SOME COMMENTS ON THINNING ALGORITHMS FOR 3-D IMAGES

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Abstract

The problem of defining *simple points* in a two-dimensional binary image was extensively discussed in the literature. This definition is more complicated for 3-d images due to the introduction of a new topological component, namely, the tunnel. Recently, some simple-point characterizations for these images have been proposed. In this paper we show the equivalences between these new characterizations and discuss the implementation problems concerned with their use in thinning algorithms. An extension of the 2-d gray-level thinning algorithm to the 3-d case is also presented.

1 INTRODUCTION

1.1 Basic notions

Let p and q be two points in $\mathbb{Z}^3$ denoted by $p = (p_x, p_y, p_z)$ and $q = (q_x, q_y, q_z)$. We say that p and q, are 26-adjacent if $|p_x - q_x| \leq 1$, $|p_y - q_y| \leq 1$ and $|p_z - q_z| \leq 1$; 18-adjacent if $|p_x - q_x| + |p_y - q_y| + |p_z - q_z| \leq 2$ and p and q are 26-adjacent; 6-adjacent if $|p_x - q_x| + |p_y - q_y| + |p_z - q_z| \leq 1$.We call the set of all points α-adjacent to p the α-adjacency of p and denote it by $N^*_\alpha(p)$, $\alpha \in \{6, 18, 26\}$. We also note $N_\alpha(p) \cup \{p\}$ as $N_\alpha(p)$.

A *binary image* or simply *image* is the four-tuple $\Im = (\mathbb{Z}^n, \beta, \omega, B)$ in which each point in $\mathbb{Z}^n$ is assigned a value 1 (black point) or 0 (white point). The black points represent the *objects* and the white points represent the *background*. Two black points are said adjacent if they are β-adjacent, two white points or a white point and a black point are adjacent if they are ω-adjacent, β, $\omega \in \{6, 18, 26\}$. In order to avoid connectivity paradoxes (see [1]) we use $\beta \neq \omega$. The set B is the set of all black points in $\Im$. We will discuss only 3-d images, i.e., images with $n = 3$.

A *path* in $X \subset \mathbb{Z}^3$ is a sequence $< x_1, \ldots, x_k >$ such that each $x_i \in X$ and x_j is adjacent to x_{j+1}, $1 \leq j < k$. A *black path* of X is a path with only black points. Two black points p and q are *connected* when there is a black path $< p = x_1, \ldots, x_k = q >$. The equivalence classes of the relation *is connected* are called the *black components of* X. *White components* are defined analogously.

Besides black components, cavities and tunnels are two other topological structures very common in 3-d images. A cavity is a set of bounded white connected components (e.g., a hollow ball). There is no formal definition for tunnel but we can say that it is a ring-like structure which can not be transformed to a single point by a sequence of elementary homotopic deformations [2]. Despite the lack of formal definition, we can compute the number of tunnels [3].

The sets of black components, cavities and tunnels of X are noted here as $\mathcal{C}(X)$, $\mathcal{H}(X)$, $\mathcal{T}(X)$, respectively. We also note the set of components of X adjacent to p as $\mathcal{C}^p(X)$. The Euler characteristic of X is given by $\chi(X) = \mathcal{C}(X) + \mathcal{H}(X) - \mathcal{T}(X)$. In the following sections we assume $\Im = (\mathbb{Z}^3, 26, 6, B)$ and $p \in B$ unless we state the contrary.

2 TOPOLOGY IN 3-D THINNING ALGORITHMS

The main requirement for a thinning algorithm is topological preservation. The skeleton obtained from thinning must contain the same number of components, cavities and tunnels as the original image. Thinning algorithms must guarantee that no component, cavity or tunnel will be added or removed during the operation. This is achieved by iterative removal of the black points of the foreground whose removal does not change the topology. This points are called *simple* points.

There are some characterizations for 3-d simple points and a few thinning algorithms based on these characterizations. As we will elsewhere, these characterizations consider mainly topological properties in the elementary cubic grid given by a point and its 26-neighborhood.

2.1 Simple point characterizations

One of the major problems related to 3-d simple point characterizations concerns the occurrence of tunnels. To illustrate such a problem, let us consider the characterization in [4] which states that a point p satisfying $\#\mathcal{C}(N_{26}(p)) = \#\mathcal{C}(N^*_{26}(p))$ can be defined as a simple point, $\#$ stands for the cardinality. Tsao and Fu showed later that this characterization does not ensure topological preservation [4]. In this sense, the first effectively correct characterization was given by Morgenthaler (see Kong et al [1] for more details):

Characterization 1 (Morgenthaler) *A point $p \in B$ is simple if, and only if, the following conditions hold:*

1-a. $\#\mathcal{C}(B \cap N_{26}(p)) = \#\mathcal{C}(B \cap N^*_{26}(p))$

1-b. $\#\mathcal{C}(\overline{B} \cap N_{26}(p)) = \#\mathcal{C}(\overline{B} \cap N^*_{26}(p))$

1-c. $\chi(B \cap N_{26}(p)) = \chi(B \cap N^*_{26}(p))$

Tsao and Fu [4] presented a simple point characterization based on the Euler characteristic and on two 2-d checking windows orthogonal to the considered point. The thinning algorithm proposed in [4] calculates efficiently the Euler characteristic using a method proposed by Lobregt et al in [5]. To state this characterization, we define for all point $p = (p_x, p_y, p_z) \in \mathbb{Z}^3$ the points $north(p) = (p_x + 1, p_y, p_z)$ and $south(p) = (p_x - 1, p_y, p_z)$. Analogously, we define the points $east(p)$, $west(p)$, $up(p)$ and $down(p)$. We also define the three two-dimensional planes orthogonal to p as $plane_\phi(p) = \{q = (q_x, q_y, q_z) | q_\phi = p_\phi\}$ and the $3{\times}3$ checking windows as $cw_\phi(p) = plane_\phi(p) \cap N_{26}(p)$, where $\phi \in \{x, y, z\}$ is one of the axis in $\mathbb{Z}^3$. We can define $plane_y(p)$ and $plane_z(p)$ in the same way. We also say that p is in direction α when $\alpha(p) \in \overline{B}$ to $\alpha \in \{north, south, east, west, up, down\}$. Note that at least one of these points is white and $p \in B$, p is a border point.

Characterization 2 (Tsao and Fu) *A border point $p \in B$ is simple, if and only if, the following conditions hold:*

2-a. $\chi(N_{26}(p) \cap B) = \chi(N^*_{26}(p) \cap B)$

2-b. the removal of p does not change the connectivity of at least two of the three 3x3 checking windows orthogonal to p. The planes to be checked depend on the directions of p, i.e, they depend on the white 6-neighbors of p and are the following: $cw_y(p)$ and $cw_z(p)$ when p is in the north or south direction ; $cw_x(p)$ and $cw_z(p)$ when p is in the east or west direction; $cw_x(p)$ and $cw_y(p)$ when p is in the up or down direction.

Bertrand and Malandain proposed in [6] a simple point characterization based on topological numbers (number of components in a neighborhood of a point). A complete thinning algorithm based on this characterization can be found in [2]. This characterization can be stated as follows:

Characterization 3 (Bertrand and Malandain) *A point $p \in B$ is simple if, and only if, the following conditions hold:*

3-a. $\#\mathcal{C}^p(B \cap N^*_{26}(p)) = 1$

3-b. $\#\mathcal{C}^p(\overline{B} \cap N^*_{18}(p)) = 1$

Saha et al [7] presented a simple point characterization and an algorithm to implement it based on tables. This characterization can be described as follows:

Characterization 4 (Saha et al) *The point $p \in B$ is simple if, and only if, the following conditions hold:*

4-a. $\#(\overline{B} \cap N^*_6(p)) \geq 1$

4-b. $\#(B \cap N^*_{26}(p)) \neq 0$

4-c. $\#\mathcal{C}(B \cap N^*_{26}(p)) = 1$

4-d. $\overline{B} \cap N^*_6(p)$ *is connected in* $\overline{B} \cap N^*_{18}(p)$

Another simple point characterization was proposed by Lee et al. This characterization and its efficient checking algorithm are discussed in [8]:

Characterization 5 (Lee et al) *A point $p \in B$ is simple if, and only if, the following conditions hold:*

5-a. $\#\mathcal{C}(B \cap N^*_{26}(p)) = 1$

5-b. $\chi(B \cap N_{26}(p)) = \chi(B \cap N^*_{26}(p))$

2.2 Equivalence between characterizations

Characterization 1 is widely accepted [1, 8] as effectively correct in the sense that it classifies all the points of an image as simple or non-simple. On the other hand, there have been discussions concerned with the correctness of some other characterizations [8, 9]. In this section we establish some formal equivalences between these characterizations. Here, two characterizations are equivalent if they classify equally a given set of points as simple or non-simple.

Theorem 1 *Characterization 1 is equivalent to characterization 3.*

Theorem 2 *Characterization 1 is equivalent to characterization 4.*

Theorem 3 *Characterization 1 is equivalent to characterization 5.*

Theorems 1 to 3 means that the characterizations proposed by Morgenthaler [1], Bertrand and Malandain [6], Saha et al [7] and Lee et al [8] are all equivalent (see proofs in the appendix).

Theorem 4 *Characterization 2 is not equivalent to characterization 1, but characterization 2 ensures topological preservation as established by Morgenthaler in characterization 1 (see [1]).*

Theorem 4 states that characterization 2 classifies some simple points as non-simple. Nevertheless, all the non-simple points, as defined by characterization 1, are correctly classified.

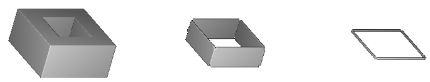
3 SOME GEOMETRIC CONSIDERATIONS

Simple point characterizations are commonly associated to thinning algorithms which reduce the structural representation of the original image. In this case, the simple points are those pixels/voxels which can be removed from the image without altering its topology.

In 2-d case and from a geometric point of view, thinning results in an skeleton composed of curves and lines. The skeletons obtained from 3-d thinning can be *surfaces skeletons* or *axis skeletons*. Surfaces skeletons are composed of thinned surfaces and curves while axis skeletons present only thinned curves (see figure In order to get one of these two skeletons, thinning algorithms avoid the removal of *end points*. End points are surface points to surface skeletons and curve points to axis skeletons. The thinning algorithms proposed in [4, 2, 8] present different definitions of *end points*.

4 GRAY-LEVEL IMAGES AND THINNING

The thinning operation is also available for 2-d gray-level images, and can be extended to the 3-d case. A *gray-level image* is a function $f : \mathbb{Z}^n \mapsto \mathbb{Z}^+$ which associates, to each point $p \in \mathbb{Z}^n$, a value in $\mathbb{Z}^+$ named the *gray level of p*. Once again, n is restricted to the set



(a) (b) (c)

Figure 1: *Example of object (a), a surface skeleton (b) and an axis skeleton (c).*

{2, 3}. Morphologically, the well-known thinning operation can be described as follows [19]. Let $T = (T_1, T_2)$ be a two-phase structuring element and f be a 2-d image. The thinning of f by T is given by:

$$g(p) = \begin{cases} Max[f(q)] & \text{iff, for } q \in T_2,\ r \in T_1, \\ & Max[f(q)] < f(p) \leq Min[f(r)] \\ f(p), & \text{otherwise.} \end{cases}$$

Obviously, we have to choose a class of structuring element T which preserves topology (homotopic structuring elements). This thinning operation can be easily extended to the 3-d case if we consider the right structuring elements preserving topological information. The minimum size of such structuring elements is $3 \times 3 \times 3$, the size of the 26-neighborhood necessary to preserve topology. Unlike the 2-d case, it is very difficult to define a general class of homotopic structuring elements among the 2^{26} possible configurations of 3-d neighborhood. Instead, we can adopt an strategy based on thresholds of image f and on simple point characterizations. A *thresholding* of f at level λ defines the set $B_\lambda = \{p \in \mathbb{Z}^n : f(p) \geq \lambda\}$. Let $\lambda = f(p)$ and B_λ be a thresholding of f. Then, if $T = B_\lambda \cap N_{26}(p)$ for each point p, we can write the 3-d thinning to gray-level image as:

$$g(p) = \begin{cases} Max_{q \in \overline{T} \cap N_{26}(p)}[f(q)] & \text{iff p is simple in } T \\ f(p), & \text{otherwise.} \end{cases}$$

The structuring element defined by the thresholding operation can also be used to determine end points according to binary end point definitions. The gray-level thinning algorithm is related to a very powerful morphological method used in image segmentation, namely, the watershed of a function [10].

5 OPTIMIZATION TECHNIQUES

5.1 Labeling effective points

During thinning operation, topological and geometric characteristics may be verified several times for all points of the image. Nevertheless, only border points are removed at each iteration. thus, avoiding unnecessary topological checking constitutes a simple way to speed up a thinning algorithm.

A point p $\in$ B is said effective, at iteration $i > 1$, if it has at least one point removed in $N^*_{26}(p)$, at iteration $i-1$. Since all current foreground points were considered as non-simple at iteration $i-1$, only effective points can be classified as simple at iteration i.

Labeling effective points consists in labeling as effective the points in the $N_{26}(p)$ neighborhood, during the removal of p. At the next iteration, only labeled points will be checked by the thinning operation. The complexity of the whole algorithm does not change since we introduce only one step before topological checking (the label verification), and another one before deletion (labeling at most 26 effective points). Both of these steps take time $O(1)$.

5.2 Configurations dictionary

Another way to avoid topological and geometric checking of a point is to have a dictionary containing the state of the points at earlier iterations. This dictionary can be used to avoid checking points with identical neighborhoods.

The only information we need to determine the state of a point p is based on the color of the points in $N_{26}(p)$. Thus, we can define a binary word δ_p indicating, in some fixed order, the colors (black=1 and white=0) of the points in $N_{26}(p)$. This binary representation can be used as a key to access the dictionary words.

To achieve efficient results in the implementation of the dictionary AVL-trees [11] and a variation of hashing tables were considered as data structures. The former is quite simple to define. The key δ_p is used to access an AVL-tree which has, at each node, the boolean information of the state (removable or not) of any point p. For the hashing table, the hashing function of a point p is given by a 12-bit binary word composed of the colors of the points in $N_{18}(p)/N_6(p)$. Collisions are stored in an AVL-tree associated to each position of the table. This structure reduces the size of the AVL-tree and, consequently, the execution time required to the operations over it.

It is important to note that, in practice, we need to limit the size of the dictionary to some constant, since the arbitrary growth of the dictionary can lead to a pathological case where all the 2^{26} possible configurations have to be stored. Note that computing the hashing function spends constant time and inserting or searching operations in an AVL-tree takes time $O(\log n)$, where n is the number of elements of the tree. As the size of this tree is upper bounded by a constant, consulting or updating the dictionary takes time $O(1)$.

6 Conclusion

We presented in this work some equivalences between simple point characterizations for 3-d images.We discussed the problem of implementing these characterizations in thinning algorithms,and proposed a very general method to improve their performance in terms of execution time. Also, an example of extending the 2-d gray-level thinning to the 3-d case, based on these characterizations, was given.

Result proofs

Lemma 1 *Condition 1-a holds for a point $p \in B$ if, and only if, condition 3-a also holds for p.*

PROOF: Because points $p \in B$ are adjacent to all points in $N^*_{26}(p)$ (since we consider 26-adjacency to black points) there is only one component in $B \cap N_{26}(p)$.

($\Rightarrow$) Suppose that 1-a holds, i.e., $\#\mathcal{C}(B \cap N^*_{26}(p)) = \#\mathcal{C}(B \cap N_{26}(p)) = 1$. As the only component in $B \cap N_{26}(p)$ is adjacent to p, we have $\#\mathcal{C}^p(B \cap N^*_{26}(p)) = 1$ and, consequently, 3-a holds for p. ($\Leftarrow$) In a converse way, the reader can easily show from 3-a that 1-a holds. ∎

Theorem 1 *We can rewrite theorem 1 as follows. Condition 1-a, 1-b and 1-c hold for a point p if, and only if, conditions 3-a and 3-b also hold for p.*

PROOF: ($\Rightarrow$) Suppose that 1-a, 1-b and 1-c hold for p. By 1-c and the Euler characteristic definition we have: $\#\mathcal{C}(B \cap N_{26}(p)) + \#\mathcal{H}(B \cap N_{26}(p)) - \#\mathcal{T}(B \cap N_{26}(p)) = \#\mathcal{C}(B \cap N^*_{26}(p)) + \#\mathcal{H}(B \cap N^*_{26}(p)) - \#\mathcal{T}(B \cap N^*_{26}(p))$. By 1-b and the discussion in lemma 1, we have: $1 + \#\mathcal{H}(B \cap N_{26}(p)) - \#\mathcal{T}(B \cap N_{26}(p)) = 1 + \#\mathcal{H}(B \cap N_{26}(p)) - \#\mathcal{T}(B \cap N^*_{26}(p)) \Rightarrow \#\mathcal{T}(B \cap N_{26}(p)) = \#\mathcal{T}(B \cap N^*_{26}(p))$.

Since p is black there is no space for tunnels in $B \cap N_{26}(p)$. So, $\#\mathcal{T}(B \cap N^*_{26}(p)) = \#\mathcal{T}(B \cap N_{26}(p)) = 0$. Saha et al showed in [7] that $\#\mathcal{T}(B \cap N^*_{26}(p)) = 0$ if, and only if, the set $\overline{B} \cap N^*_6(p)$ is 6-connected in the set $\overline{B} \cap N^*_{18}(p)$, i.e., $\#\mathcal{C}(\overline{B} \cap N_6(p)) = 1$, since 6-adjacency is considered for white points. With this result and the fact that $\#\mathcal{T}(B \cap N^*_{26}(p)) = 0$ we have $\#\mathcal{C}(\overline{B} \cap N^*_{18}(p)) = 1$ and, as p is adjacent to any white point in $\overline{B} \cap N^*_6(p)$ and these points are connected, we have that $\#\mathcal{C}(\overline{B} \cap N^*_{18}(p)) = 1$ and, hence, 3-b holds for p. From lemma 1, we have that 3-a holds.

($\Leftarrow$) Suppose that 3-a and 3-b hold for a point p. By 3-a and lemma 1, we have 1-a. Thus, we know that there is only one component adjacent to p in $\overline{B} \cap N^*_{18}(p)$. This is possible only if $\overline{B} \cap N^*_6(p) \neq$, i.e., we have at least one white point 6-adjacent to p (in other words, p has to be a border point). The only way to have a cavity in $B \cap N^*_{26}(p)$ is to consider $N_6(p) \subset B$. But, as there is at least one white point in $N^*_6(p)$, $N^*_6(p) \not\subset B$. Thus, we have no cavity in $B \cap N^*_{26}(p)$. As p is black, there is no space for cavities in $B \cap N_{26}(p)$. So, $\#\mathcal{H}(B \cap N^*_{26}(p)) = \#\mathcal{H}(B \cap N_{26}(p)) = 0$ and 1-b holds for p. By Saha et al [7] we have that $\#\mathcal{T}(B \cap N^*_{26}(p)) = 0$. By the previous discussion, $\#\mathcal{T}(B \cap N_{26}(p)) = 1$. From this, 1-a and 1-b, we conclude that 1-c holds for p.

We can use the results of theorem 1 to prove theorem 2. Since we know that characterizations 1 and 3 are equivalent, we can write theorem 2 as follows:

Theorem 2 *Conditions 3-a and 3-b hold for a point p if, and only if, conditions 4-a, 4-b, 4-c and 4-d also hold for p.*

PROOF: The proof is trivial. Condition 3-a is equivalent to conditions 4-b and 4-c; condition 3-b is equivalent to conditions 4-a and 4-d. ∎

Theorem 3 *Theorem 3 can be written as follows. Conditions 1-a, 1-b and 1-c hold for p if, and only if, conditions 5-a and 5-b also hold for p.*

PROOF: By the previous discussion (proof of theorem 1) we know that condition 1-a is equivalent to condition 5-a and condition 1-c is equivalent to condition 5-b. We have to prove only that conditions 5-a and 5-b imply condition 1-b. In fact, this means that condition 1-b is always valid if 1-a and 1-c are valid. This result was proved by Tsao and Fu to images with $\beta = 26$ and $\omega = 6$ (our case here), as reported in [1]. ∎

Lemma 2 *Characterization 1 is not equivalent to characterization 2.*

PROOF: In this case it is just necessary to show an example of a point that can be classified as simple by characterization 1 and non-simple by characterization 2. Such an example is shown in figure 2. ∎

Figure 2: *Example of a point p (grid central point) considered as simple by characterization 1 and non-simple by characterization 2. Note that checking windows $cw_y(p)$ and $cw_z(p)$ are disconnected when the central point is removed.*

Lemma 3 *Let $p \in B$ be a border point. Conditions 2-a and 2-b hold for p only if conditions 4-a to 4-d, also hold for p.*

PROOF: Let p be a north point such that 5-a and 5-b hold for p.

We start showing by contradiction that condition 4-d holds for p. To do that we suppose 4-d does not hold for p. Thus, $\exists q \in \overline{B} \cap N_6^*(p)$ such that there is no 6-path $\pi =< north(p), \ldots, q >$, with all points in $\overline{B} \cap N_{18}^*(p)$. Since each of the 6-neighbors of p differs from p in only one coordinate, it is easy to see that $N_6^*(p) \subset cw_y(p) \cup cw_z(p)$. At least one of these planes contains both $north(p)$ and q. Suppose, with no loss of generality, that $cw_y(p)$ is such a plane. From condition 2-a, there is only one black component in $cw_y(p) - \{p\}$. In figure 3 the points $north(p)$ and q divide $cw_y(p)$ into two partitions S' and S''. By 2-b there is just one black component in $cw_y(p) - \{p\}$ which proves the existence of one, and only one 6-connected path in $cw_y(p) - \{p\}$ leading from $north(p)$ to q. We have a contradiction here and, thus, condition 4-d is valid when p is a north point. The proofs for the other directions of p are quite similar.

Conditions 4-a, 4-b, 4-c are easily verified from the above hypothesis and are left to the reader. ∎

Now we can use theorems 2 and 3 to state theorem 4 as follows.

Theorem 4 *Conditions 2-a and 2-b hold for p only if conditions 4-a to 4-d also hold for p, but the converse is not true.*

PROOF: The first part of the statement comes directly from lemma 3 and theorem 2. The converse is not true because of lemma 2. ∎

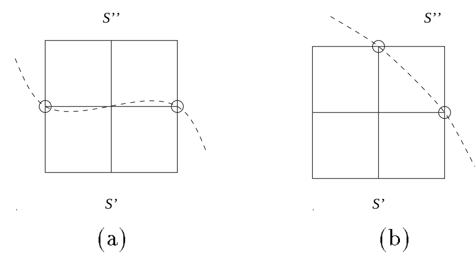


Figure 3: *Possible configurations to white points north(p) and q in $cw_y(p)$.*

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