Acyclic Clique-Interval Graphs

R. Zucchello
D. Mat. - Fac. C. Exactas
UNLP - Argentina

R. Dahab
IC - UNICAMP

Relatório Técnico IC–98-40

Outubro de 1998

Acyclic Clique-Interval Graphs

R. Zucchello
D. Mat. - Fac. C. Exactas
UNLP - Argentina

R. Dahab*
IC - UNICAMP

October 30, 1998

Abstract

The class of acyclic clique-interval (ACI) graphs is introduced as the class of those graphs $G=(V,E)$ whose cliques are intervals (chains) of an acyclic order on the vertex set V. The class of ACI graphs is related to the classes of proper interval graphs, tree-clique graphs and to the class DV (intersection graphs of directed paths of a directed tree). Compatibility between a graph and an acyclic order is defined, ACI graphs are characterized in terms of it and some special sets of vertices are found by means of the acyclic compatible order. ACI graphs are also characterized in terms of the dual hypergraph of the hypergraph of all cliques of G. Results concerning substitution and reduction preserving the ACI status are established. A strong necessary condition for a graph to be an ACI graph is also given.

*Research partially supported by Pronex-Finep grant no. 107/97

1 Introduction

We deal with finite undirected simple (i.e. without loops or multiple edges) graphs. For a graph $G = (V(G), E(G))$, $V(G)$ and $E(G)$, or simply V and E, are the vertex set and the edge set of G, respectively. A *clique* of a graph G is either a subset of $V(G)$ that induces a maximal complete subgraph of G or, if no confusion arises, this maximal complete subgraph. The *covering graph* ([12]) of an ordered set D is the graph whose vertices are the elements of D and whose edges are those pairs $\{a, b\}$ of elements $a, b \in D$ such that a covers b or b covers a. For acyclic order we mean an order such that its covering graph contains no cycle.

Definition 1.1 *A graph G is an* acyclic clique-interval graph *(an* ACI *graph for short) if there is an acyclic order α on the set $V(G)$ such that every clique of G is an interval of $(V(G),\alpha)$.*

ACI will also denote the collection of all the acyclic clique-interval graphs. An *interval graph* ([5]) has been defined as the intersection graph of a family of intervals of the real line (or of any total order); if we add the requirement that no interval properly contains another one, we obtain a *proper interval graph* ([5, 6]). These graphs have been characterized ([6, 10, 13]) as those graphs $G = (V, E)$ such that their cliques are intervals of $(V, \leq)$ where $\leq$ is a total order on V. It can inmediately be seen that the class of proper interval graphs is included in the class of ACI graphs, and that this inclusion is strict (the wheel W_4, for example, is an ACI graph but it is not a proper interval graph). Another generalization of proper interval graphs is given in [6]: *tree-clique graphs* are those (connected) graphs for which there is a spanning tree T such that every clique of G induces a subtree of T. Tree-clique graphs appeared independently in [15] under the name of *expanded trees*. As the covering graph of an acyclic order is a tree (a forest if not connected), it is easy to see that the class of ACI graphs is included in the class of tree-clique graphs. This inclusion is strict too: consider, for instance, any odd wheel W_n with n ≥ 5: it is a tree-clique graph but it is not an ACI graph. Moreover, ACI graphs may be characterized as follows:

Theorem 1.2 *Let G be a connected graph. Then, G is an ACI graph if and only if there is a directed spanning tree T of G such that every clique of G induces a directed path in T.*

Proof. If G is a connected ACI graph, there is an order on $V(G)$ satisfying the required conditions. The covering graph of this order is a spanning tree of G. Direction on T is given as it follows: (x, y) is an arc in the directed tree T iff y covers x in the order. Any interval in the order is a directed path in T and the same holds for the cliques of G. Conversely, if T is a directed spanning tree of G satisfying the hypothesis, let us define the binary relation R on $V(G)$ as it follows: xRy iff there is a directed $(x - y)$-path P on T. This is an order on $V(G)$ (R is a reflexive relation considering each vertex as a directed path of null length). Suppose that R has an interval I that is not a chain; then I contains at least two vertices z, w that are incomparable and uRz, wRv, where u and v are respectively the least and greatest elements of I. Then, disregarding the direction of the edges, there is a cycle in T, which is a contradiction. Therefore R is an acyclic order on $V(G)$, and y covers x iff (x, y)

is an arc of the directed tree T. Then, as every clique C induces a directed path of T, C is a chain of the order. □

The previous theorem allows us to consider in an ACI graph G either an acyclic order on $V(G)$ or a directed spanning tree of G satisfying the respective property for the cliques of G. Gavril ([4]) defines that a graph belongs to the class DV if it is an intersection graph of directed paths of a directed tree. ACI graphs are also related to these graphs. For a graph G, let $K(G)$ denote its *clique graph*, that is, the intersection graph of the cliques of G. Considering K as a function defined on a class of graphs in [7], Gutierrez shows that $K(ACI) = DV$ and also $K(DV) = ACI$. For this reason, ACI graphs are called *dually DV graphs* in [11]. In Section 2 we define a compatibility between a graph and an acyclic order in a similar way that a compatibility with a total order has already been defined. We characterize ACI graphs in terms of this compatibility and show a necessary condition for a graph to be ACI in terms of induced proper interval graphs. We also show in Section 2 how some independent (dominating, irredundant) sets ([1, 2]) in an ACI graph can be found by means of its compatible order. In Section 3 we study the behaviour of ACI graphs wrt substitution and reduction ([3, 5, 9]): given an ACI graph G we find the conditions under which it is possible to apply substitution (resp. reduction) to G such that the resulting graph preserves the ACI property. In Section 4 an ACI graph G is characterized in terms of properties of the dual hypergraph of the hypergraph of all cliques of G. In Section 5 we show a strong necessary condition (which is not sufficient) for a graph to be an ACI graph: G does not contain induced odd cycles of length k $\geq$ 5.

2 Compatibility between a graph and an acyclic order. Special sets of vertices in an ACI graph

Roberts([14]) defined compatibility between a graph and a total order and showed that a graph is compatible with a total order if and only if it is a proper interval graph. In this section we define compatibility between a graph and an acyclic order, just changing in the existing definition "total order" by "acyclic order". We characterize ACI graphs in terms of this compatibility, and give a necessary condition for a graph to be an ACI graph in terms of induced proper interval graphs. We show how to find some distinguished sets of vertices, which are described below, in an ACI graph, using the compatible acyclic order.

Definition 2.1 *Let $G = (V(G), E(G))$ be a graph and let α be an acyclic order on $V(G)$. G and α are* **compatible** *if for every pair of vertices x, z such that $xz \in E(G)$, the following conditions are satisfied: (i) x, z are α-comparable ; (ii) for every vertex y such that $x\alpha y\alpha z$ (or $z\alpha y\alpha x$), then $xy, yz \in E(G)$.*

Theorem 2.2 *A graph G is an ACI graph if and only if there is an acyclic order α on V(G) such that G and α are compatible.*

Proof. Let G be an ACI graph. Then there is an acyclic diagram D for G (we mean: for the ordered set $V(G)$) and every clique C of G is a chain in D. Consider an edge of G;

then both its end vertices belong to at least a clique C of G. As C is a chain in D, then all the vertices in C are comparable. Let x, y, z be vertices in this chain such that y follows x and z follows y in the chain. As y is also a vertex of clique C, we have that $xy, yz \in E(G)$. Let α be the order corresponding to diagram D; we obtain that G and α are compatible. Conversely, let G be compatible with an order α on $V(G)$; we must show that every clique of G is a chain of $(V(G),\alpha)$. Let C be a clique of G; as every pair of vertices in C are incident, by 2.1*(i)* vertices in C are all pairwise α-comparable, and therefore included in a chain I. By 2.1*(ii)*, the underlying set in any chain $[v, w]_\alpha$ for vw in $E(G)$, induces a complete subgraph of G. There is a first element x and a last element y of C on I, so there cannot be a vertex u on I, such that $x\alpha u\alpha y$ and u does not belong to C. Then $[x, y]_\alpha$ is the interval of $(V(G),\alpha)$ containing exactly the vertices of C. This completes the proof. □

Theorem 2.3 *Let G be an ACI graph and let α be an acyclic order compatible with G. Then for every interval I of $(V(G), \alpha)$, the underlying set of vertices of I induces a proper interval graph.*

Proof. Let I be an interval of $(V(G), \alpha)$. Suppose that x, y, z are vertices on I such that $x\alpha y\alpha z$. If $xz \in E(G)$, then, by Theorem 2.2, we also have $xy, yz \in E(G)$. Let G' be the subgraph of G induced by (the underlying set of vertices in the interval) I. As I is a chain of $(VG), \alpha)$, order α restricted to I is a total order; thus G' is compatible in the sense given in [6, 10, 14] with a total order. Therefore G' is a proper interval graph. □

In [1, 2] *independent (stable)* and *dominating* sets of a graph are defined. The *open neighborhood* of a vertex u is $N(u) = \{v \in V(G); uv \in E(G)\}$ and its *closed neighborhood* is $N[u] = \{u\} \cup N(u)$. In [2] they define that a vertex $x \in B \subseteq V(G)$ is *redundant* in B if $N[x] \subseteq N[B - \{x\}]$. This notion arises from problems in communications networks: any vertex that may receive a communication from some vertex in B, may also be informed from some vertex in $B - \{x\}$; thus x may be removed from B without affecting the totality of accessible vertices. They call a set of vertices *irredundant* if it contains no redundant vertex. In the next results, due to Berge and to Cockayne and Hedetniemi, respectively, and in the following one, maximality and minimality of sets refer to inclusion order:

Theorem 2.4 *[1, 2] If X is a maximal independent set, then X is a minimal dominating set.*

Theorem 2.5 *[2] If X is a minimal dominating set, then X is a maximal irredundant set.*

In Lemma 2.6 some independent sets in an ACI graph are obtained considering the maximal or minimal elements of the order compatible with the graph. In Corollary 2.8, it is shown how, under certain conditions, a maximal independent (minimal dominating, maximal irredundant) set of an ACI graph G can be found, also considering the acyclic order compatible with G.

Lemma 2.6 *Let G be an ACI graph without isolated vertices and let α be an acyclic order compatible with G. Then, any subset of maximal (respectively minimal) elements of $(V(G),\alpha)$ is an independent set of G.*

Proof. Let $A \subseteq M^+$, where M^+ is the set of maximal elements of $(V(G), \alpha)$. As every pair of vertices in A is non-comparable, by compatibility of G and α , they are non-adjacent vertices in G; thus A is an independent set of G. The same is obtained for the set M^- of minimal elements. □

With the same argument, the previous result can be generalized to any set of all pairwise α-incomparable vertices of G. In any graph G, let $(\mathcal{S}, \subseteq)$ be the class of all the independent sets of $V(G)$, ordered by inclusion. It is easy to prove that $(\mathcal{S}, \subseteq$ is a lower semilattice, in which the empty set is its least element, followed by all the singletones, followed by all the 2-elements sets $\{a, b\}$ such that ab is not an edge of G, and so on, until the class of all maximal independent sets of G. In some ACI graphs the sets M^-, M^+ may be in this class among possibly some other maximal independent sets such that some of their elememts are maximal and some others are minimal of the ordered set $(V(G), \alpha)$. We ask: Under which conditions is it possible to join all the maximal and all the minimal elements of $(V(G), \alpha)$ such that this union is a maximal independent set? Next theorem gives an answer to this question.

Theorem 2.7 *Let G be a graph compatible with an acyclic order α on $V(G)$ which contains a chain $I = [x, y] = \bigcap_{C clique of G} C \neq \emptyset$ (the intersection of all the intervals corresponding to all the cliques in G) and let every clique contain at least a simplicial vertex, such that for every clique C, all its simplicial vertices either follow y or else precede x. Then, the set M of all maximal and minimal elements of $(V(G), \alpha)$ is a maximal independent set of G.*

Proof. By Lemma 2.6, both M^- and M^+ are independent sets of G. In M^+ there is exactly one simplicial vertex of each clique whose simplicial vertices follow vertex y, and in M^- there is exactly one of each clique whose simplicial vertices precede x; by hypothesis, the sets of cliques considered respectively for M^- and M^+ are disjoint. Moreover, for every pair m, u of vertices, $m \in M^+$ and u $\in M^-$, mu $\notin E(G)$, because they belong to different cliques, $C^+ \neq C^-$, and only to them as they are simplicial vertices. Let $M = M^+ \cup M^-$; then M is an independent set of G. Let w be any vertex in $V(G) - M$; then, there is a clique C of G to which w belongs and there is a vertex r in M such that $wr \in E(G)$. So, no vertex can be added to M without losing independence. We conclude that M is a maximal independent set of G. □

Corollary 2.8 *The set M is also a minimal dominating set and a maximal irredundant set of G.*

This is inmediate by 2.4 and 2.5.

Corollary 2.9 *The independent set M is also a maximum independent set (with respect to cardinality).*

Proof. Suppose that N is an independent set with more vertices than M. By construction, M contains one and only one vertex of each clique of G. Thus, the cardinality of M is the number of cliques of G, and as N has more vertices than this number, at least two of them must belong to the same clique, which contradicts the independence of N. Then, M is maximum. $\square$

3 Substitution and Reduction in ACI graphs

Let H be a graph with vertex set $V(H) = \{x_1, ..., x_p\}$ and let $\mathcal{F}$ be a family of p pairwise disjoint graphs $G_i = (V_i, E_i)$ for $i = 1, \ldots, p$. In [3, 9], it is defined the *substitution* of $\mathcal{F}$ on the graph H as the graph whose vertex set is $\bigcup_{i=1,\ldots,p} V_i$ and whose edge set is the union of the set $\bigcup_{i=1,\ldots,p} E_i$ with the set $\{xy; x \in V_i, y \in V_j, i \neq j, ij \in E(H)\}$, denoted $H[G_1 ... G_p]$. In a graph G, a set $A \subseteq V(G)$ is called *homogeneous* [9] (*externally related* in [3]) if for every pair of vertices $x, y \in A$, $N(x) - A = N(y) - A$; that is, they all have the same neighbours "out" of A). If $W = \{x, y\}$ is a homogeneous set of a graph G and xy is an edge of G, then x and y are called *twins* of G ([9]). Let R be the binary relation on $V(G)$, defined by xRy iff x and y are twins of G, which is obviously an equivalence relation. G/R is the (simple) graph whose vertices are the R-equivalence classes, two distinct classes $R(x)$ and $R(y)$ are adjacent in G/R iff xy is an edge in G. If a graph is isomorphic to G/R, then it has no twins and it is the *reduced* graph ([14]) of G. Next theorem shows that if substitution by complete graphs is applied to an ACI graph, the resulting graph is also ACI. Then, we show that applying reduction of twins to an ACI graph G, the resulting graph is also ACI. If a finite sequence of such reductions is applied we finally obtain that the reduced graph of G is an ACI graph.

Theorem 3.1 *The class of ACI graphs is closed under substitution by complete graphs.*

Proof. Let G be an ACI graph and $v \in V(G)$. Vertex v is replaced, in the way described above, by a complete graph K such that $V(G) \cap V(K) = \emptyset$. There is an acyclic diagram for $(V(G), \alpha)$ where α is an acyclic order on $V(G)$. In this diagram, v is replaced by a chain $[a_1, ..., a_r]$, where $\{a_1, ..., a_r\} = V(K)$, in such way that if in the diagram of G, u is followed by v according to α, then u is followed by the chain $[a_1, a_r]$, and if w follows v, then w follows this chain; every vertex that is incomparable with v remains incomparable with all vertices $a_1, ..., a_r$. All these vertices of K lie in a chain in the diagram because K is a complete graph. An acyclic order on $(V(G) - \{v\}) \cup V(K)$ is obtained, that is, the graph $G[K]$ obtained by substitution of vertex v by the complete graph K is an ACI graph. If substitution by complete graphs is applied to more than one vertex of G, following the same steps as above for each of them, the substituted graph is ACI too. $\square$

For a graph G and any vertex $v \in V(G)$, let $C(v) = \{C \text{ clique of } G \; ; \; v \in C\}$

Theorem 3.2 *Let G be an ACI graph and $W = \{x, y\}$ a set of twins of G. Then, the graph G' obtained by reduction of W to a single vertex is an ACI graph.*

Proof. Assume wlog that G is connected. By Theorem 1.2, there is a directed spanning tree T of G in which every clique of G induces a directed path. As x and y are twins, then

$C(x) = C(y) = K$, W is included in all the cliques in K and W has an empty intersection with any other clique not in K. If there is only one clique C in K, then T may be chosen such that the directed path C in T contains the edge xy (with some direction). Suppose there are at least two cliques in K; since they all have the edge xy in common, then xy (with some direction) is in T, otherwise T would contain a cycle. Let G' be the graph obtained by reducing in G the set W to a single vertex. The directed edge with end vertices x and y is deleted from T, and these two vertices are identified into one in T. A directed spanning tree T' for G' is obtained. For every directed path C corresponding to a clique C in K, if C' is its corresponding clique in G', let $C - (x, y)$ be the directed path for C' in T'. For cliques of G not in K, their corresponding directed paths in T' are the same as in T. By 1.2, G' is an ACI graph. □

If G is an ACI graph, applying a finite sequence of reductions as the one described in 3.2, the reduced graph of G is obtained, and this is also an ACI graph.

Corollary 3.3 *The class ACI is closed under reduction of complete homogeneous subgraphs.*

Proof. Let G be an ACI graph and F a complete subgraph of G such that $V(F)$ is a homogeneous set of G. Then, as every two vertices in F are twins, F may be reduced to a single vertex and the resulting graph is ACI. □

4 ACI graphs and the dual hypergraph of the hypergraph of all cliques of a graph

Let $G = (V(G), E(G))$ and let $\mathcal{C}(G)$ be the set of all the cliques of G; then $\mathcal{C} = (V(G), \mathcal{C}(G))$ is the *hypergraph of cliques* of G. The *dual* hypergraph of $\mathcal{C}$ is the hypergraph $\mathcal{C}^*(G)$ (or $\mathcal{C}^*$) whose edges are the sets $C(v)$, as defined in Section 3, indexed by the vertex set $V(G)$; that is $\mathcal{C}^* = (C(v))_{v \in V(G)}$. We define the following collection Γ of finite families of sets:

Definition 4.1 *The finite family $(F_i)_{i \in I}$ is a member of Γ if there is an order θ on I such that:*

(i) For $i, j, k \in I$, if $j \in [i, k]_\theta$, then $F_i \cap F_k \subseteq F_j$;

(ii) if $F_i \cap F_j \neq \emptyset$, then i, k are θ-comparable.

Corollary 4.2 *Order θ is an acyclic order.*

Proof. Suppose order θ on I verifying *(i)* and *(ii)* has an interval which is not a chain, having i and k as least and greatest elements and h, j as non-comparable members. If j and h belong to $[i, k]_\theta$, then by *(i)*, $F_i \cap F_k$ is included in both F_j and F_h, so that they have a nonempty intersection. But since h and j are not comparable, we get a contradiction with *(ii)*. □

Theorem 4.3 *([8]) A graph G is an ACI graph if and only if $\mathcal{C}^*(G)$ belongs to Γ*

Proof. Let G be an ACI graph and α a corresponding acyclic order on $V(G)$. Let $v_j \in [v_i, v_k]_\alpha$, and let $C \in C(v_i) \cap C(v_k)$. C is a chain in order α, and v_i, v_k belong to C. Then, also $v_j \in C$, which means that $C \in C(v_j)$. Thus $C(v_i) \cap C(v_k) \subseteq C(v_j)$ and 4.1.*(i)* holds. Let $C(v_i) \cap C(v_k)$ be non-empty; then C belongs to this intersection, $v_i, v_k \in C$ and C is a chain of $(V(G), \alpha)$; so, they are α-comparable. Conversely, let $\mathcal{C}^*(G) = (C(v))_{v \in V(G)}$ be a family of the collection Γ. Let θ be the order on $V(G)$; by 4.2 it is acyclic. Let C be a clique of G; for every pair $u, v \in C$, we have $C(u) \cap C(v) \neq \emptyset$, because C belongs to each one of them. By 4.1 *(ii)*, this nonempty intersection implies that u, v are θ-comparable. As this happens for every pair of vertices in C, it is a chain and G is an ACI graph. □

5 Forbidden cycles in an ACI graph

In this section we prove if G is an ACI graph, then G has no induced cycles of odd length n, $n \geq 5$. Let G be an ACI graph and, wlog, connected. By 1.2, let T be a directed spanning tree of G such that every clique in G induces a directed path in T. Let $Z = (z_1, z_2, ..., z_k)$ be an induced cycle of G, where $k \geq 4$ and let $\mathcal{C}(Z) = \{C_1, C_2, ..., C_k\}$ be a set of cliques of G containing edges $z_1z_2, z_2z_3, ..., z_kz_1$ of Z. When no confusion arises, we refer to the directed path of T containing z_i, z_{i+1} also as C_i.

Lemma 5.1 *Each C_i contains no vertices of Z other than z_i and z_{i+1}.*

Proof. This is immediate, as otherwise Z would not be an induced cycle. □

Lemma 5.2 *Any directed path of T contains at most two vertices of Z.*

Proof. Suppose that there exists a directed path P of T containing distinct vertices z_i, z_j, z_m of Z, such that $z_iRz_jRz_m$ with respect to order R associated with T, described in the proof of 1.2. Let T', T'' be subtrees of T such that $T' \cap T'' = \{z_j\}$ and $T' \cup T'' = T$ with $z_i \in T'$ and $z_m \in T''$. Consider the arc A of Z with end vertices z_i and z_m and not containing z_j. Then, there are vertices z', z'' consecutive in A, such that $z' \in T'$ and $z'' \in T''$. The clique C of $\mathcal{C}(Z)$ containing z', z'' corresponds to a path in T that does not contain z_j. The union of this path with P must contain a cycle, a contradiction to the fact that T is a tree. Therefore, no such path P may exist. □

Lemma 5.3 *Let z_i, z_j be two vertices of Z, such that i and j have the same parity. Then, there is no directed path in T joining z_i and z_j.*

Proof. Either for all vertices z_i with i of the same parity, the path in T joining z_i and z_{i+1}, is directed from z_i towards z_{i+1}, or else, for all of them, it is directed from z_{i+1} to z_i, otherwise there would be a directed path in T containing three vertices of Z, which contradicts 4.2. Suppose now, wlog, that for z_i and z_j there exists a directed (z_i, z_j)-path in T. As it is pointed out above, there is either a (z_j, z_{j+1}) or a (z_{i-1}, z_i) directed path in T. In any case, there is a directed path in T containing three vertices of Z, a contradiction. □

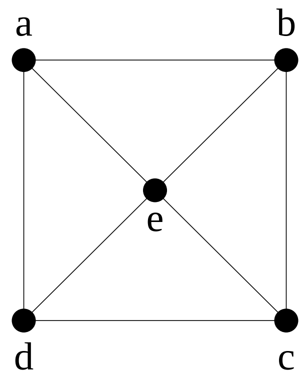

Figure 1: A graph with an even cycle that is not ACI

.

Theorem 5.4 *The length k of cycle Z is even.*

Proof. If k is odd, then, by 4.3, no directed path in T joins z_1 and z_k. But, since they are adjacent in Z, there is a directed path, namely C_k, in T, joining them. Then k must be even. □

This condition about induced cycles is not sufficient for a graph to be ACI, as Figure 1 illustrates.

References

[1] C. Berge, *Theory of Graphs and its Applications*, Methuen, London (1962)

[2] B. Bollobas, E. J. Cockayne, Graph-Theoretic Parameters Concerning Domination, Independence, and Irredundance, *J. of Graph Th.* 3 (1979) 241-249.

[3] M. Chein, M. Habib, M.C. Maurer, Partitive Hypergraphs, *Discrete Math* 37 (1981) 35-50

[4] F. Gavril, A Recognition Algorithm for the Intersection Graphs of Directed Paths in Directed Trees,*Discrete Math* 13 (1975) 237-249.

[5] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, NY, 1980.

[6] M. Gutierrez, *Grafos de Intervalos Propios y Grafos Arbóreos*, Tesis Doctoral, Facultad de Ciencias Exactas, Univ Nac de La Plata,1994.

[7] M.Gutierrez, Intersection Graphs and Clique Application, to appear in *Graphs and Combinatorics.*

[8] M. Gutierrez, R. Zucchello. Grafos ACI: una generalización de los grafos de intervalos propios. XLVI Reunión Anual de Comunicaciones Científicas de la Unión Matemática Argentina (1996).

[9] M. Habib, *Substitution des Structures Combinatoires. Theorie et Algorithmes,*These Univ Pierre et Marie Curie, 1980.

[10] H. Maehara, On Time Graphs, *Discrete Math* 32 (1980) 281-289.

[11] E. Prisner, J. L.Swarcfiter, Recognizing clique graphs of directed and rooted path graphs, preprint (1998).

[12] I. Rival, The Diagram, *Graphs and Order, Proceedings of the NATO Advanced Study Institute*, Banff, Canada (1984) 103-136.

[13] F. S. Roberts, On Nontransitive Indifference, *J. of Mathematical Psychology* 7 (1970) 243-258.

[14] F. S. Roberts, On the Compatibility Between a Graph and a Simple Order,*J. of Combinatorial Th.* 11 (1971) 28-38

[15] J. L. Szwarcfiter, C. F. Bornstein, Clique Graphs of Chordal Graphs and Path Graphs, *Siam J. Discrete Math.* Vol. 7 No. 2 (1994) 331-336.