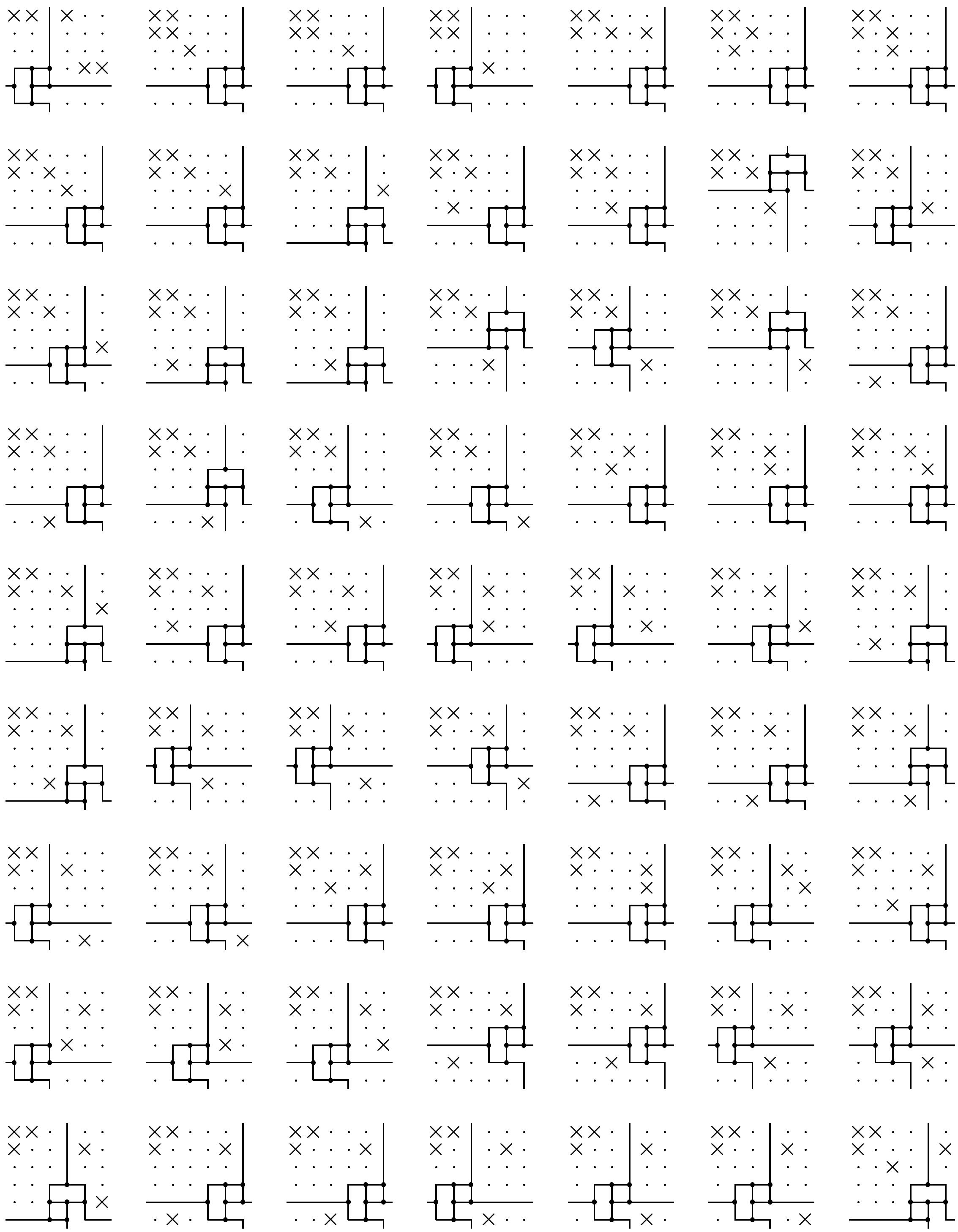

Figure 21: $\phi(C_6 \times C_6) \geq 6$.

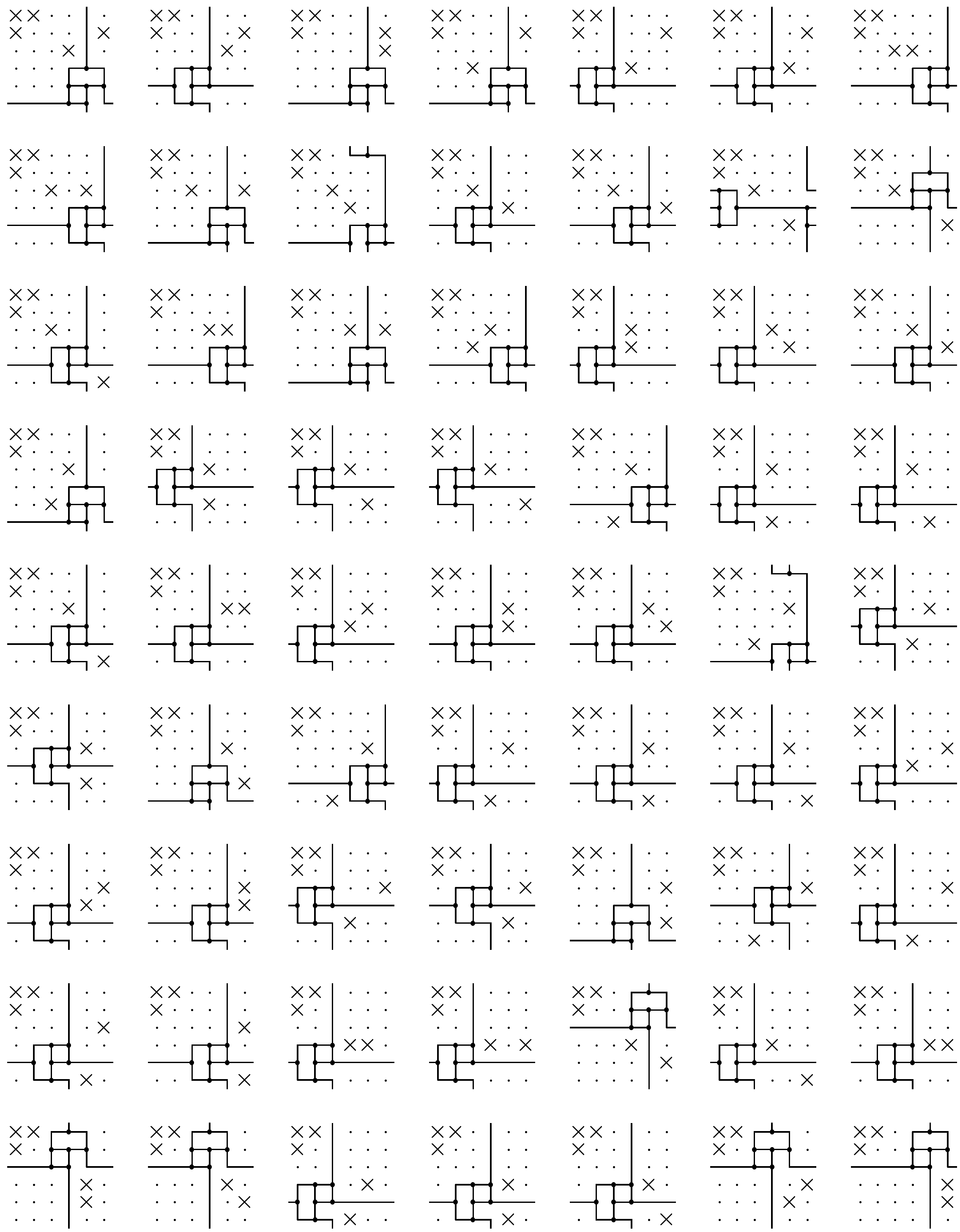

Figure 22: $\phi(C_6 \times C_6) \geq 6$.

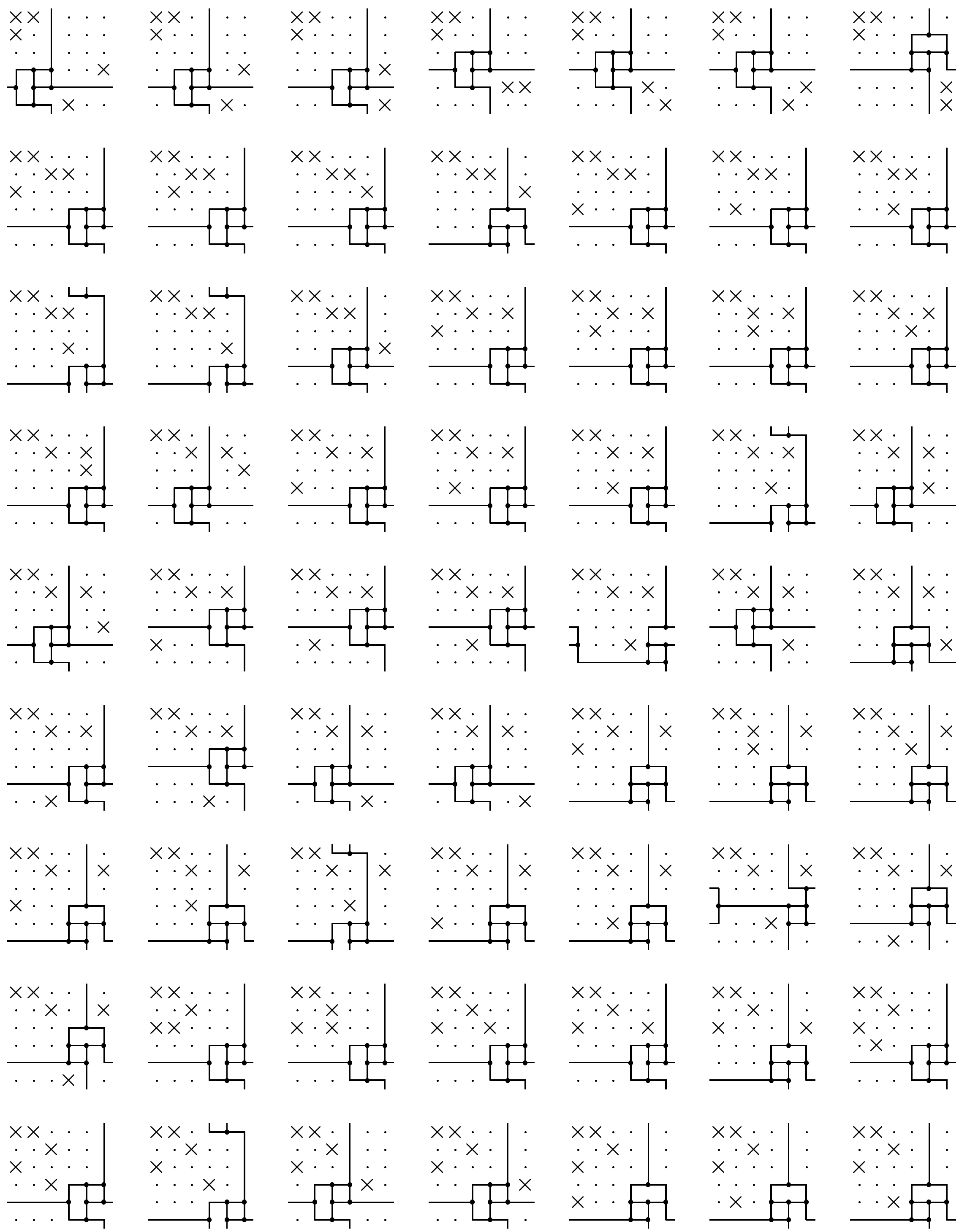

Figure 23: $\phi(C_6 \times C_6) \geq 6$.

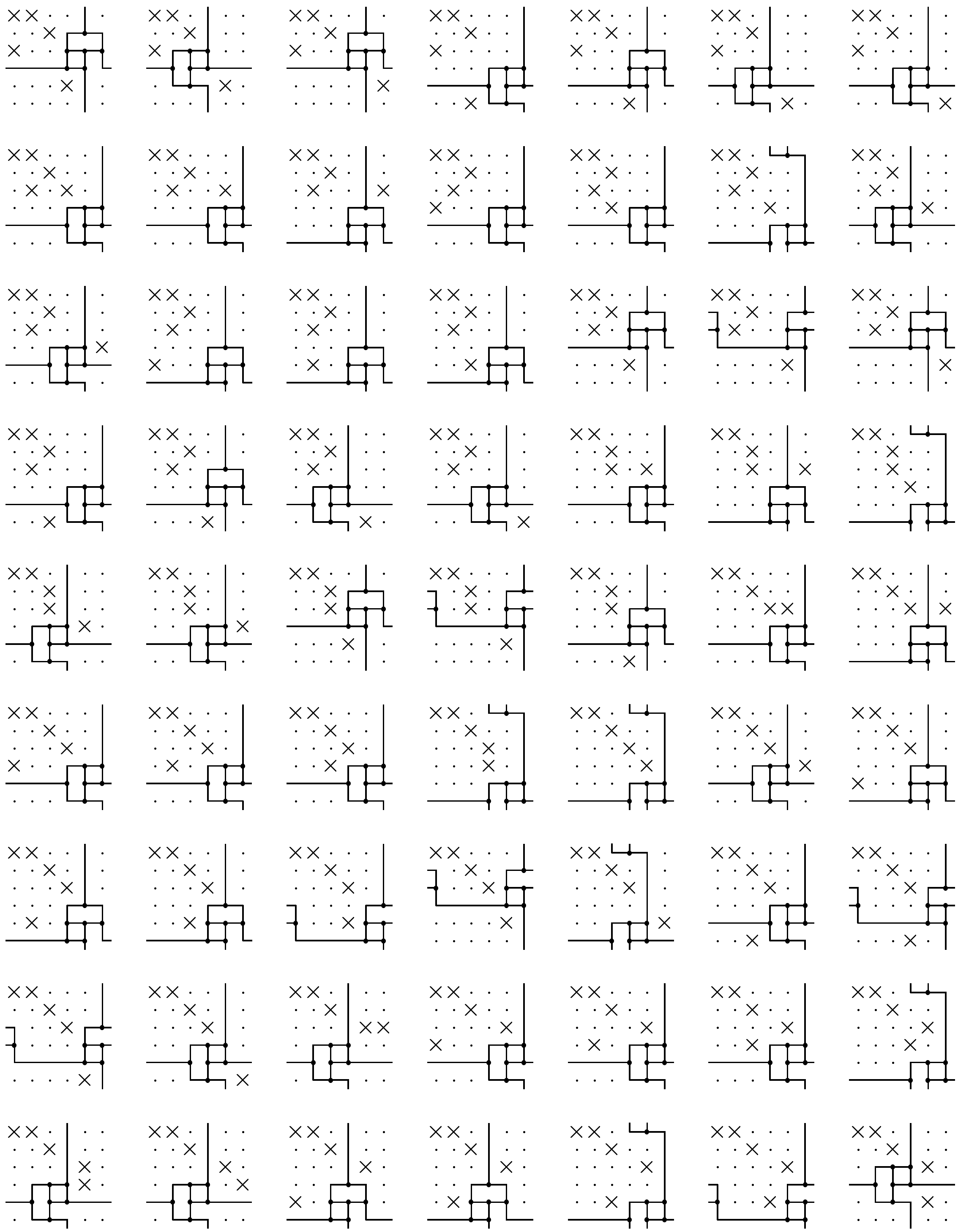

Figure 24: $\phi(C_6 \times C_6) \geq 6$.

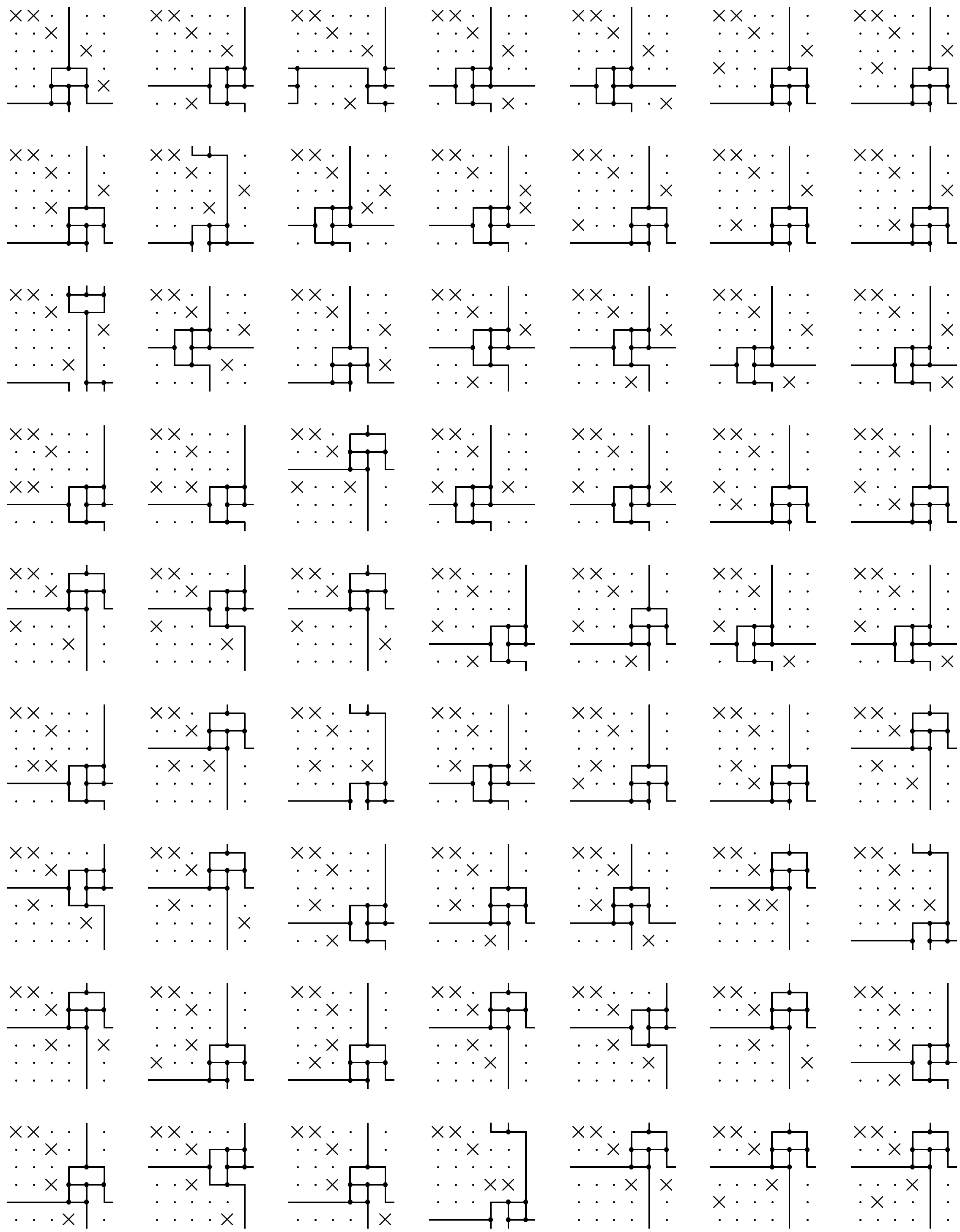

Figure 25: $\phi(C_6 \times C_6) \geq 6$.

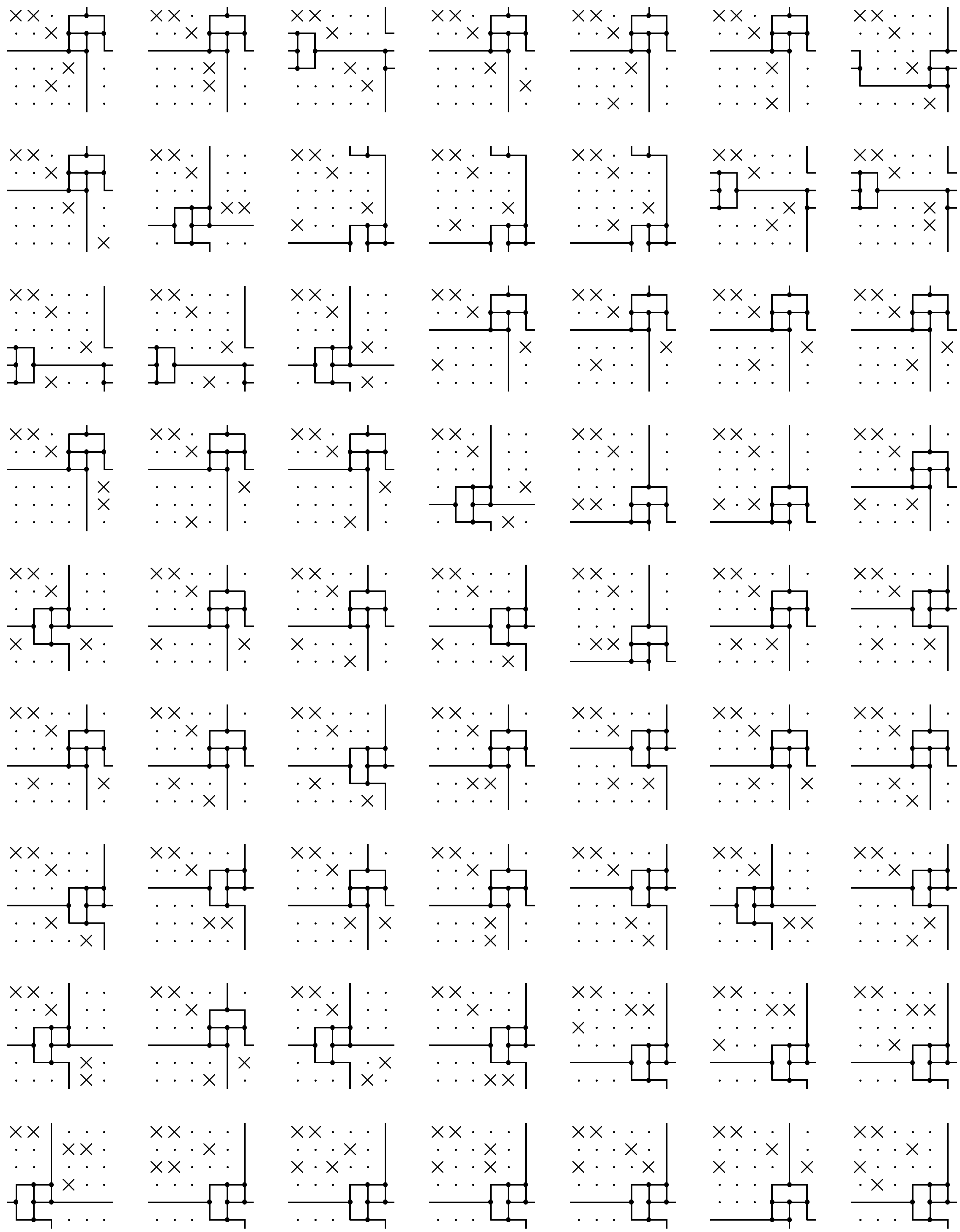

Figure 26: $\phi(C_6 \times C_6) \geq 6$.

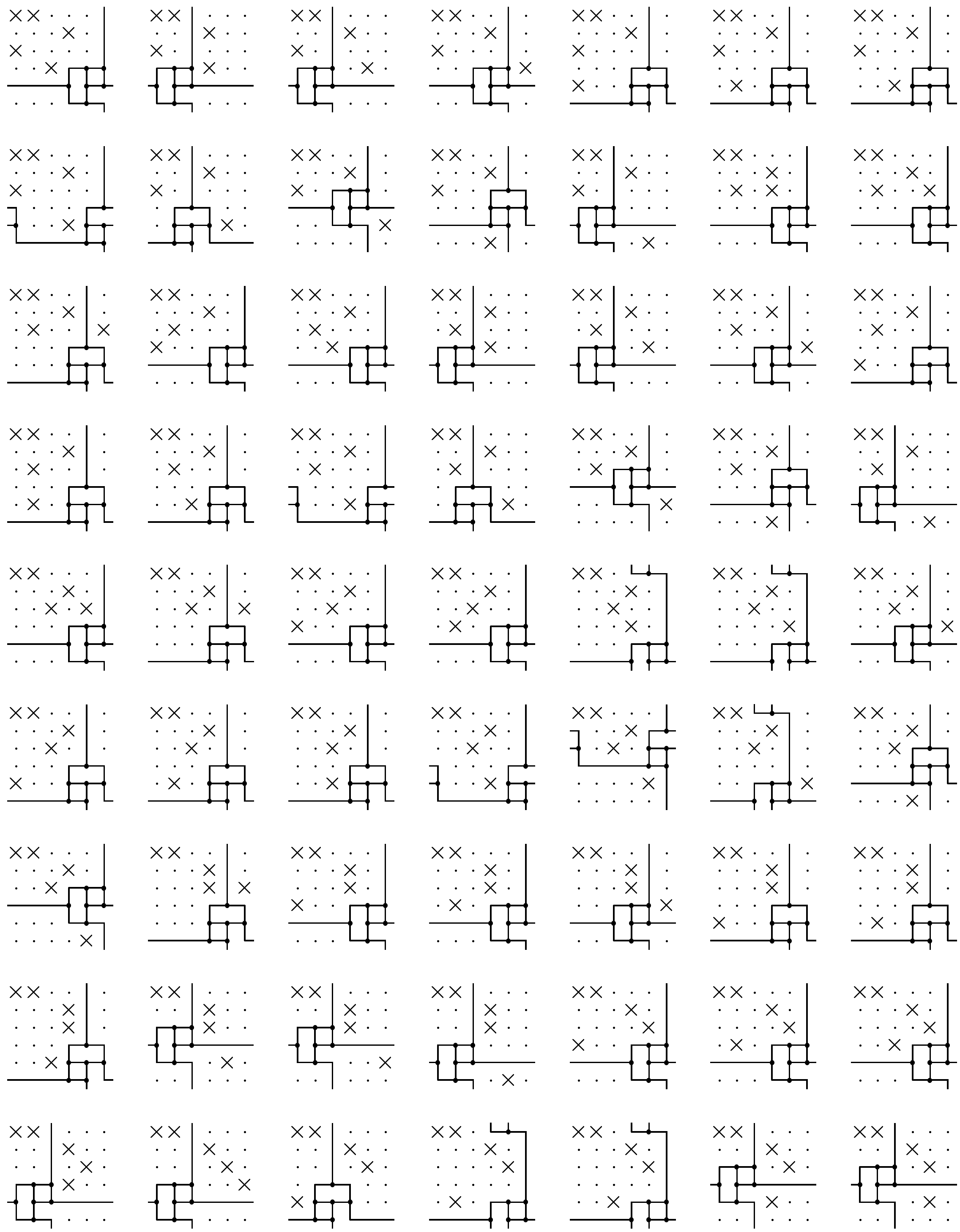

Figure 27: $\phi(C_6 \times C_6) \geq 6$.

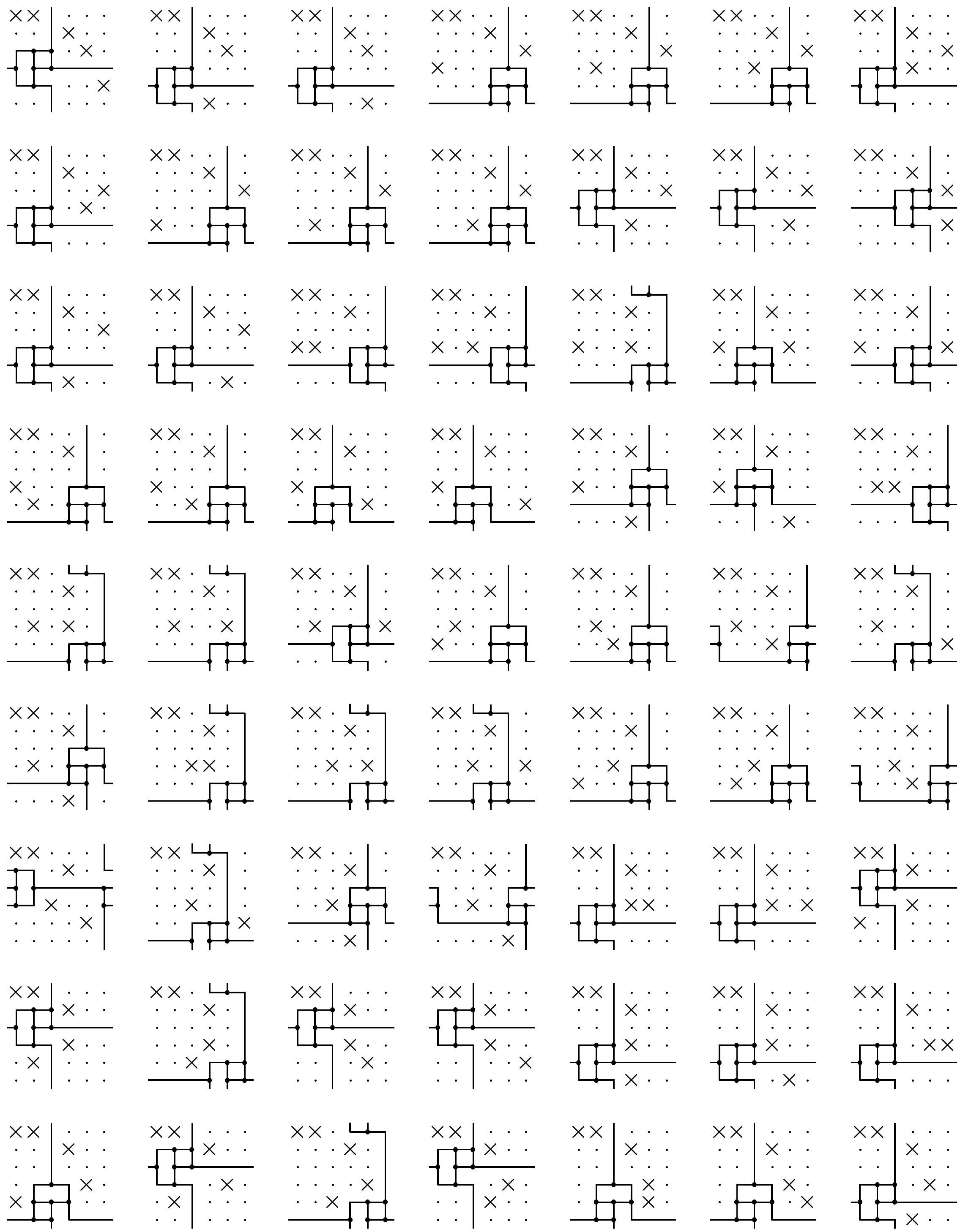

Figure 28: $\phi(C_6 \times C_6) \geq 6$.

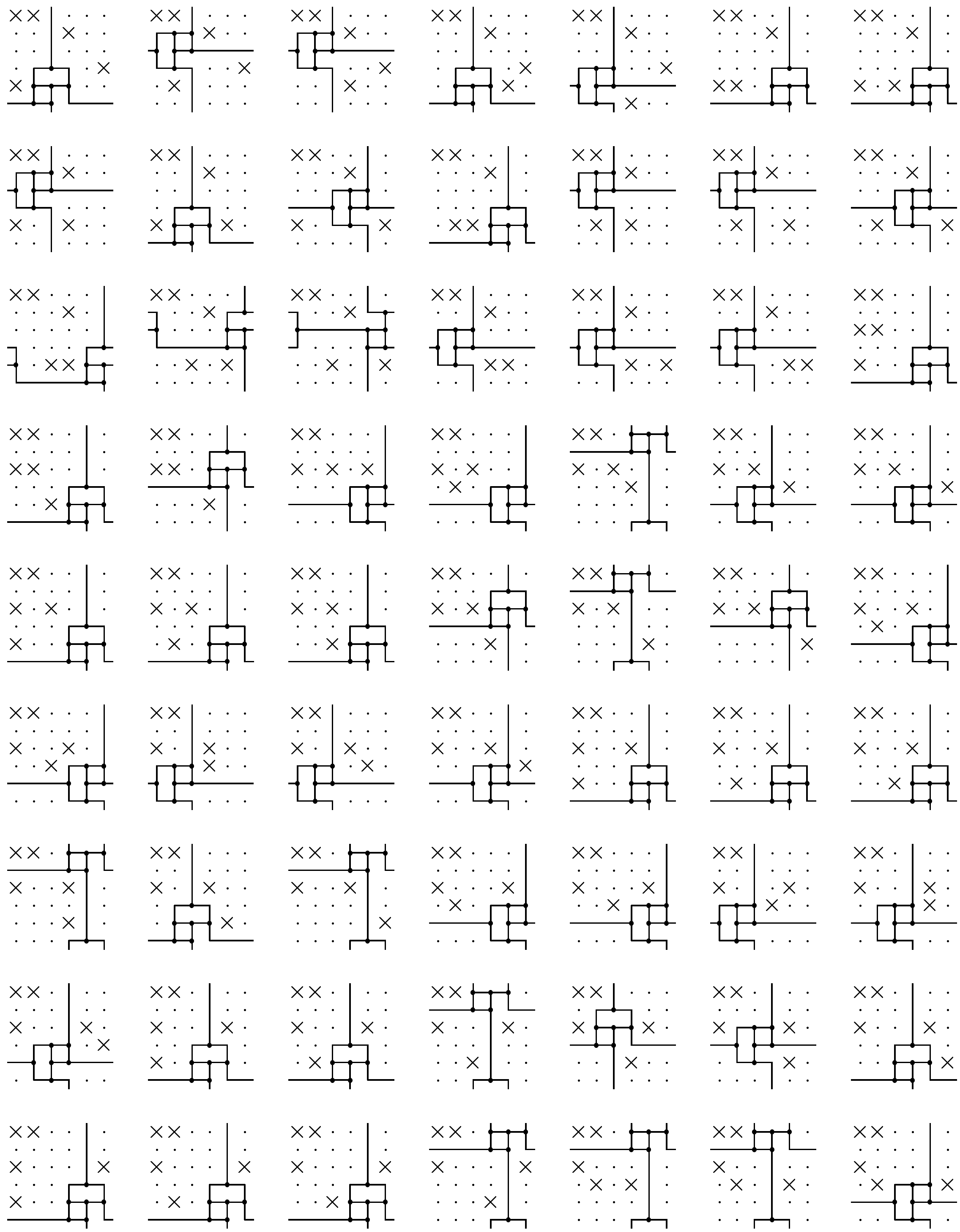

Figure 29: $\phi(C_6 \times C_6) \geq 6$.

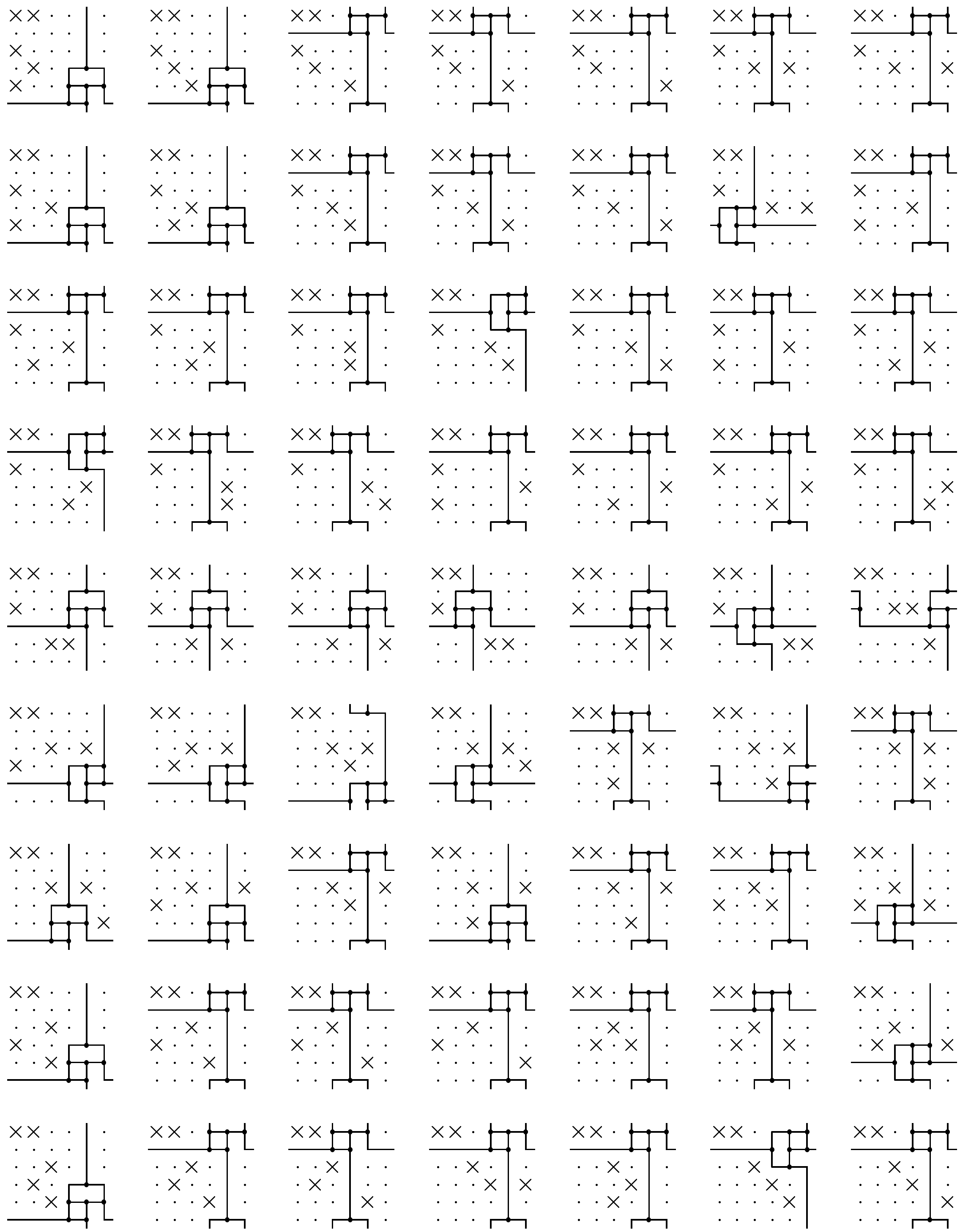

Figure 30: $\phi(C_6 \times C_6) \geq 6$.

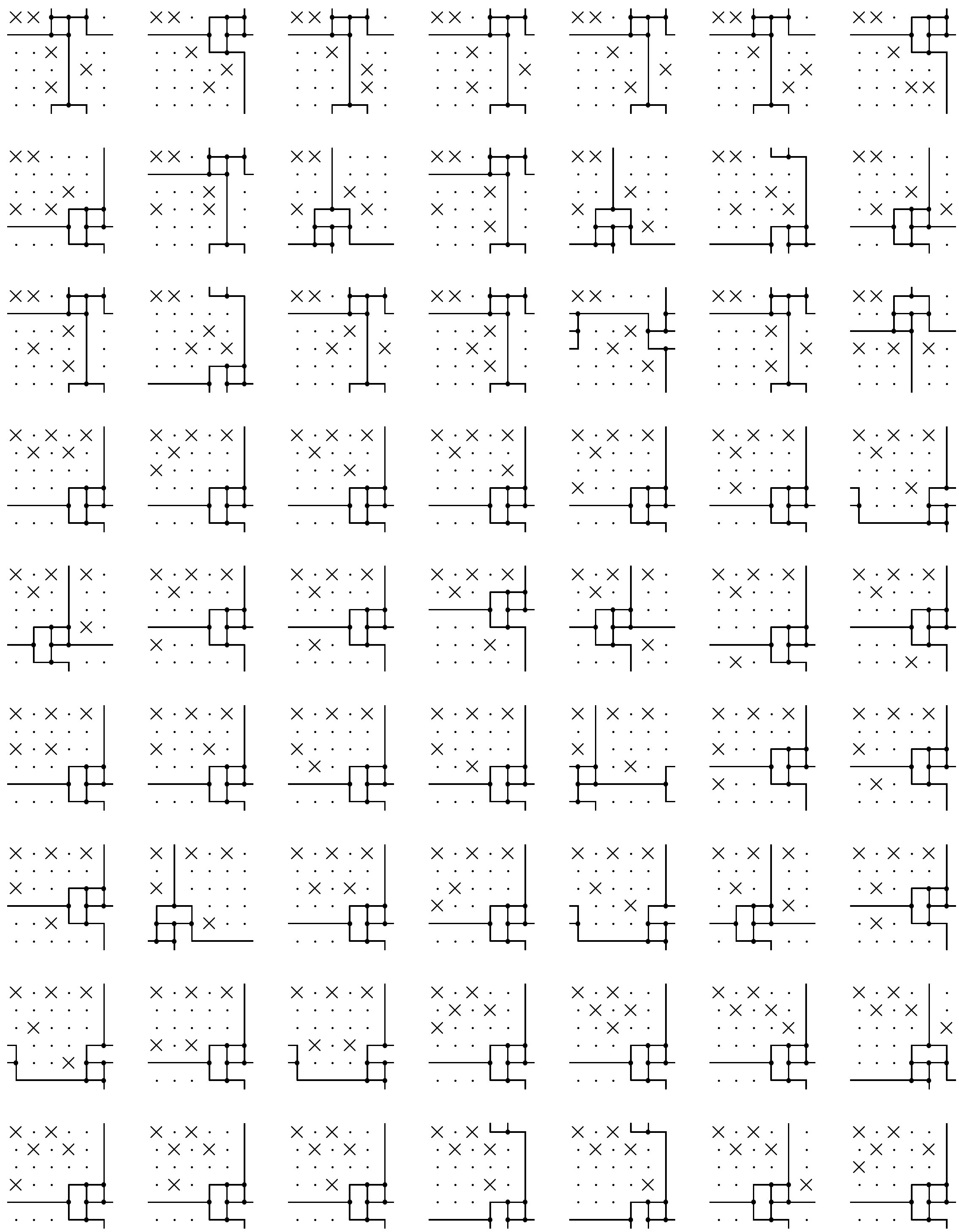

Figure 31: $\phi(C_6 \times C_6) \geq 6$.

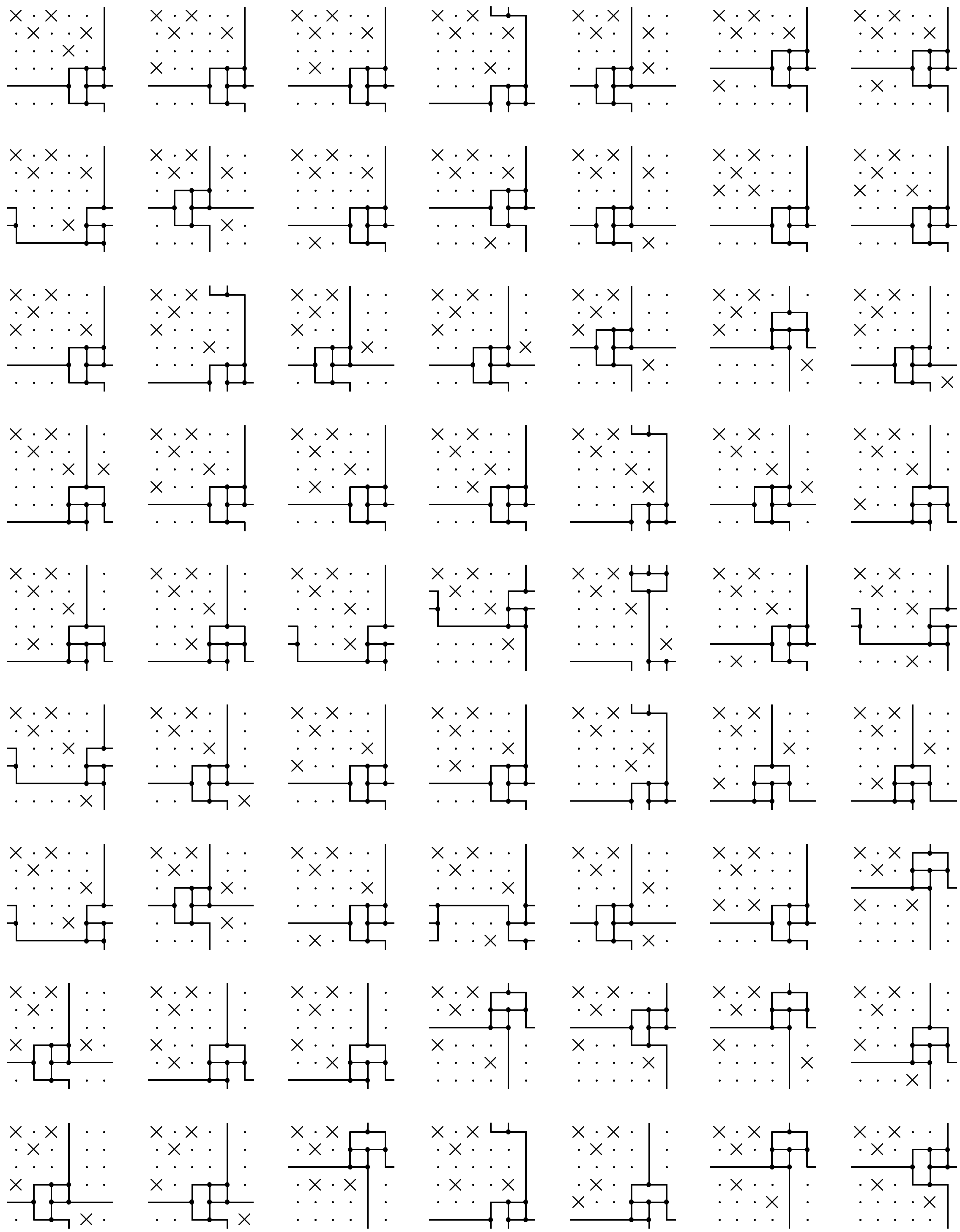

Figure 32: $\phi(C_6 \times C_6) \geq 6$.

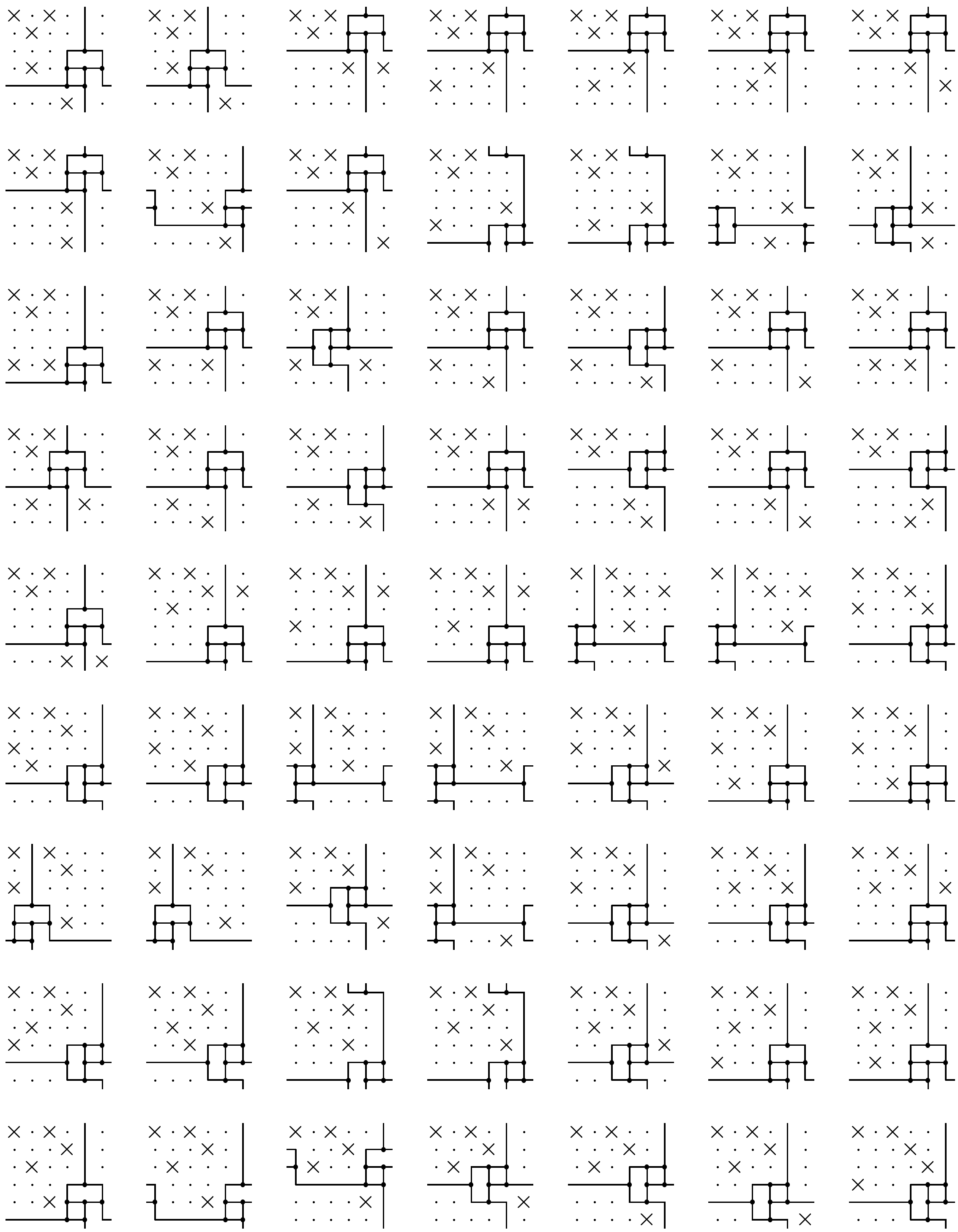

Figure 33: $\phi(C_6 \times C_6) \geq 6$.

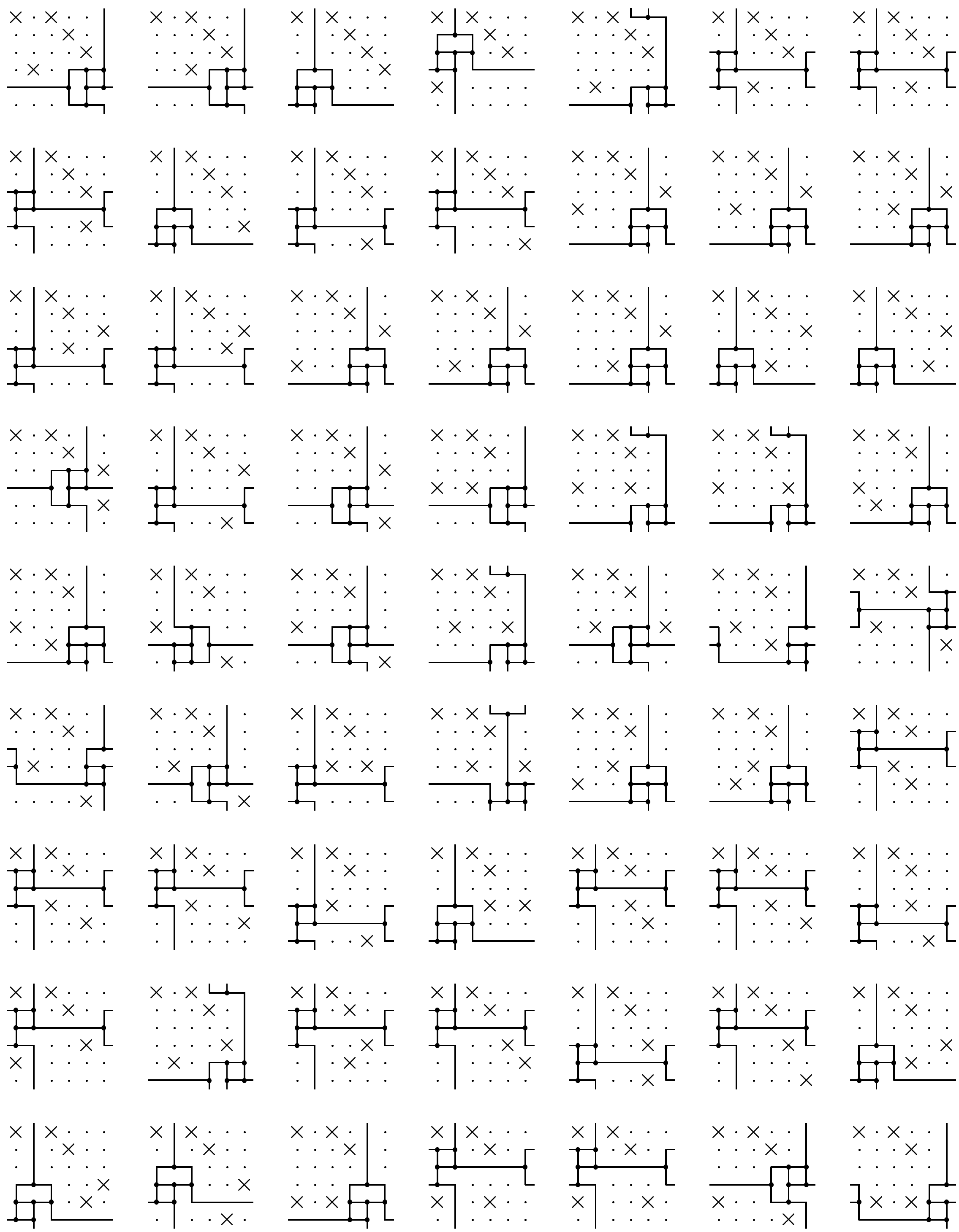

Figure 34: $\phi(C_6 \times C_6) \geq 6$.

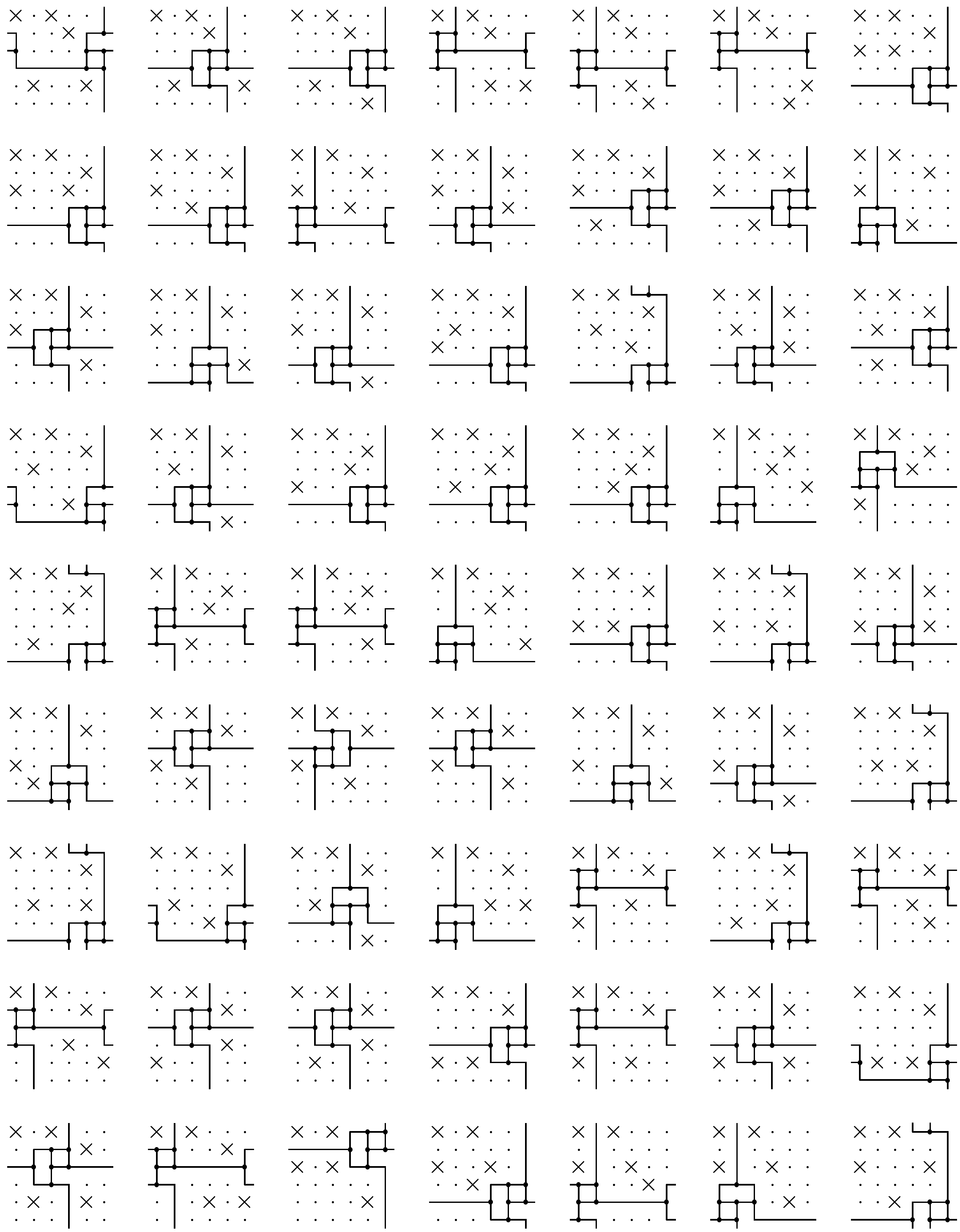

Figure 35: $\phi(C_6 \times C_6) \geq 6$.

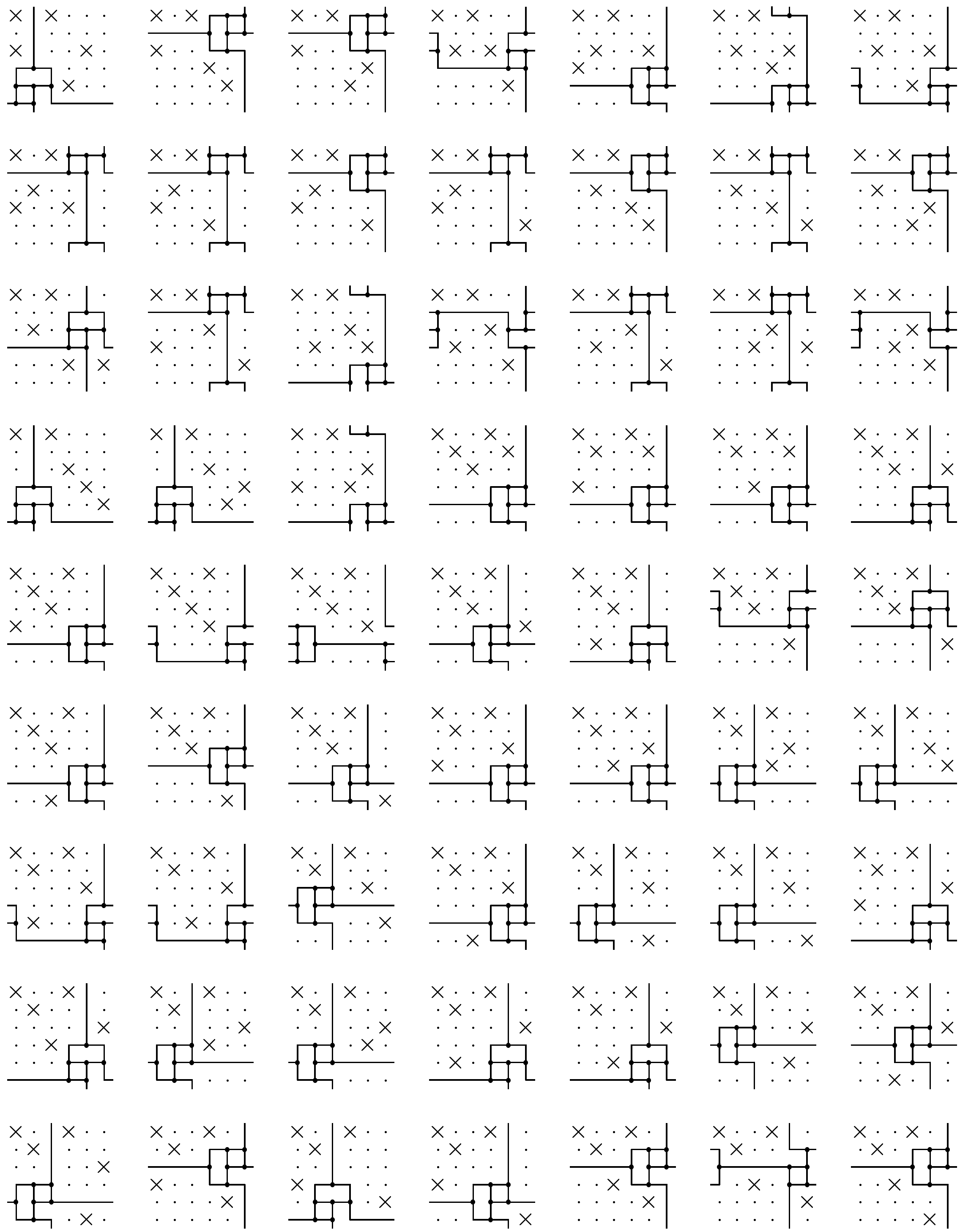

Figure 36: $\phi(C_6 \times C_6) \geq 6$.

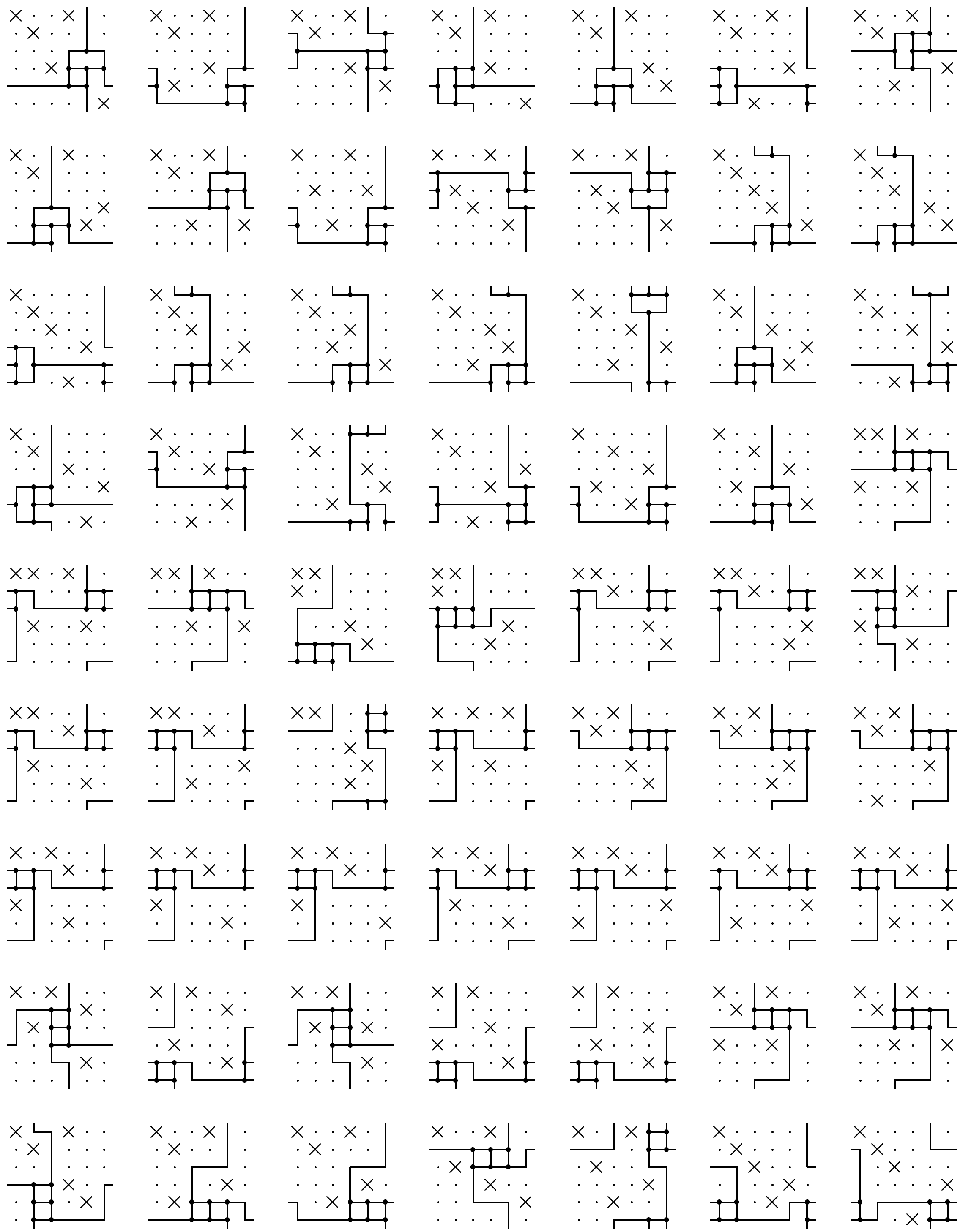

Figure 37: $\phi(C_6 \times C_6) \geq 6$.

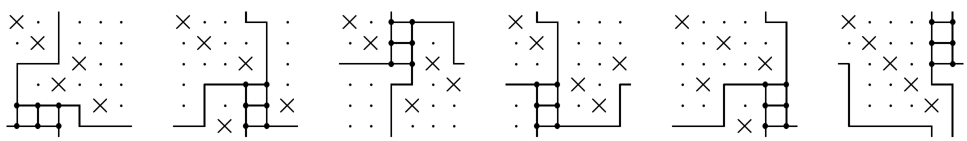

Figure 38: $\phi(C_6 \times C_6) \geq 6$.