Exact Algorithms for Circles on the Sphere

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Abstract

We develop exact algorithms for geometric operations on general circles and circular arcs on the sphere, using integer homogeneous coordinates. The algorithms include testing a point against a circle, computing the intersection of two circles, and ordering three arcs out of the same point. These operations allow robust manipulation of maps on the sphere, providing a reliable framework for GIS, robotics, and other geometric applications.

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1 Introduction

A *spherical map* is a partition of the sphere's surface in three kinds of *elements*: *vertices* (points), *edges* (circles and arcs of circles, not necessary geodesic ones), and *faces* (open regions). Most maps handled in geographical information systems (GIS) are of this type: examples include geodesic polygons, latitude-longitude grids, polygonal maps under stereographic projection, sattelite images, etc. [6].

Our ultimate goal is to develop an exact representation of spherical maps, and robust algorithms to operate on them. Here we describe a basic ingredient of that goal: an exact representation of points, circles and circular arcs on the sphere, and exact formulas for various geometric operations on such objects. In particular, we give exact formulas for computing the points of intersection of two circles, the location of a point with respect to a circle, the ordering of circular arcs around a point and the circular ordering of three points on a circle. We have implemented and tested all these algorithms in Modula-3 [7], using the *GNU Multiple Precision* (GMP) arithmetic library [4].

Exactness may seem a pointless luxury in GIS applications, since GIS data is by its nature approximate, and approximate results are sufficient for all practical purposes. However, most geometric algorithms used in GIS, such as point location and map overlay, become much more complex and prone to failure if their basic operations are subject to rounding errors, no matter how small [5, 11, 10]. Consider for example a distributed application that cuts a map into smaller submaps, handles each piece to a separate processor, and combines the partial results into a single map. If the cutting step is exact, the final step needs only to identify common boundary edges between the partial results, and remove them. The task becomes much harder if the cutting step is affected

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by rounding errors: the partial results may overlap, or may be separated by gaps. The pasting operation is then almost impossible to specify, let alone to implement [2, 1].

The representation that we define here allow us to represent any spherical map with arbitrary accuracy, without increasing the number of elements of the original map; and also to compute exactly the overlay [9] of two such maps. Moreover, the resulting maps can be overlaid again and again without intermediate rounding or simplification steps. We note that most geometric operations used in GIS (region intersection, point location, feature extraction, clipping etc) can be reduced to map overlay.

To be sure, there are many geometric operations that cannot always be carried out exactly in our model: for instance, constructing a circle through three vertices, or centered at one vertex. But such objects can still be approximated in our model with arbitrary accuracy, and then exactly manipulated by our algorithms.

2 Spherical geometry

2.1 Circles on the sphere

Let $\mathbb{S}^2$ be the *unit sphere* of $\mathbb{R}^3$, that is, the sphere with unit radius centered at the origin O of $\mathbb{R}^3$. In this paper, we are concerned with oriented circles on the sphere $\mathbb{S}^2$, which we call *S-circles*.

Note that every S-circle corresponds to the intersection of $\mathbb{S}^2$ with an oriented plane. More precisely, given an S-circle c there is a unique oriented plane α such that $c = \alpha \cap \mathbb{S}^2$ and vice-versa. We say that α is the *supporting plane* of c, and write $\alpha = \mathrm{spln}(c)$. Conversely, we also say that c is the S-circle defined by α, and write $c = \mathrm{scrc}(\alpha)$. By definition, the *positive sense of travel* along $\mathrm{scrc}(\alpha)$ is the one that agrees with the orientation of the plane α. See figure 1.

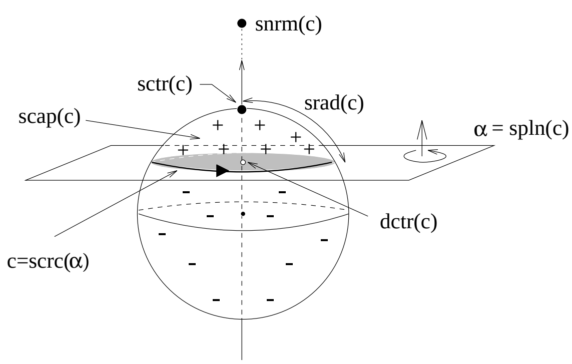

Figure 1: Elements of an S-circle.

Thus, we can unambiguously represent an S-circle by the coefficients $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ of the oriented plane α: four real numbers such that a point p with homogeneous coordinates $[w, x, y, z]$ is on the plane α if and only if $\alpha_0 w + \alpha_1 x + \alpha_2 y + \alpha_3 z = 0$; and, moreover, p is on the *positive* (resp. *negative*) *side* of α if and only if $\alpha_0 w + \alpha_1 x + \alpha_2 y + \alpha_3 z$ is positive (resp. negative)[1].

[1] We use the notation of Stolfi [8] where the homogeneous coordinates of a point (x, y, z) of $\mathbb{R}^3$ are $[1, x, y, z]$ with positive scalar multiples identified; that is, for $\lambda > 0$, $[w, x, y, z]$ and $[\lambda w, \lambda x, \lambda y, \lambda z]$ are the same point. In particular, the point $[0, x, y, z]$ is assumed to be at infinity in the direction of the Cartesian vector (x, y, z).

The plane α defines an S-circle (i.e., intersects the sphere $\mathbb{S}^2$) if and only if $\alpha_0^2 \leq \alpha_1^2+\alpha_2^2+\alpha_3^2$. In this case, the S-circle defined by α will be denoted by $((\alpha_0, \alpha_1, \alpha_2, \alpha_3))$. If $\alpha_0^2 = \alpha_1^2+\alpha_2^2+\alpha_3^2$, then α is tangent to $\mathbb{S}^2$, and $\mathrm{scrc}(\alpha)$ reduces to a point.

The direction orthogonal to $\mathrm{spln}(c)$ and pointing into its positive side is the *normal* of c, denoted $\mathrm{snrm}(c)$. It is convenient to think of $\mathrm{snrm}(c)$ as the point at infinity $[0, \alpha_1, \alpha_2, \alpha_3]$.

An S-circle c divides $\mathbb{S}^2$ into two open regions, one of which may be empty. By definition, the *positive cap* of c is the part of $\mathbb{S}^2$ that lies on the positive side of $\mathrm{spln}(\alpha)$. Also, an S-circle has two centers: the *S-center* on its positive cap, and the *D-center* on its supporting plane. Their homogeneous coordinates are:

$$\begin{aligned}\mathrm{sctr}(c) &= \left[\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}, \alpha_1, \alpha_2, \alpha_3\right]\\ \mathrm{dctr}(c) &= \left[\alpha_1^2+\alpha_2^2+\alpha_3^2, -\alpha_0\alpha_1, -\alpha_0\alpha_2, -\alpha_0\alpha_3\right]\end{aligned}$$

In particular, if c is a single point, its positive cap is either empty or the whole $\mathbb{S}^2$, and $\mathrm{dctr}(c) = \pm\,\mathrm{sctr}(c)$.

The length of the spherical arc that joins $\mathrm{sctr}(c)$ to any point on c is the *S-radius* of c, given by $\mathrm{srad}(c) = \arccos\left(-\alpha_0/\left(\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}\right)\right) \in [0 \,\_\_\, \pi]$. Conversely, the S-circle c whose S-center is $p = [p_0, p_1, p_2, p_3]$ and whose S-radius is $\theta \in [0 \,\_\_\, \pi]$ is $c = ((-p_0 \cos\theta, p_1, p_2, p_3))$.

2.2 Arcs of S-circles

Two distinct points p and q on an S-circle $c = \mathrm{scrc}(\alpha)$ divide it into two connected parts called S-arcs. See figure 2. By definition, *the S-arc of c from p to q*, denoted by $\mathrm{sarc}(p, q, c)$, is the set of points encountered as we move from p to q on c, along its positive sense of travel. Given $A = \mathrm{sarc}(p, q, c)$, we write $c = \mathrm{scrc}(A)$, $p = \mathrm{org}(A)$ and $q = \mathrm{dst}(A)$.

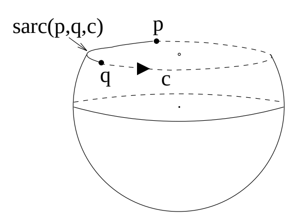

Figure 2: The S-arc of c determined by p and q.

Geometrically, $\mathrm{sarc}(p, q, c)$ corresponds to the part of c that lies on the positive side of the plane $\beta = p \vee q \vee \mathrm{snrm}(c)$, assuming $\vee$ as defined by Stolfi [8].

3 Exact spherical geometry

3.1 Rational points

A point is said to be *rational* if all its Cartesian coordinates are rational numbers. Note that the set of rational points is a dense subset of $\mathbb{R}^3$.

Since a point does not change when its homogeneous coordinates are scaled by a positive factor, any rational point can be represented by four integers.

3.2 Exactly representable points of $\mathbb{S}^2$

There are two sets of points of $\mathbb{S}^2$ that have obvious exact representations. One is the set $\mathcal{A}$ consisting of the rational points of $\mathbb{S}^2$, that is

$$\mathcal{A} = \{[a_0, .. a_3] \mid a_1^2 + a_2^2 + a_3^2 = a_0^2 \text{ and } (a_0, .. a_3) \in \mathbb{Z}^4\}$$

Another is the set $\mathcal{B}$ consisting of the points $p = [w, x, y, z]$ whose homogeneous coordinates x, y, and z are rational. Since p lies on $\mathbb{S}^2$, the weight coordinate w can be determined by the formula $w = \sqrt{x^2 + y^2 + z^2}$, and can be left implicit.

$$\mathcal{B} = \{[b_0, .. b_3] \mid b_1^2 + b_2^2 + b_3^2 = b_0^2 \text{ and } (b_1, b_2, b_3) \in \mathbb{Z}^3\}$$

Note that $\mathcal{B}$ is the radial projection onto $\mathbb{S}^2$ of all rational points of $\mathbb{R}^3$. Obviously, $\mathcal{A} \subset \mathcal{B}$.

3.3 Rational Planes and S-circles

We say that a plane α is *rational* if its coefficients $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ are rational—or, equivalently, integers. An S-circle is *rational* if its supporting plane is rational.

We propose to consider only rational S-circles and S-arcs thereof. This is not a significant restriction for practical purposes, since, in the Hausdorff metric, the set of rational S-circles is dense in the set of all S-circles. In fact, let $a = (\!(a_0, a_1, a_2, a_3)\!)$ be any S-circle, and let ϵ be a real number between 0 and 1. One can verify that

$$b = \left(\!\!\left(\left\lceil \frac{a_0}{\delta} \right\rceil, \left\lceil \frac{a_1}{\delta} \right\rceil, \left\lceil \frac{a_2}{\delta} \right\rceil, \left\lceil \frac{a_3}{\delta} \right\rceil\right)\!\!\right)$$

where $\delta = \epsilon^2 \sqrt{a_1^2 + a_2^2 + a_3^2}/36$, is a rational S-circle such that $\|a - b\| \leq \epsilon$ in the Hausdorff metric.

Now, let $c = (\!(c_0, .. c_3)\!)$ be a rational S-circle. By formulas of section 2.1, we have $\text{sctr}(c) \in \mathcal{B}$, $\text{dctr}(c) \in \mathcal{A}$ and $\text{snrm}(c) \in \mathcal{A}$, so these elements can be computed exactly. Also, given any $p \in \mathcal{B}$, and any $q \in \mathcal{A}$, there is a rational S-circle with S-center p passing through q.

Finally, given three $\mathcal{A}$-points p, q and r, the S-circle passing through these points is $\text{scrc}(\alpha)$ where $\alpha = p \vee q \vee r$. The coefficients of α are 3×3 determinants on the coordinates of p, q and r [8] and therefore are rational numbers. Note that the circle passing through three $\mathcal{B}$-points may not be rational; fortunately, this operation is not necessary for map overlay.

3.4 Exact intersection of rational S-circles

An essential step in the map overlay operation is to compute the intersection of two S-circles—which, in general, is either empty or consists of two points. To make this operation unambiguous, we define *the (canonical) meeting point* of two S-circles a and b as being the point $a \wedge b$ where a crosses b from its positive side into its negative side. See figure 3(a).

Unfortunately, this point may not be rational, and not even in $\mathcal{B}$. For instance, $(\!(1, 2, 2, 2)\!) \wedge (\!(1, 2, -2, 2)\!) = [4\ ,\ -1 + \sqrt{7}\ ,\ 0\ ,\ -1 - \sqrt{7}\ ]$; since the ratio $(-1 - \sqrt{7})/(-1 + \sqrt{7})$ is not rational, this point is not in $\mathcal{B}$.

To circumvent this difficulty, we observe that $a \cap b$ is equal to the intersection of the line $l = \text{spln}(a) \cap \text{spln}(b)$ and $\mathbb{S}^2$. See figure 3(b). Thus, the problem of representing $a \cap b$ reduces to that of representing the line l. Moreover, if we consider l to be oriented, we can use its orientation to unambiguously select $a \wedge b$.

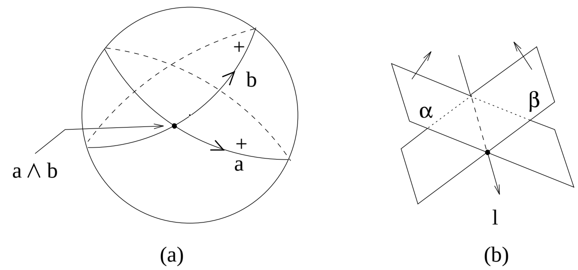

Figure 3: Intersection of two S-circles.

3.5 Plücker coefficients of a line

By definition, given two (oriented) planes α and β, their oriented intersection $\alpha \wedge \beta$ is the line $\alpha \cap \beta$, oriented in the direction $normal(\alpha) \times normal(\beta)$—see figure 3(b). That line can be unambiguously represented by the six coefficients $\langle l_{01}, l_{02}, l_{12}, l_{03}, l_{13}, l_{23} \rangle$ where $l_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$. These six numbers are the *Plücker coefficients* of l [8, 3]. We write $l = \langle l_0, l_1, l_2, l_3, l_4, l_5 \rangle$, with the determinants implicitly ordered as above.

These numbers are not independent: a non-zero sextuple $\langle l_0, ..l_5 \rangle$ represents a line of $\mathbb{R}^3$ if and only if $l_0 l_5 - l_1 l_4 + l_2 l_3 = 0$. On the other hand, the Plücker coefficients of a line are homogeneous, that is, $\langle l_0, ..l_5 \rangle$ and $\langle \lambda l_0, ..\lambda l_5 \rangle$ are the same line, for any $\lambda > 0$.

The *direction* of a finite line l is the point at infinity $\text{dir}(l) = [0, l_5, -l_4, l_2]$. Thus the lines $l = \langle l_0, ..l_5 \rangle$ and $m = \langle -l_0, ..-l_5 \rangle$, which are incident to the same points, can be distinguished by their orientation. We say that m is the *opposite* of l, denoted by $m = \neg l$.

Let $p = [p_0, ..p_3]$ and $q = [q_0, ..q_3]$ be two points, $\alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ and $\beta = \langle \beta_0, \beta_1, \beta_2, \beta_3 \rangle$ be two planes and $l = \langle l_0, ..l_5 \rangle$ be a line. The basic geometric operations $\vee$ (join) and $\wedge$ (meet), as defined by Stolfi [8], involving lines are given by these formulas:

$$
\begin{aligned}
\alpha \wedge \beta = &\ \langle \alpha_0\beta_1 - \alpha_1\beta_0, \alpha_0\beta_2 - \alpha_2\beta_0, \alpha_1\beta_2 - \alpha_2\beta_1, \\
&\ \alpha_0\beta_3 - \alpha_3\beta_0, \alpha_1\beta_3 - \alpha_3\beta_1, \alpha_2\beta_3 - \alpha_3\beta_2 \rangle \\
l \wedge \alpha = &\ [-l_2\alpha_3 + l_4\alpha_2 - l_5\alpha_1, l_1\alpha_3 - l_3\alpha_2 + l_5\alpha_0, \\
&\ -l_0\alpha_3 + l_3\alpha_1 - l_4\alpha_0, l_0\alpha_2 - l_1\alpha_1 + l_2\alpha_0] \\
p \vee l = &\ \langle\ l_0p_1 + l_1p_2 + l_3p_3, -l_0p_0 + l_2p_2 + l_4p_3, \\
&\ -l_1p_0 - l_2p_1 + l_5p_3, -l_3p_0 - l_4p_1 - l_5p_2 \rangle \\
p \vee q = &\ \langle p_2q_3 - p_3q_2,\ p_3q_1 - p_1q_3,\ p_0q_3 - p_3q_0, \\
&\ p_1q_2 - p_2q_1,\ p_2q_0 - p_0q_2,\ p_0q_1 - p_1q_0 \rangle
\end{aligned}
\tag{1}
$$

Note that these formulas return all-zero tuples when the operation is not defined; for instance, $l \wedge \alpha = [0, 0, 0, 0]$ when l lies on the plane α.

The line l passes through the origin O if and only if $l_0 = l_1 = l_3 = 0$. More generally, the Euclidean distance between l and O is

$$
\text{dist}(l, O) = \sqrt{\frac{l_0^2 + l_1^2 + l_3^2}{l_2^2 + l_4^2 + l_5^2}} \tag{2}
$$

The *bissector* of l is the plane $\text{bisect}(l)$ perpendicular to l, passing through O and oriented such that $\text{dir}(l)$ is on its positive side; its coefficients are $\langle 0, l_5, -l_4, l_2 \rangle$.

A line is said to be *rational* if its Plücker coefficients are rational or, equivalently, integers. Formulas (1) imply that the intersection of two rational planes is a rational line, and so is the line passing through two rational points.

3.6 Enter and exit points of a line

With the above conventions, the meeting point $a \wedge b$ of two S-circles is the point $\mathrm{ext}(l)$ where the oriented line $l = \mathrm{spln}(a) \wedge \mathrm{spln}(b)$ exits the sphere $\mathbb{S}^2$. See figure 4.

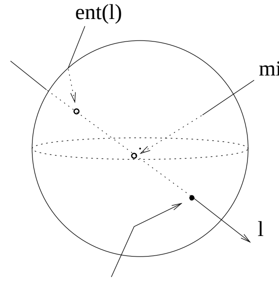

Figure 4: The points $\mathrm{ent}(l)$, $\mathrm{mid}(l)$ and $\mathrm{ext}(l)$.

The point where l enters the sphere will be denoted $\mathrm{ent}(l)$. The midpoint of $\mathrm{ent}(l)$ and $\mathrm{ext}(l)$ is $\mathrm{mid}(l) = l \wedge \mathrm{bisect}(l)$, which is the point of l closest to the origin O. If $l = \langle l_0, ..l_5 \rangle$,

$$\begin{aligned}
\mathrm{ext}(l) &= [\, \mu, \; -l_1l_2 - l_3l_4 + l_5\sqrt{\delta}, \\
&\qquad l_0l_2 - l_3l_5 - l_4\sqrt{\delta}, \\
&\qquad l_0l_4 + l_1l_5 + l_2\sqrt{\delta} \;] \\
\mathrm{mid}(l) &= [\, \mu, \; -l_1l_2 - l_3l_4, \\
&\qquad l_0l_2 - l_3l_5, \\
&\qquad l_0l_4 + l_1l_5 \;] \\
\mathrm{ent}(l) &= [\, \mu, \; -l_1l_2 - l_3l_4 - l_5\sqrt{\delta}, \\
&\qquad l_0l_2 - l_3l_5 + l_4\sqrt{\delta}, \\
&\qquad l_0l_4 + l_1l_5 - l_2\sqrt{\delta} \;]
\end{aligned} \tag{3}$$

where $\mu = l_2^2 + l_4^2 + l_5^2$ and $\delta = \mu - (l_0^2 + l_1^2 + l_3^2)$.

Observe that $\mathrm{ent}(l) = \mathrm{ext}(l)$ if and only if l is tangent to $\mathbb{S}^2$. By formula (2), this happens if and only if $l_2^2 + l_4^2 + l_5^2 \;=\; l_0^2 + l_1^2 + l_3^2$.

3.7 Sub-rational points of $\mathbb{S}^2$

From the previous sections, we can conclude that the meeting points of two rational S-circles belong to the set of *sub-rational* points of $\mathbb{S}^2$

$$\mathcal{C} = \{\, \mathrm{ext}(l) \mid l \text{ is a rational line} \,\}$$

Note that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$. Every point p of $\mathcal{C}$ can be represented exactly by the six integer coefficients of any rational line l such that $p = \mathrm{ext}(l)$. The relationship between $\mathcal{A}$ and $\mathcal{C}$ is clarified by the following theorems (whose proofs are given in appendix A):

Theorem 1 *Let $l = \langle l_0, l_1, l_2, l_3, l_4, l_5 \rangle$ be a rational line. Then, $\mathrm{ext}(l)$ is in $\mathcal{A}$ if and only if $(l_2^2 + l_4^2 + l_5^2) - (l_0^2 + l_1^2 + l_3^2)$ is a perfect square.*

Theorem 2 *A point p lies in $\mathcal{A}$ if and only if there are two distinct rational lines l and m such that* $\mathrm{ext}(l) = p = \mathrm{ext}(m)$.

Theorem 3 *A point p of $\mathbb{R}^3$ is in $\mathcal{C}$ if and only if $p = [a_0, a_1 + b_1\sqrt{c}, a_2 + b_2\sqrt{c}, a_3 + b_3\sqrt{c}\,]$ where a_0, a_1, a_2, a_3, b_1, b_2, b_3, and c are integers satifying*

(i) $a_0 \neq 0$ *and* $b_1^2 + b_2^2 + b_3^2 \neq 0$

(ii) $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$

(iii) $a_0(a_0 - c) = a_1^2 + a_2^2 + a_3^2$

In that case, $p = \mathrm{ext}(l)$ where $l = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_0b_3,\ a_1b_2 - a_2b_1,\ -a_0b_2,\ a_0b_1\rangle$.

3.8 Canonical representation of points

Theorem 2 says that a point p belongs to $(\mathcal{C} \setminus \mathcal{A})$ if and only there is a unique rational line l such that $p = \mathrm{ext}(l)$. In that case, we denote l by $\mathrm{stab}(p)$. On the other hand, if $p \in \mathcal{A}$, there are infinitely many lines such that $p = \mathrm{ext}(l)$. In particular, for any rational point q inside $\mathbb{S}^2$, the line $q \vee p$ will have this property. In this case, we define $\mathrm{stab}(p)$ as being the line $O \vee p$.

The geometric algorithms become much simpler if we consistently represent every point $p \in \mathcal{C}$ by the integer Plücker coefficients of the canonical line $\mathrm{stab}(p)$, and if these coefficients are *reduced* by eliminating all common factors. Then, for example, testing the equality of two points reduces to comparing their 6-tuples. We implement this idea by two procedures:

---

procedure *Apt*($p : \mathcal{C}$*-point*) : $\mathcal{A}$*-point*;
  {If $p \in \mathcal{A}$ returns the homogeneous coordinates of p;
   otherwise returns $[0, 0, 0, 0]$}
  $\delta \leftarrow p_2^2 + p_4^2 + p_5^2 - (p_0^2 + p_1^2 + p_3^2)$;
  if δ is a perfect square **then**
    return $[\mu, -p_1p_2 - p_3p_4 + p_5\sqrt{\delta},$
        $p_0p_2 - p_3p_5 - p_4\sqrt{\delta},$
        $p_0p_4 + p_1p_5 + p_2\sqrt{\delta}\,]$
  else return $[0, 0, 0, 0]$

---

procedure *Stab*($p : \mathcal{C}$*-point*) : $\mathcal{C}$*-point*;
  {Returns $\mathrm{stab}(p)$, the canonical representation of p}
  $ap \leftarrow Apt(p)$;
  if $ap = [0, 0, 0, 0]$ **then return** *Reduce*(p)
  else return *Reduce*($O \vee ap$)

---

4 Robust algorithms

4.1 Relative position of a point and a plane

Another important predicate is the position of a point $p \in \mathcal{C}$ relative to a rational plane α, denoted by $\mathit{SideOf}(p, \alpha)$.

To implement this test, we consider three cases. If p is rational, then $\mathit{SideOf}(p, \alpha)$ can be written as $\mathit{Apt}(p) \diamond \alpha$, where $\diamond$ is the point-plane test as defined by Stolfi [8]. Otherwise, let $l = \mathrm{stab}(p)$ and $q = l \wedge \alpha$. If q is inside $\mathbb{S}^2$, then $\mathrm{ext}(l)$ and the rational point $\mathrm{dir}(l)$ are on the same side of α. On the other hand, if q is outside $\mathbb{S}^2$, then $\mathrm{ext}(l)$ and $\mathrm{ent}(l)$ are on the same side of α, and so is $\mathrm{mid}(l)$ (which is rational).

---

procedure $\mathit{SideOf}(p : \mathcal{C}\text{-point};\ \alpha : \text{Plane}) : \text{Sign};$

  {Returns the relative position of a $\mathcal{C}$-point p and a plane α}

  $ap \leftarrow \mathit{Apt}(p);$

  if $ap \neq [0, 0, 0, 0]$ **then return** $ap \diamond \alpha$

  else

    $l \leftarrow \mathrm{stab}(p); \quad q \leftarrow l \wedge \alpha;$

    $\sigma \leftarrow \mathrm{sign}(q_1^2 + q_2^2 + q_3^2 - q_0^2);$

    $\mu \leftarrow \mathrm{mid}(l) \diamond \alpha; \quad \delta \leftarrow \mathrm{dir}(l) \diamond \alpha;$

    if $\sigma = -1$ **then return** δ

    elsif $\sigma = 1$ **then return** μ

    else if $\mu = \delta$ **then return** μ **else return** 0

Note that this operation also allow us to test the position of p relative to a rational S-circle c, by evaluating $\mathit{SideOf}(p, \mathrm{spln}(c))$.

5 Circular order

5.1 Ahead and behind

Many algorithms for circles on the sphere rely on the *circular ordering* of points along a circle. This concept can be defined in terms of a basic predicate $\otimes_c(p, q)$ that takes two points p and q on an S-circle c, and returns $+1$ if q is *ahead of* p or -1 if q is *behind* p. See figures 5(a,b).

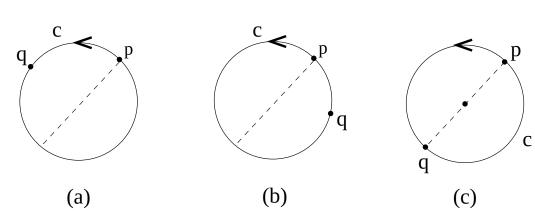

Figure 5: (a) $\otimes_c(p, q) = +1$, (b) $\otimes_c(p, q) = -1$, (c) $q = \oslash_c(p)$.

This predicate describes the direction of the "natural" or "shortest" route from p

to q. For each point p on the circle c we must define the "diametrally opposite" point $\oslash_c(p)$. See figure 5(c). Then the "shortest" path from p to q is the one that does not go through $\oslash_c(p)$. Note that the direction of the shortest path is ambiguous when $q = p$, or $q = \oslash_c(p)$; in those cases, we define $\otimes_c(p, q)$ as 0 (indeterminate). As one can see, $\otimes_c(q, p) = \otimes_{\neg c}(p, q) = -\otimes_c(p, q)$.

Observe that the predicate $\otimes_c(p, q)$ corresponds to checking whether the line $p \vee q$ is oriented positively or negatively with respect to the axis of the S-circle c, which is defined as the line $O \vee \mathrm{snrm}(c)$. Thus, $\otimes_c(p, q)$ is the the orientation of the tetrahedron $O, \mathrm{snrm}(c), p, q$. Therefore,

$$\otimes_c(p,q) = \mathrm{sign}\begin{vmatrix} 1 & 0 & 0 & 0 \\ c_0 & c_1 & c_2 & c_3 \\ p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \end{vmatrix} = \mathrm{sign}\begin{vmatrix} c_1 & c_2 & c_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}$$

Since the points p or q may not be rational, we cannot compute the determinant exactly. Instead, let l be $\mathrm{stab}(p)$, let α be the $\mathrm{spln}(c)$, and let $\beta = \langle \beta_0, \beta_1, \beta_2, \beta_3 \rangle$ be the plane $l \vee \mathrm{snrm}(c)$.

If β passes through O, then $\otimes_c(p, q)$ reduces to $\mathit{SideOf}(q, \beta)$. See figure 6(a). On

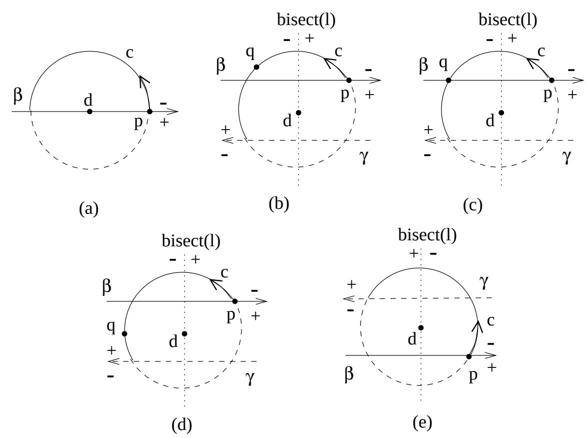

Figure 6: Ahead/Behind of two points on an S-circle; $d = \mathrm{dctr}(c) = (O \vee \mathrm{snrm}(c)) \wedge \mathrm{spln}(c)$

the other hand, if O is not on β, let γ be the reflection of the plane β across the origin O, that is, $\gamma = \langle \beta_0, -\beta_1, -\beta_2, -\beta_3 \rangle$. Then, $\oslash_c(p) = \mathrm{ext}(\alpha \wedge \gamma)$. Firstly, suppose that O is on the positive side of β. Then $\otimes_c(p, q)$ is $+1$ if and only if

- q is on the negative side of β (figure 6(b)); or
- q is on β and $p \neq q$ (figure 6(c)); or
- q is on the positive side of β and on the positive side of γ and also, on the negative side of the plane $\mathrm{bisect}(l)$ (figure 6(d)).

Secondly, if O is on the negative side of β, as in figure 6(e), we can reduce this case to the previous one, using the identity $\otimes_c(p, q) = \otimes_{\neg c}(q, p)$. Note that inverting c

corresponds to inverting β and γ. The complete algorithm is

---

procedure *Ahead*$(c : \text{Plane};\ p, q : \mathcal{C}\text{-point}) : \text{Sign};$
  $\{\text{Computes } \otimes_c(p, q)\}$
  $\beta \leftarrow \mathit{Stab}(p) \vee \text{snrm}(c);\quad d_c \leftarrow \mathit{SideOf}(O, \beta);$
  if $d_c = 0$ **then return** $-\mathit{SideOf}(q, \beta);$
  case $d_c \cdot \mathit{SideOf}(q, \beta)$ **of**
  -1 : $d_m \leftarrow +1;$
   0 : **if** $p = q$ **then** $d_m \leftarrow 0$ **else** $d_m \leftarrow +1$
  $+1$: **if** $\mathit{SideOf}(q, \text{bisect}(l)) \geq 0$
    then $d_m \leftarrow -1$
    else
      $\gamma \leftarrow \langle \beta_0, -\beta_1, -\beta_2, -\beta_3 \rangle;$
      $d_m \leftarrow dc \cdot \mathit{SideOf}(q, \gamma)$
  return $d_c \cdot d_m$

5.2 Circular order of three points on an S-circle

The concept of circular order can be formalized by a predicate $\otimes_c(p, q, r)$ that returns $+1$ if the three points p, q, r occur in that order along the oriented circle c; -1 if they occur in the opposite order; and 0 if any two of the points coincide. In other words, $\otimes_c(p, q, r)$ is the orientation $\Delta(p, q, r, \text{snrm}(c))$ of the triangle pqr relative to $\alpha = \text{spln}(c)$, that is

$$\otimes_c(p, q, r) = \text{sign} \begin{vmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \\ r_0 & r_1 & r_2 & r_3 \\ c_0 & c_1 & c_2 & c_3 \end{vmatrix}$$

Again, since p, q, or r may not be rational, we cannot compute the determinant exactly. Thus, we will determine $\otimes_c(p, q, r)$ reducing it to three calls of the ahead/behind predicate $\otimes_c$.

Let u be a point on the plane α. The *signature* of u relative to the triangle pqr and the plane α is the sign sequence $\sigma_0\sigma_1\sigma_2$, where $\sigma_0 = \Delta(u, q, r, c)$, $\sigma_1 = \Delta(p, u, r, c)$, $\sigma_2 = \Delta(p, q, u, c)$ [8]. Given the orientation of the triangle pqr relative to α, each σ_i tells whether the point u and the ith vertex of the triangle are both on the same side of the line determined by the other two vertices. See figure 7. In particular, if u is inside the triangle pqr, its signature is '$+++$' or '$---$', depending on the triangle's orientation relative to α.

Moreover, let c be a circle passing through the points p, q, and r. It can be seen that the signature of any point u inside that circle is dominated by the orientation of the triangle pqr relative to α, that is $\Delta(p, q, r, \alpha) = \text{sign}(\sigma_0 + \sigma_1 + \sigma_2)$. In particular, if we use $u = \text{dctr}(c)$ we have $\sigma_0 = \Delta(\text{dctr}(c), q, r, \text{snrm}(c))$. Since $\text{dctr}(c) \vee \text{snrm}(c) = O \vee \text{snrm}(c)$ then we can replace $\text{dctr}(c)$ with O in this formula. Therefore, $\sigma_0 = \Delta(O, q, r, \text{snrm}(c)) = \otimes_c(q, r)$. By a similar reasoning, we get $\sigma_1 = \otimes_c(r, p)$ and $\sigma_2 = \otimes_c(p, q)$. The test then

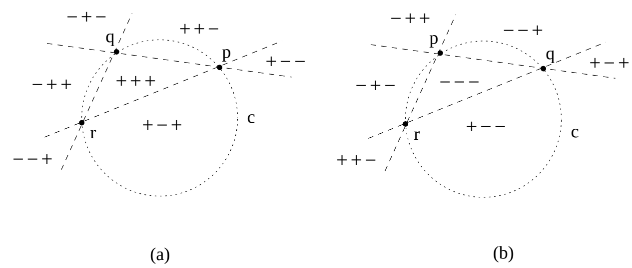

Figure 7: Signatures of a point relative to the triangle p, q, and r on the plane α.

becomes

procedure *COrder*$(c : \text{S-circle};\ p, q, r : \mathcal{C}\text{-point}) :$ Sign;
{Computes $\otimes_c(p, q, r)$}
return $\text{sign}(Ahead(c, p, q) + Ahead(c, q, r) + Ahead(c, r, p))$

5.3 Circular ordering of three arcs around a common origin

In the construction of spherical maps, a frequent subproblem is to compute the order in which three S-arcs leave a common origin vertex. That is, given three S-arcs $A = \text{sarc}(v, p, a)$, $B = \text{sarc}(v, q, b)$ and $C = \text{sarc}(v, r, c)$, we wish to compute $\otimes_v(A, B, C)$, the circular ordering in which the three arcs cross a sufficiently small S-circle centered at v. See figure 8. Informally the result $\otimes_v(A, B, C)$ should be $+1$ if the order is counterclockwise, as seen from $\text{dir}(O \vee v)$; -1 if it is clockwise; and 0 if any two of the arcs are parts of the same oriented circle.

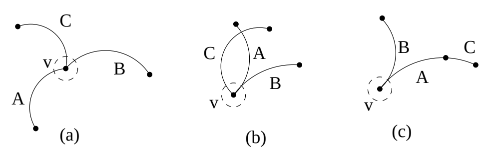

Figure 8: Circular order of three S-arcs.

An essential step of this algorithm is to compute the circular order $\otimes_l(\alpha, \beta, \gamma)$ of three planes α, β and γ around a common line l. We solve this problem by the same principle used to compute $\otimes_c(p, q, r)$, namely we reduce $\otimes_c$ to three ahead/behind tests, each involving two of the planes and the line l. In this case, the ahead/behind predicate $\otimes_l(\alpha, \beta)$ reduces to testing whether $\alpha \wedge \beta$ is the line l or its opposite; the "diametral opposite" of α is then the plane $\neg\alpha$.

Now, to compute $\otimes_v(A, B, C)$, we only need to order the directions in which the three arcs leave the point v. However, we must overcome two difficulties: the direction vectors may not be rational, and two distinct arcs may leave the point v in the same direction (as in figure 8(b)).

If v is not a rational point, then, by theorem 2, v is the exit point of a unique rational line l, which is not tangent to the sphere. In that case, the supporting planes of the three arcs must contain l; and the order of the arcs around v is the same as the order of the three planes around l, i.e. $\otimes_l(\mathrm{spln}(a), \mathrm{spln}(b), \mathrm{spln}(c))$.

If v is a rational point, the three planes may not have a common line. In that case, let σ be the (rational) plane tangent to $\mathbb{S}^2$ at v oriented so that $\mathrm{dir}(O \vee v)$ is on its positive side. So, the direction of A at v is $d_A = \mathrm{dir}(\mathrm{spln}(a) \wedge \sigma)$; and similarly for B and C. If d_A, d_B and d_C are distinct then it suffices to determine the circular order of these directions around $O \vee v$.

To complete the algorithm, we need to consider the case of two or more arcs leaving v in the same direction. In that case, we must break the tie by comparing the curvatures of the arcs—which is equivalent to ordering their planes around the common tangent line. In particular, if all three directions are equal, the result is simply the circular order of the three planes around the common tangent line. If only two directions coincide, it suffices to check the ahead/behind order of the two corresponding planes around their common tangent line.

---

procedure *ArcOrder*$(A, B, C : \text{S-arc}) : \text{Sign}$;
  {Computes $\otimes_v(A, B, C)$, where $v = \mathrm{org}(A) = \mathrm{org}(B) = \mathrm{org}(C)$}
  $\alpha \leftarrow \mathrm{spln}(\mathrm{scrc}(A))$;
  $\beta \leftarrow \mathrm{spln}(\mathrm{scrc}(B))$;
  $\gamma \leftarrow \mathrm{spln}(\mathrm{scrc}(C))$;
  $v \leftarrow \mathrm{org}(A)$; $p \leftarrow Apt(v)$;
  if $p = [0, 0, 0, 0]$ **then return** $\otimes_{\mathrm{stab}(v)}(\alpha, \beta, \gamma)$
  else
    {σ is the plane tangent to $\mathbb{S}^2$ at p }
    $\sigma \leftarrow \langle -p_0, p_1, p_2, p_3 \rangle$;
    $d_A \leftarrow \mathrm{dir}(\alpha \wedge \sigma)$;
    $d_B \leftarrow \mathrm{dir}(\beta \wedge \sigma)$;
    $d_C \leftarrow \mathrm{dir}(\gamma \wedge \sigma)$
    if $d_A \neq d_B$ **and** $d_B \neq d_C$ **and** $d_C \neq d_A$
    then return $\otimes_{\sigma \wedge \Omega_2}(d_A, d_B, d_C)$
    elsif $d_A = d_B = d_C$
    then return $\otimes_{d_A}(\alpha, \beta, \gamma)$
    else {only two tangents are equal.}
      while $d_A \neq d_B$ **do**
        $(\alpha, \beta, \gamma) \leftarrow (\beta, \gamma, \alpha)$;
        $(d_A, d_B, d_C) \leftarrow (d_B, d_C, d_A)$
      return $\otimes_{d_A}(\alpha, \beta)$

---

6 Conclusions

In this paper, we developed a basic collection of exact algorithms for manipulating circles and circular arcs on the sphere. These algorithms are exact, in the sense that they only use integer arithmetic. We have implemented all these algorithms in Modula-3 [7], using the *GNU Multiple Precision* (GMP) arithmetic library [4] providing a reliable framework for GIS, robotics, and other geometric applications.

7 Acknowlegdements

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A Appendix: Proofs of all theorems

Theorem 1 *Let $l = \langle l_0, l_1, l_2, l_3, l_4, l_5 \rangle$ be a rational line. Then,* $\mathrm{ext}(l)$ *is in $\mathcal{A}$ if and only if $(l_2^2 + l_4^2 + l_5^2) - (l_0^2 + l_1^2 + l_3^2)$ is a perfect square.*

PROOF: By formula (3),

$$\begin{aligned}\mathrm{ext}(l) \;=\; [\;\; \mu, \;\; & -l_1 l_2 - l_3 l_4 + l_5\sqrt{\delta}, \\ & l_0 l_2 - l_3 l_5 - l_4\sqrt{\delta}, \\ & l_0 l_4 + l_1 l_5 + l_2\sqrt{\delta} \;\;]\end{aligned}$$

where $\mu = l_2^2 + l_4^2 + l_5^2$ and $\delta = \mu - (l_0^2 + l_1^2 + l_3^2)$.

Obviously, if δ is a perfect square then $\mathrm{ext}(l) \in \mathcal{A}$.

Conversely, suppose that δ is not a perfect square. Since l intersects $\mathbb{S}^2$, we know by formula (2) that $(l_2, l_4, l_5) \neq (0, 0, 0)$. Then, one of the homogeneous coordinates of $\mathrm{ext}(l)$ have the form $a_i + l_i\sqrt{\delta}$, with a_i rational and $l_i \neq 0$. Since μ is rational the ratio $\mu/(a_i + l_i\sqrt{\delta})$ is irrational which implies that $p \notin \mathcal{A}$. □

Lemma 4 *Two lines $l = \langle l_0, .. l_5 \rangle$ and $m = \langle m_0, .. m_5 \rangle$ intersect if and only if*

$$l_0 m_5 - l_1 m_4 + l_2 m_3 + l_3 m_2 - l_4 m_1 + l_5 m_0 = 0$$

In that case, if the lines are not parallel,

$$\begin{aligned}l \cap m = \;\; [\; \kappa_{24}^2 + \kappa_{25}^2 + \kappa_{45}^2, \;\; & (l_3 m_2 - l_1 m_4 + l_5 m_0)\kappa_{24} - \kappa_{15}\kappa_{25} - \kappa_{35}\kappa_{45}, \\ & (l_0 m_5 + l_3 m_2 - l_4 m_1)\kappa_{25} + \kappa_{04}\kappa_{24} + \kappa_{34}\kappa_{45}, \\ & (l_0 m_5 - l_1 m_4 + l_2 m_3)\kappa_{45} - \kappa_{02}\kappa_{24} - \kappa_{12}\kappa_{25} \;]\end{aligned} \tag{4}$$

where

$$\kappa_{ij} = \begin{vmatrix} l_i & l_j \\ m_i & m_j \end{vmatrix}$$

Theorem 2 *A point p lies in $\mathcal{A}$ if and only if there are two distinct rational lines l and m such that* $\mathrm{ext}(l) = p = \mathrm{ext}(m)$.

PROOF: First, suppose that $p \in \mathcal{A}$. Let q_1 and q_2 be two distinct points in $\mathcal{A}$ such that $q_1 \neq p \neq q_2$. Then, the two lines $l = q_1 \vee p$ and $m = q_2 \vee p$ are rational, are distinct and $\mathrm{ext}(l) = p = \mathrm{ext}(m)$.

Conversely, let l and m be two distinct rational lines such that $\mathrm{ext}(l) = p = \mathrm{ext}(m)$. If $l \neq \neg m$ then formula (4) shows that $p = l \cap m$ is rational. On the other hand, if $l = \neg m$ then $\mathrm{ext}(l) = \mathrm{ext}(m)$ implies that l and m are tangent to $\mathbb{S}^2$. In that case $p = \mathrm{mid}(l)$ which is rational. □

Theorem 3 *A point p of $\mathbb{R}^3$ is in $\mathcal{C}$ if and only if $p = [a_0, a_1 + b_1\sqrt{c}, a_2 + b_2\sqrt{c}, a_3 + b_3\sqrt{c}\,]$ where a_0, a_1, a_2, a_3, b_1, b_2, b_3, and c are integers satisfying*

(i) $a_0 \neq 0$ *and* $b_1^2 + b_2^2 + b_3^2 \neq 0$

(ii) $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$

(iii) $a_0(a_0 - c) = a_1^2 + a_2^2 + a_3^2$

In that case, $p = \text{ext}(l)$ *where* $l = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_0b_3,\ a_1b_2 - a_2b_1,\ -a_0b_2,\ a_0b_1\rangle$

PROOF: Let p be a point in $\mathcal{C}$. Thus, there is a rational line $l = \langle l_0, .. l_5\rangle$ such that $\text{ext}(l) = p$. Choosing

$$\begin{array}{rclrcl} a_0 &=& l_2^2 + l_4^2 + l_5^2 & b_1 &=& l_5 \\ a_1 &=& -l_1l_2 - l_3l_4 & b_2 &=& -l_4 \\ a_2 &=& l_0l_2 - l_3l_5 & b_3 &=& l_2 \\ a_3 &=& l_0l_4 + l_1l_5 & c &=& (l_2^2 + l_4^2 + l_5^2) - (l_0^2 + l_1^2 + l_3^2) \end{array}$$

we have

(i) $a_0 = b_1^2 + b_2^2 + b_3^2 \neq 0$, since $l_2 = l_4 = l_5 = 0$ only for lines at infinity—formula (2).

(ii) $a_1b_1 + a_2b_2 + a_3b_3 = l_5(-l_1l_2 - l_3l_4) - l_4(l_0l_2 - l_3l_5) + l_2(l_0l_4 + l_1l_5) = 0$

(iii)
$$\begin{array}{rclcl} a_1^2 &=& (-l_1l_2 - l_3l_4)^2 &=& l_1^2l_2^2 + l_3^2l_4^2 + l_1l_2l_3l_4 + l_1l_2l_3l_4 \\ a_2^2 &=& (l_0l_2 - l_3l_5)^2 &=& l_0^2l_2^2 + l_3^2l_5^2 - l_0l_2l_3l_5 - l_0l_2l_3l_5 \\ a_3^2 &=& (l_0l_4 + l_1l_5)^2 &=& l_0^2l_4^2 + l_1^2l_5^2 + l_0l_1l_4l_5 + l_0l_1l_4l_5 \end{array}$$

so

$$\begin{array}{rl} a_1^2 + a_2^2 + a_3^2 = & l_0^2(l_2^2 + l_4^2) + l_1^2(l_2^2 + l_5^2) + l_3^2(l_4^2 + l_5^2) + l_2l_3(l_1l_4 - l_0l_5) \\ & +l_0l_5(l_1l_4 - l_2l_3) + l_1l_4(l_2l_3 + l_0l_5) \end{array}$$

and, applying the Plücker condition to the last three terms, we have

$$\begin{array}{rcl} a_1^2 + a_2^2 + a_3^2 &=& l_0^2(l_2^2 + l_4^2) + l_1^2(l_2^2 + l_5^2) + l_3^2(l_4^2 + l_5^2) + l_2l_3(l_2l_3) \\ && +l_0l_5(l_0l_5) + l_1l_4(l_1l_4) \\ &=& l_0^2(l_2^2 + l_4^2 + l_5^2) + l_1^2(l_2^2 + l_4^2 + l_5^2) + l_3^2(l_2^2 + l_4^2 + l_5^2) \\ &=& (l_0^2 + l_1^2 + l_3^2)(l_2^2 + l_4^2 + l_5^2) \\ &=& a_0(a_0 - c) \end{array}$$

Moreover, by formula (3), $p = \text{ext}(l) = [a_0, a_1 + b_1\sqrt{c}, a_2 + b_2\sqrt{c}, a_3 + b_3\sqrt{c}\,]$.

Conversely, let $a_0, \ldots, a_3$, b_1, b_2, b_3, and c be integers such that (i), (ii) and (iii) are satisfied. Let $l = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_0b_3,\ a_1b_2 - a_2b_1,\ -a_0b_2,\ a_0b_1\rangle$. By condition (i), $a_0 \neq 0$ and there exists $i \in \{1, 2, 3\}$ such that $b_i \neq 0$. Therefore, $a_0b_i \neq 0$ which implies that at least one component of l is not zero. Moreover,

$$l_0l_5 - l_1l_4 + l_2l_3 = a_0b_1(a_2b_3 - a_3b_2) - (-a_0b_2)(a_3b_1 - a_1b_3) + a_0b_3(a_1b_2 - a_2b_1) = 0$$

which implies that the Plücker condition is satisfied. Thus, l is a valid rational line. By formula (3),

$$\text{ext}(l) = \left[\mu,\ -l_1l_2 - l_3l_4 + l_5\sqrt{\delta},\ l_0l_2 - l_3l_5 - l_4\sqrt{\delta},\ l_0l_4 + l_1l_5 + l_2\sqrt{\delta}\,\right]$$

where

$$\mu = l_2^2 + l_4^2 + l_5^2 = a_0^2b_3^2 + a_0^2b_2^2 + a_0^2b_1^2 = a_0^3$$

and $\delta = \mu - (l_0^2 + l_1^2 + l_3^2)$. Now

$$\begin{array}{rcl} l_0^2 + l_1^2 + l_3^2 &=& (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &=& a_2^2b_3^2 + a_3^2b_2^2 - 2a_2a_3b_2b_3 + a_1^2b_3^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_2^2 \\ && +a_2^2b_1^2 - 2a_1a_2b_1b_2 \\ &=& a_1^2(b_2^2 + b_3^2) + a_2^2(b_1^2 + b_3^2) + a_3^2(b_1^2 + b_2^2) \\ && -2a_3b_3(a_2b_2 + a_1b_1) - 2a_1a_2b_1b_2 \end{array}$$

Using condition (ii), we get

$$\begin{aligned} l_0^2 + l_1^2 + l_3^2 &= a_1^2(b_2^2 + b_3^2) + a_2^2(b_1^2 + b_3^2) + a_3^2(b_1^2 + b_2^2) + 2a_3^2 b_3^2 - 2a_1 a_2 b_1 b_2 \\ &= a_1^2(b_1^2 + b_2^2 + b_3^2) + a_2^2(b_1^2 + b_2^2 + b_3^2) + a_3^2(b_1^2 + b_2^2 + b_3^2) \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) = a_0(a_1^2 + a_2^2 + a_3^2) \end{aligned}$$

Therefore $\delta = a_0(a_0^2 - a_1^2 - a_2^2 - a_3^2)$.
Since

$$\begin{aligned} -l_1 l_2 - l_3 l_4 &= -(a_3 b_1 - a_1 b_3) a_0 b_3 - (a_1 b_2 - a_2 b_1)(-a_0 b_2) \\ &= -a_0 a_3 b_1 b_3 + a_0 a_1 b_3^2 + a_0 a_1 b_2^2 - a_0 a_2 b_1 b_2 \\ &= -a_0 b_1 (a_3 b_3 + a_2 b_2) + a_0 a_1 (b_1^2 + b_3^2) \\ &= a_0 a_1 b_1^2 + a_0 a_1 (b_1^2 + b_3^2) \\ &= a_0 a_1 (b_1^2 + b_2^2 + b_3^2) \\ &= a_0^2 a_1 \end{aligned}$$

then

$$\begin{aligned} -l_1 l_2 - l_3 l_4 - l_5 \sqrt{\delta} &= a_0^2 a_1 + a_0 b_1 \sqrt{a_0(a_0^2 - a_1^2 - a_2^2 - a_3^2)} \\ &= a_0^2 \left(a_1 + b_1 \sqrt{\frac{a_0(a_0^2 - a_1^2 - a_2^2 - a_3^2)}{a_0^2}} \right) \\ &= a_0^2 (a_1 + b_1 \sqrt{c}) \end{aligned}$$

Analogously, the third and fourth homogeneous coordinates of $\operatorname{ext}(l)$ are respectively,

$$\begin{aligned} l_0 l_2 - l_3 l_5 - l_4 \sqrt{\delta} &= a_0^2 (a_2 + b_2 \sqrt{c}) \\ l_0 l_4 + l_1 l_5 + l_2 \sqrt{\delta} &= a_0^2 (a_3 + b_3 \sqrt{c}) \end{aligned}$$

So,

$$\operatorname{ext}(l) = \left[a_0^3,\ a_0^2(a_1 + b_1\sqrt{c}),\ a_0^2(a_2 + b_2\sqrt{c}),\ a_0^2(a_3 + b_3\sqrt{c}) \right]$$

By condition (i), we know that $a_0 \neq 0$. Then we have

$$\operatorname{ext}(l) = \left[a_0,\ a_1 + b_1\sqrt{c},\ a_2 + b_2\sqrt{c},\ a_3 + b_3\sqrt{c} \right]$$

□