Ear decompositions of matching covered graphs

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Abstract

Ear decompositions of matching covered graphs are important for understanding their structure. By exploiting the properties of the dependence relation introduced by Carvalho and Lucchesi in [2], we are able to provide simple proofs of several well-known theorems concerning ear decompositions. Our method actually provides proofs of generalizations of these theorems. For example, we show that every matching covered graph G different from K_2 has at least Δ edge-disjoint removable ears, where Δ is the maximum degree of G. This shows that any matching covered graph G has at least $\Delta!$ different ear decompositions, and thus is a generalization of the fundamental theorem of Lovász and Plummer establishing the existence of ear decompositions. We also show that every brick G different from K_4 and $\overline{C}_6$ has $\Delta - 2$ edges, each of which is a removable edge in G, that is, an edge whose deletion from G results in a matching covered graph. This generalizes a well-known theorem of Lovász. Using this theorem, we give a simple proof of another theorem due to Lovász which says that every non-bipartite matching covered graph has a canonical ear decomposition, that is, one in which either the third graph in the sequence is an odd-subdivision of K_4 or the fourth graph in the sequence is an odd-subdivision of $\overline{C}_6$. Our method in fact shows that every non-bipartite matching covered graph has a canonical ear decomposition which is optimal, that is one which has as few double ears as possible. Most of these results appear in the Ph.D.thesis, [3], of the first author written under the supervision of the second author.

1 Introduction

A **matching covered graph** is a non-trivial connected graph in which every edge is in some perfect matching. There is an extensive literature on matching covered graphs. For a history of the subject and for all terminology and notation not defined here, we refer the reader to Lovász and Plummer [6], Lovász [7], or Murty [9].

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Two types of decompositions of matching covered graphs, namely tight cut decompositions and ear decompositions, have played significant roles in the development of the subject. For example, both these decompositions are essential ingredients in Lovász's chracterization of the matching lattice [7]. Here we use tight cut decompositions in our study of ear decompositions. For all the relevant definitions and theorems concerning the tight cut decomposition of a matching covered graph, we refer the reader to the sources cited above. We recall below some of the basic definitions related to ear decompositions.

Let G' be a subgraph of a graph G and let P be a path in $G - E(G')$. A **single ear** of G' is a path P of odd length in G such that (i) both the ends of P are in $V(G')$, and (ii) P is internally disjoint from G'. The following theorem provides a decomposition of bipartite matching covered graphs.

Theorem 1.1 *Given any bipartite matching covered graph G, there exists a sequence*

$$G_1 \subset G_2 \subset \ldots \subset G_r = G$$

of matching covered subgraphs of G where (i) $G_1 = K_2$, and (ii) for $1 \leq i \leq r-1$, G_{i+1} is the union of G_i and a single ear of G_i.

The sequence $G_1, G_2, \ldots, G_r$ of subgraphs of G with the above properties is an ear decomposition of G in which each member of the sequence is a matching covered subgraph of G. Such decompositions do not exist for non-bipartite matching covered graphs. For example, K_4 has no ear decomposition as in Theorem (1.1). However, every matching covered graph has an ear decomposition with a slight relaxation of the above definition. For describing that decomposition, we require the notion of a double ear.

A **double ear** of a matching covered graph G' is a pair of vertex disjoint single ears of G'. An **ear** of G' is either single ear or double ear of G'.

An **ear decomposition** of a matching covered graph G is a sequence

$$G_1 \subset G_2 \subset \ldots \subset G_r = G$$

of matching covered subgraphs of G where (i) $G_1 = K_2$, and (ii) for $1 \leq i \leq r-1$, G_{i+1} is the union of G_i and an ear (single or double) of G_i. The following fundamental theorem was established by Lovász and Plummer. See [6].

Theorem 1.2 (The two-ear Theorem) *Every matching covered graph has an ear decomposition.*

There are various proofs of this theorem. The original proof due to Lovász and Plummer is somewhat involved. Later on, Little and Rendl [10] gave a simpler proof. Recently, another simple proof was given by Szigeti [11].

A matching covered subgraph H of a graph G is **nice** if every perfect matching of H extends to a perfect matching of G. It is easy to see that all the subgraphs in an ear decomposition of a matching covered graph must be nice matching covered subgraphs of the graph. In all the previous approaches, an ear decomposition of a matching covered graph

G is established by showing how a nice matching covered proper subgraph G_i of G can be extended to a nice matching covered subgraph G_{i+1} by the addition of a single or double ear.

Our approach is to build an ear decomposition of a matching covered graph in the reverse order. To state our theorem precisely, we need to define the notion of a removable ear in a matching covered graph.

Let G be a matching covered graph, and let P be a path in G. Then, P is said to be a **removable single ear** in G if (i) P is an odd path which is internally disjoint from the rest of the graph, and the graph obtained from G by deleting all the edges and internal vertices of P is matching covered. (A removable ear of length one is **a removable edge**.) The notion of a **removable double ear** in G is similarly defined. A **removable ear** in G is either a single or double ear which is removable.

In trying to establish the existence of ear decompositions with special properties, it is often convenient to find the subgraphs in the ear decomposition in the reverse order starting with $G_r = G$. Thus, after obtaining a subgraph G_i in the sequence which is different from K_2, we find a suitable removable ear (single or double) and obtain G_{i-1} from G_i by removing that ear from G_i. For example, to show that a matching covered graph G has an ear decomposition, it suffices to show that every matching covered graph different from K_2 has a removable ear. This is our approach. We in fact prove the folowing more general theorem.

Theorem 1.3 *Any matching covered graph G different from K_2 has Δ removable ears (single or double).*

Clearly Theorem(1.2) is a special case of Theorem (1.3). In fact (1.3) shows that every matching covered graph G has at least $\Delta!$ different ear decompositions.

Suppose that

$$G_1 \subset G_2 \subset \ldots \subset G_r = G$$

is an ear decomposition of a matching covered graph G. If a member G_i of the sequence is obtained from its predecessor G_{i-1} by adding a single (double) ear, then we shall say that G_i is obtained from G_{i-1} by the addition of a single (double) ear. In any ear decomposition of a non-bipartite matching covered graph G, there must be at least one double ear addition. Furthermore, if G_i is the first member of an ear decomposition which is obtained by a double ear addition[1], then G_1, ...,G_{i-1} are bipartite and G_i, ...,$G_r = G$ are non-bipartite.

In any ear decomposition, by definition, G_1 is K_2, and G_2 is an even circuit. It is not difficult to check that if G_3 is non-bipartite, then it must in fact be an odd subdivision of K_4, and if G_3 is bipartite and G_4 is non-bipartite, then G_4 must be an odd subdivision of $\overline{C}_6$. We shall refer an ear decomposition

$$G_1 \subset G_2 \subset \ldots \subset G_r = G$$

of a non-bipartite matching covered graph G as a **canonical ear decomposition** if either its third member G_3 or its fourth member G_4 is non-bipartite. The following fundamental theorem was proved by Lovász in 1983 [5].

[1]We only consider ear decompositions which are fine. By this we mean that a double ear (P_1, P_2) is added to G_i to obtain G_{i+1} only if neither $G_i + P_1$, nor $G_i + P_2$ is matching covered.

Theorem 1.4 (The canonical ear decomposition Theorem) *Every non-bipartite matching covered graph G has a canonical ear decomposition*

In [5], Lovász gave several applications of the above theorem to extremal problems in graph theory. Later on he used it to deduce the following theorem which was crucial for his work on the matching lattice.

Theorem 1.5 (The removable edge Theorem) *Every brick different from K_4 and $\overline{C}_6$ has a removable edge.*

Lovász's original proof of Theorem (1.4) was quite involved. A simpler proof was given by Carvalho and Lucchesi [1]. Our approach here shall be to prove Theorem (1.5) first and then to derive (1.4) from it. We shall in fact prove the following generalization of (1.5).

Theorem 1.6 *Every brick different from K_4 and $\overline{C}_6$ has at least $\Delta - 2$ removable edges.*

A proof of the above theorem was first given by Carvalho and Lucchesi in [2]. The proof we give here is along the same lines, but is somewhat simpler.

Any two ear decompositions of a bipartite matching covered graph have the same length. However, the length of an ear decomposition of a non-bipartite matching covered graph depends on the number of double ears used in the decomposition. An **optimal** ear decomposition of a matching covered graph G is one which uses as few double ears as possible. Optimal ear decomposition of G have been used by Carvalho and Lucchesi [4] for determining a basis of the matching lattice of G. In this connection, they proved the following generalization of (1.4).

Theorem 1.7 (Optimal canonical ear decomposition Theorem) *Every non-bipartite matching covered graph G has an optimal ear decomposition which is canonical.*

One merit of our approach is that we are able to prove the above theorem from (1.5) directly by a simple inductive argument.

In the next section, we recall the definition of the dependence relation that we alluded to in the abstract, and state some of its salient properties. In section 3, we give a proof of Theorem (1.3). In section 4, we give a proof of Theorem (1.6). In section 5, we give proofs of Theorems (1.4) and (1.7).

2 A dependence relation

Let G be a matching covered graph, and let e and f be any two edges of G. Then e **depends** on f, or e **implies** f, if every perfect matching that contains e also contains f. (Equivalently, e depends on f if e is not admissible in $G - f$.)

We shall write $e \Rightarrow f$ to indicate that e depends on f. Clearly, $\Rightarrow$ is reflexive and transitive. It is convenient to visualize $\Rightarrow$ in terms of the digraph it defines on the set of edges of G. (See Figure 1 for an illustration.) A study of this relation, as shown by its many

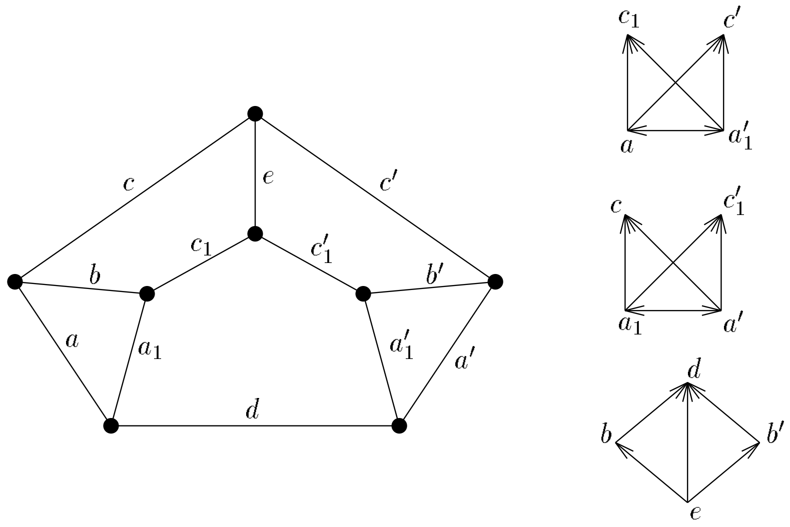

Figure 1:

important uses in Carvalho's thesis [3], turns out to be very useful for understanding the structure of matching covered graphs.

The folowing lemmas can be proved easily using Tutte's theorem.

Lemma 2.1 (See [9]) *Let G be a graph which has a perfect matching. Then an edge e of G is admissible if, and only if, there is no barrier which contains both the ends of e.*

Lemma 2.2 *Let G be a matching covered graph, and let e and f be any two edges of G. Then, the relation $e \Rightarrow f$ holds if, and only if, there exists a barrier B of $G - f$ such that (i) both the ends of e are in B, and (ii) the ends of f lie in different components of $G - f - B$.*

Proof: If $G - f$ has a barrier B satisfying the above properties, then clearly $e \Rightarrow f$. Suppose now that $e \Rightarrow f$, or that e is not admissible in $G - f$. Clearly, $G - f$ has a perfect matching. Therefore, by (2.1), $G - f$ has a barrier B such that both ends of e are in B. On the other hand, since e is admissible in G, B cannot be a barrier in G itself. It follows that the ends of f lie in different components of $G - f - B$. ■

We shall say that two edges e and f are **mutually dependent** if $e \Rightarrow f$ and $f \Rightarrow e$. In this case we shall write $e \Leftrightarrow f$. Clearly $\Leftrightarrow$ is an equivalence relation on $E(G)$. By identifying the vertices in the equivalence classes in the digraph representing dependence relation ($\Rightarrow$) on $E(G)$, we obtain the digraph $D(G)$ representing the dependence relation ($\Rightarrow$) on the set of equivalence classes. This digraph is clearly acyclic. The sources in this digraph are called *minimal classes*. If e is an edge, then a source Q in the component of $D(G)$ containing e (or more precisely containing the equivalence class containing e) is said to be a **minimal class induced by** e. The following lemma is an immediate consequence of this definition.

Lemma 2.3 *If Q is a minimal class, then every edge not in Q is admissible in $G - Q$. (Thus, if $G - Q$ is connected, then $G - Q$ is matching covered)*

The equivalence classes in a brick have some attractive properties:

Lemma 2.4 *Let G be a brick and let e and f be two edges of G such that $e \Leftrightarrow f$. Then, $G - e - f$ is bipartite. Moreover, both ends of e lie in one part of the bipartition, and both ends of f lie in the other part of the bipartition.*

Proof: By Lemma (2.2), $G - f$ has a barrier B such that both ends of e are in B, and the two ends of f are in different components of $G - f - B$.

Suppose that there exists another edge e' which has both its ends in B. Let M be a perfect matching in G which contains the edge e'. Simple counting shows that $f \in M$, and $e \notin M$. But this contradicts the hypothesis that $e \Leftrightarrow f$. Therefore, e is the only edge with both ends in B.

Suppose now that $G - e - f$ is not bipartite. Then, some component H of $G - e - B$ is non-trivial. As G is a brick, $\nabla(V(H))$ is not a tight cut in G. Hence, there exists a perfect matching M of G such that $|M \cap \nabla(V(H))| \geq 3$. A simple counting argument shows $|M \cap \nabla(V(H))| = 3$, $f \in M$, and $e \notin M$. Again, this contradicts the hypothesis. Therefore $G - e - f$ is bipartite.

Conversely, if $G - e - f$ is bipartite, then clearly $e \Leftrightarrow f$. ■

Lemma 2.5 *Let G be a brick and let Q be an equivalence class of G. Then, $|Q| \leq 2$. Moreover, if $|Q| = 2$, then $G - Q$ is bipartite.*

Proof: If possible, let e, f, and g, be any three edges in Q. Then, by Lemma 2.4, $E \setminus \{e, f\}$ and $E \setminus \{e, g\}$ are coboundaries of G. The symmetric difference of these two coboundaries is $\{f, g\}$. But the symmetric difference of any two coboundaries of a graph is a coboundary of the graph. Thus, $\{f, g\}$ is a coboundary of G. This is a contradiction because G is 3-connected. ■

In a general matching covered graph, the equivalence classes can be arbitrarily large. However, the following result can be proved using an argument similar to the one used in the proof of (2.5):

Lemma 2.6 (Carvalho-Lucchesi) *Let G be a matching covered graph and let Q be a minimal class of G. If Q does not include an edge cut of size 2, then $|Q| \leq 2$. (In particular, if G is a 3-edge-connected, then $|Q| \leq 2$ for any minimal class Q of G.)*

The proof given by Carvalho and Lucchesi of Theorem (1.3) in [3] is based on the above lemma.

3 Removable ears in matching covered graphs

We shall first consider the cases in which G is either a brick or is a brace on at least six vertices. We shall then be able to prove the theorem in general by induction using the tight cut decomposition procedure.

Lemma 3.1 *Let G be a brick. Then G has Δ removable ears.*

Proof: Let v be a vertex of degree Δ of G, and let $e_1,...,e_\Delta$, be the edges incident with v. Let $Q_1,...,Q_\Delta$ be minimal equivalence classes induced by $e_1,...,e_\Delta$, respectively. It follows from the definitions that, for each i, $1 \leq i \leq \Delta$, there exists a perfect matching M_i, such that M_i contains the edges in Q_i, and the edge e_i (because, Q_i, being induced by e_i, depends on e_i). It follows that the M_i are distinct, and hence, the Q_i are distinct. But, by (2.3), $G - Q_i$ is matching covered for each i, and by (2.5) $|Q_i| \leq 2$ for $1 \leq i \leq \Delta$. ■

Lemma 3.2 *Let H be a brace on six or more vertices. Then, every edge e of H is removable.*

Proof: Let H be a brace with bipartion (U, W), where $|U| = |W| \geq 3$. By definition, for any subset X of U (or of W), $0 < |X| < |U| - 1$, we have $|N(X)| \geq |X| + 2$.

Let e be any edge of H. Then, $|N_{H-e}(X)| \geq |X| + 1$, for each subset X of U (or of W), $0 < |X| < |U| - 1$. Suppose now that there exists a subset X of U (or of W), $|X| = |U| - 1$, such that $|N_{H-e}(X)| \leq |X|$. As $H - e$ has a perfect matching, we must in fact have $|N_{H-e}(X)| = |X|$. Then, there exists a vertex $w \in W$ such that $|N_{H-e}(w)| = 1$. Consequently, $|N_H(w)| \leq 2$. This is not possible because H is a brace.

Therefore $|N_{H-e}(X)| \geq |X| + 1$ for every proper subset X of U. Hence $H - e$ is a matching covered graph. ■

Proof of Theorem (1.3): It is easy to verify the statement of the theorem if $|V(G)| \leq 4$. Let G be a matching covered graph on six or more vertices. If G is either a brick or a brace, then the validity of the theorem follows from Lemmas (3.1) and (3.2). To prove the theorem in general, assume inductively that any matching covered graph G' with $|E(G') + |V(G')| < |E(G) + |V(G)|$ has $\Delta(G')$ edge-disjoint removable ears. We shall first deal with a few simple cases.

First consider the case in which G has two parallel edges, say e and f. Then, $G - f$ is matching covered and has fewer edges than G. By induction, $G - f$ has at least $\Delta(G - f)$ edge-disjoint removable ears, say $Q_1, Q_2, ..., Q_{\Delta(G-f)}$. If e is not in any one of these $\Delta(G - f)$ ears, then they are removable ears in G as well, and in addition, $\{e\}$ and $\{f\}$, are also removable in G. On the other hand, if e is one of the Q_i, say $e \in Q_1$, then $\{e\}$, $\{f\}$, $Q_2, ..., Q_{\Delta(G-f)}$ are removable ears in G. Since $\Delta(G - f) \geq \Delta(G) - 1$, it follows that G has at least $\Delta(G)$ edge-disjoint removable ears. Thus we may assume that G is simple.

Now consider the case in which G has two adjacent vertices, say v and w, of degree two. Let u be the neighbour of v different from w, and t be the neighbour of w different from v. If u is the same vertex as t, then it would be a cut vertex of G, which is impossible. Therefore u and v are distinct, and $P = (u, v, w, t)$ is a path of length three which is internally disjoint from the rest of the graph. Obtain the graph G' from G by deleting v and w and joining u and t by a new edge e. Clearly, G' has fewer vertices than G, and $\Delta(G') = \Delta(G)$. Therefore, by induction, G' has Δ removable ears. Any removable ear of G' that does not include the edge e is also a removable ear of G. And, if Q' is a removable ear of G' which includes e, we can obtain a removable ear Q of G from Q' be replacing e by the path P. It follows that G has at least $\Delta(G)$ edge-disjoint removable ears. Thus we may assume that no two of its vertices of degree two are adjacent.

We divide the rest of the proof into two main cases depending on whether or not G is bicritical. If G is bicritical, and is not a brick, then G has a 2-separation cut. This is the first case we consider.

Case 1. G is bicritical and $\{v_1, v_2\}$ is a 2-separation of G: Firstly note that if there is an edge f joining v_1 and v_2 in G, then $G - f$ is also bicritical. In particular, f is removable in G. Moreover, if Q is a removable ear of $G - f$, then Q is also a removable ear of G. Thus, if there is an edge joining v_1 and v_2, we can delete it and apply induction to obtain the required result. So, we may assume that there is no such edge in G.

Express G, in the usual manner, as the union of two edge-disjoint graphs H_1, H_2 with $V(H_1) \cup V(H_2) = \{v_1, v_2\}$. Let $e = v_1v_2$ be a new edge. Then, $G_1 = H_1 + e$, and $G_2 = H_2 + e$ are both matching covered. Furthermore, $\Delta(G_1), \Delta(G_2) \geq 3$, and thus neither G_1 nor G_2 is either K_2.

Let v be a vertex of degree Δ in G. Without loss of generality, we may assume either that $v \in V(G_1) \setminus \{v_1, v_2\}$ or that $v = v_1$. Let us first suppose that $v \in V(G_1) \setminus \{v_1, v_2\}$. Then, by induction, G_1 has at least Δ edge-disjoint removable ears, and at most one of them contains the edge e, and G_2 has at least three edge-disjoint removable ears, and at most one of them contains the edge e. Thus, G has at least $\Delta - 1 + 2 = \Delta + 1$ edge-disjoint removable ears. Now suppose that $v = v_1$, and let Δ_1 and Δ_2 be the degrees of v_1 in H_1 and H_2, respectively. Then, the degrees of v_1 in G_1 and G_2, respectively, are $\Delta_1 + 1$ and $\Delta_2 + 1$. By induction, G_1 has at least $\Delta_1 + 1$ edge-disjoint removable ears, and at most one of them contains the edge e, and G_2 has at least $\Delta_2 + 1$ edge-disjoint removable ears, and at most one of them contains the edge e. Thus, G has at least $\Delta_1 + \Delta_2 = \Delta$ edge-disjoint removable ears. ■

To complete the proof, we must consider the case in which G is not bicritical. In this case, G has at least one non-trivial barrier, and hence has non-trivial tight cuts which are barrier cuts. We shall apply the induction hypothesis to graphs obtained by contracting the shores of suitably chosen barrier cuts. In this part of the proof, we shall find it convenient to use the notation described below, and the lemma that follows.

If $C = \nabla(S)$ is a tight cut of G, then we shall denote the graph obtained by shrinking S to a single vertex s by by $G\{S, s\}$; if the name of the vertex s is irrelevant, we shall simply write $G\{S\}$. (Thus, $G\{S\}$, and $G\{\overline{S}\}$ are the two C-contractions of G.) Similarly, if S_1, S_2,...,S_k are k disjoint subsets of the vertex set of G, then we shall denote the graph obtained from G by shrinking each of the subsets S_1, S_1,...,S_k to single vertices by $G\{S_1, S_2, ..., S_k\}$

Lemma 3.3 *Let G be a matching covered graph. Let $C = \nabla(S)$ be a non-trivial tight cut of G and let Q be a removable ear of $G\{\overline{S}\}$. If either Q is edge-disjoint from C, or if $E(Q) \cap C = \{e\}$ and e is removable in $G\{S\}$, then Q is also removable in G.*

With the aid of this lemma, we can now deal of the remaining case in which G has a non-trivial barrier. It is convenient to first consider the case in which G has a barrier of size two. Note that if G has a vertex u of degree two, with v and w as its neighbours, then $\{v, w\}$ is a barrier of size two of G. Furthermore, since G does not have adjacent vertices of degree two, neither v nor w has degree two. Thus, if G has a barrier of size two, then it has one in which neither vertex has degree two in G.

Case 2.1. G has barriers of size two: In this case, as noted above, there exists a barrier of size two in which neither vertex has degree two. Let $\{v_1, v_2\}$ be such a barrier, and let S_1 and S_2 be the vertex sets of the two components of $G \setminus \{v_1, v_2\}$.

Suppose one of S_1, S_2, say S_2 is a singleton. Then there are at least two edges from each of v_1 and v_2 to S_1. So all edges in $\nabla(S_1)$ are multiple edges in $G\{S_1\}$, and hence are removable in $G\{S_1\}$. Furthermore, $G\{\overline{S_1}\}$ is different from K_2 and has a vertex whose degree in $G\{\overline{S_1}\}$ is at least $\Delta(G)$. By induction hypothesis, $G\{\overline{S_1}\}$ has at least Δ removable ears. By Lemma (3.3), they are also removable in G.

In general, for $i, j = 1, 2$, if there is more than one edge from v_i to S_j, then all these edges would be multiple edges in $G\{S_j\}$, and hence would be removable edges of $G\{S_j\}$. Thus, since both v_1 and v_2 have degree at least three, the number of edges in the cut $\nabla(S_1)$ which are not removable in $G\{S_1\}$, plus the number of edges of $\nabla(S_2)$ which are not removable in $G\{S_2\}$ is at most two. By considering the cuts $\nabla(S_1)$, $\nabla(S_2)$, using the induction hypothesis, and applying Lemma (3.3) twice, we can deduce that G has at least Δ removable ears. ■

Case 2.2. G is not bicritical, and the size of a smallest non-trivial barrier in G is at least three.: Let B be a minimal non-trivial barrier of G. Let S_1, S_2, ..., S_b, where $b = |B|$, be the vertex sets of the odd components of $G - B$. The minimality of B implies that the bipartite graph $H = G\{S_1, S_2, ..., S_k\}$ obtained by shrinking S_1, S_2, ..., S_b to single vertices is a brace. Furthermore, since $|B| \geq 3$, H is a brace on six or more vertices. Therefore, by Theorem (3.2), every edge of H is removable.

Consider the graphs $G\{\overline{S_1}\}$, $G\{\overline{S_2}\}$, ..., $G\{\overline{S_b}\}$. It is easy to see that none of these graphs can be K_2.

Let v be a vertex of degree Δ in G. Without loss of generality, v is either in B or in S_1. First suppose that v is in B, and let Δ_1, Δ_2,..., Δ_b be the numbers of edges from v to S_1, S_2, ..., S_b, respectively. Then, clearly, $\Delta(G\{\overline{S_1}\}) \geq \Delta_1$, $\Delta(G\{\overline{S_2}\}) \geq \Delta_2$,..., $\Delta(G\{\overline{S_b}\}) \geq \Delta_b$. By induction, for $1 \leq i \leq k$, $G\{\overline{S_i}\}$ has at least Δ_i edge-disjoint removable ears. Since each edge of $\nabla(S_i)$ is removable in H, it follows that, for $1 \leq i \leq b$, G has at least Δ_i edge-disjoint removable ears contained in $E(G\{\overline{S_i}\})$. Thus, G has at least $\sum_{i=1}^{b} \Delta_i = \Delta$ edge-disjoint removable ears.

Now consider the case in which v is in S_1. Then, $\Delta(G\{\overline{S_1}\}) \geq \Delta(G)$. By induction, G_1 has at least $\Delta(G)$ edge-disjoint removable ears. Since each edge of $\nabla(S_1)$ is removable in H, it follows that all removable edges of $G\{\overline{S_1}\}$ are also removable in G. ∎

Remark: The above theorem implies that every matching covered graph has at least $\Delta!$ removable ears. This bound is best possible because the graph on two vertices and Δ parallel edges joining the two vertices has exactly $\Delta!$ ear decompositions.

4 Removable edges in bricks

The following lemma establishes a property of equivalent edges in bipartite matching covered graphs. It is used in the proof of Theorem (1.6).

Lemma 4.1 *Let H be a bipartite matching covered graph, and let e and f be two distinct equivalent edges of H. Then $H - \{e, f\}$ is disconnected.*

Proof: By (2.2), there is a barrier in $H - f$ which contains both the ends of the edge e. Choose a maximal barrier B in $H - f$ which contains both the ends of e. Then, each component of $H - f - B$ is critical. But, since H is bipartite, each of these components is also bipartite. It follows that all components of $H - f - B$ are trivial. If possible, let h be an edge of H which also has both its ends in B. Then, by simple counting, we can see that any perfect matching through h contains f, but not e. This contradicts the hypothesis that e and f are equivalent. Now suppose that $H - \{e, f\}$ is connected. Then, there is an even path P in $H - \{e, f\}$ connecting the two ends of e. But then $P + e$ is an odd circuit in H, which is impossible. Therefore, we conclude that $H - \{e, f\}$ is disconnected. ∎

Proof of the Theorem (1.6): Let v be a vertex of degree Δ in G. As in the proof of Theorem (3.1) there exist Δ disjoint minimal classes Q_1, Q_2,...,Q_Δ in G, each depending on an edge incident with v. Each Q_i is either a singleton (a removable edge) or a doubleton. Thus, in order to prove that there are at least $(\Delta - 2)$ singletons, it suffices to prove that there are at most two doubletons. Assume to the contrary that there are three distinct doubletons, say, $Q_1 = \{e_1, \overline{e}_1\}$, $Q_2 = \{e_2, \overline{e}_2\}$, and $Q_3 = \{e_3, \overline{e}_3\}$. We shall deduce that in this case G must either be K_4 or $\overline{C}_6$.

We know that for $1 \leq i \leq 3$, $G - Q_i$ is a bipartite matching covered graph. Let us write $H = G - Q_1$, denote the bipartition of H by (A, B). By Lemma (4.1), $H - Q_2$ and $H - Q_3$ are disconnected. Note that, for each $2 \leq i \leq 3$, each component of $H - Q_i$ is a bipartite graph with a perfect matching, and thus its partition has parts of equal cardinality.

Case 1. The two edges of Q_3 belong to different components of $H - Q_2$: In this case, the edges e_2 and $\overline{e}_2$ of Q_2 must belong to different components of $H - Q_3$. Since $H - Q_2$ is disconnected, e_2 and $\overline{e}_2$ must be cut edges of the components of $H - Q_3$. Thus, $G - (Q_1 \cup Q_2 \cup Q_3)$ has four components. Let us denote them by H_1, H_2, H_3, and H_4. Now, since G is 3-connected, we must have that the graph obtained from G by shrinking each H_j to a single vertex is a K_4 with Q_1, Q_2, and Q_3 as its perfect matchings. For any H_j, and any M_i (notation as in (3.1)), all vertices but one of H_j are matched among themselves. So

$|V(H_j)|$ is odd. It is easy to see that the larger part of the bipartition of H_j is a barrier of G. It follows that each $|V(H_j)| = 1$, and hence that $G = K_4$.

Case 2. The two edges of Q_3 belong to the same component of $H - Q_2$: Let J and J' be the components of $H - Q_2$, and let $A \cap V(J) = A_1$, $B \cap V(J) = B_1$, $A \cap V(J') = A'_1$, and $B \cap V(J') = B'_1$. We may assume without loss of generality that the edges e_3 and $\overline{e}_3$ of Q_3 have their ends in J'. In this case, the edges e_2 and $\overline{e}_2$ of Q_2 must belong to the same component of $H - Q_3$.

Let L and L' be the components of $H - Q_3$, and assume without loss of generality that J is a subgraph of L. Then, Q_2 is a 2-edge cut of L. One component of $L - Q_2$ is J, let (A_2, B_2) denote the bipartition of the other component of $L - Q_2$, where $A_2 = A \cap V(L - A_1)$, and $B_2 = B \cap V(L - B_1)$. Let (A_3, B_3) denote the bipartition of L', where $A_3 = A \cap V(L')$, and $B_3 = B \cap V(L')$. Thus, $G - (Q_1 \cup Q_2 \cup Q_3)$ has three components with bipartitions (A_1, B_1), (A_2, B_2), and (A_3, B_3). Furthermore, $|A_1| = |B_1|$, $|A_2| = |B_2|$, and $|A_3| = |B_3|$.

As in the proof of Theorem (3.1), for each i, there is a perfect matching M_i of G which contains both the edges of Q_i but is disjoint from $Q_{i'}$, $i' \neq i$. Thus, for $1 \leq i \leq 3$, and $1 \leq j \leq 3$, $|V(Q_i) \cap A_j| = |V(Q_i) \cap B_j|$, where $V(Q_i)$ is the set of end vertices of the edges in Q_i. Now, Since G is 3-connected, an edge of Q_1 must join vertices in A_1 and A_3, and the other edge of Q_1 must join vertices in B_1 and B_3. Thus, in summary, we have:

- An edge of Q_1, say e_1, joins a vertex u_1 in A_1 with a vertex v_1 in A_3, and the other edge $\overline{e}_1$ of Q_1 joins a vertex $\overline{u}_1$ in B_1 with a vertex $\overline{v}_1$ in B_3,
- an edge of Q_2, say e_2, joins a vertex u_2 in A_1 with a vertex v_2 in B_2, and the other edge $\overline{e}_2$ of Q_2 joins a vertex $\overline{u}_2$ in B_1 with a vertex $\overline{v}_2$ in A_2, and
- an edge of Q_3, say e_3, joins a vertex u_3 in A_2 with a vertex v_3 in B_3, and the other edge $\overline{e}_3$ of Q_3 joins a vertex $\overline{u}_3$ in B_2 with a vertex $\overline{v}_3$ in A_3.

Figure 2 shows all the incidences described above.

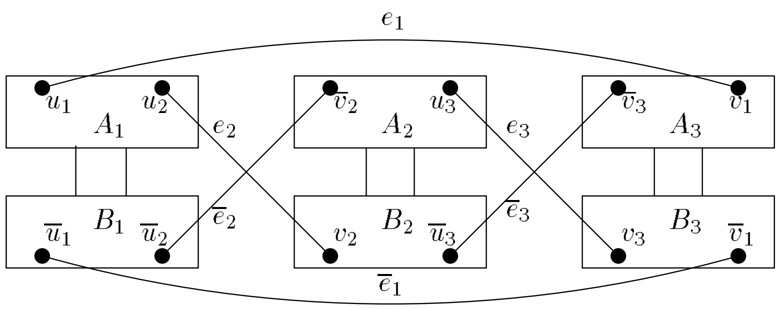

Figure 2:

Now, we shall show that all A_i and B_i are singletons, and thereby deduce that G is $\overline{C}_6$. Towards this end, we shall first show that $u_1 = u_2$. Suppose that $u_1 \neq u_2$. Since G is bicritical, $G - \{u_1, u_2\}$ has a perfect matching N. This perfect matching necessarily contains $\overline{e}_1$ and $\overline{e}_2$. Thus, $\overline{u}_1 \neq \overline{u}_2$.

Let N' denote the set of edges in N which have at least one end in $A_1 \cup B_1 \setminus \{u_1, u_2\}$. Let M_1' denote the subset of edges of M_1 matching vertices in $A_3 \cup B_3 \setminus \{v_1, \overline{v}_1\}$, and let M_2' denote the subset of edges of M_2 matching vertices in $A_2 \cup B_2 \setminus \{v_2, \overline{v}_2\}$. Then, $M = N' \cup \{e_1, e_2\} \cup M_1' \cup M_2'$ is a perfect matching of G which contains both Q_1 and Q_2. This is impossible because Q_1 is a minimal class induced by one edge incident with the vertex v of maximum degree, and Q_2 is a minimal class induced by a different edge incident with the vertex v.

Thus, $u_1 = u_2$, and similarly, $\overline{u}_1 = \overline{u}_2$. It now follows that A_1 is a barrier of G. But since G is a brick, $|A_1| = 1$. Similar arguments show that, in fact, all A_i and B_i are singletons. It follows that G is $\overline{C}_6$. ■

Remark: The above theorem is best possible. For example, the brick in Figure 1 has exactly one removable edge.

5 Canonical ear decompositions

In this section we shall present a proof of Theorem (1.7). We start with the following useful lemma.

Lemma 5.1 *Let G be a matching covered graph. Let $C = \nabla(S)$ be a non-trivial tight cut of G. Suppose that $G\{S, s\}$ has a canonical ear decomposition with d double ears and that $G\{\overline{S}, s\}$ is bipartite. Then, G also has a canonical ear decomposition with d double ears.*

Proof: If possible, choose a counter-example (G, S) such that (i) $|V(G)| + |E(G)|$ is as small as possible, and (ii) subject to (i), $|S|$ is as small as possible.

It follows from (i) that G has no multiple edges. Since C is a non-trivial, $|S|$ is at least three. Let us first consider the case in which $|S| = 3$.

Case 1 $|\mathbf{S}| = 3$: In this case, $G\{\overline{S}, s\}$ has exactly four vertices, s being one of them. Let u, v, and w be the other three vertices, where u is of degree 2 in $G\{\overline{S}, s\}$, with v and w as its neighbours.

Let $\mathcal{P}$, where

$$\mathcal{P}: \quad G_1, G_2 = G_1 + Q_1, ..., G_r = G_{r-1} + Q_{r-1}$$

be a canonical ear decomposition of $G\{S, s\}$ with d double ears. For convenience, we shall assume that the edge of G_1 is not incident with s. We shall see how an ear decomposition $\mathcal{P}'$ of G can be obtained from $\mathcal{P}$ by modifying some of the ears in it.

If P is a path in $G\{S, s\}$ which does not contain s as an internal vertex, then there is a path $\tilde{P}$ in G with $E(\tilde{P}) = E(P)$. Let us refer to $\tilde{P}$ as the path that corresponds to P. All modifications of ears except one consist of merely replacing paths in the ears in $\mathcal{P}$ by the paths in G that correspond to them. In the exceptional case, a path in an ear of $\mathcal{P}$ is modified so as to include the edges vu and uw, which are the two edges of G not in $G\{S, s\}$. A precise description of this modification is given below.

Let G_{i+1} be the first graph in $\mathcal{P}$ which contains the vertex s. Then, the path P containing s in the ear Q_i must have s as an internal vertex. Then, either (i) the two edges in P incident with s are incident with the different vertices in G, or (ii) these two edges are incident with the same vertex in G. We treat these two cases separately.

(i) Suppose that the two edges in P incident with s are incident with the different vertices in G. In this case modify P by replacing the vertex s by (v, u, w).

(ii) Suppose the two edges in P incident with s are incident with the same vertex in G, say v, then do not modify the ear Q_i (except for replacing the paths in it by the corresponding paths in G). Any subsequent ear in $\mathcal{P}$ which contains a path R incident with s would have s as an end vertex of R. Consider the first occurence of such an ear, say Q_k, where the edge of R incident with s is an edge xw incident with w in G. Modify the path R by replacing (x, w) in R by (x, w, u, v).

For $1 \leq i \leq (r-1)$, let Q'_i be the path or the pair of paths obtained by modifying Q_i according to the above specified rules. Consider the sequence

$$\mathcal{P}': \quad G'_1 = G_1, G'_2 = G'_1 + Q'_1, ..., G'_r = G'_{r-1} + Q'_{r-1}$$

of subgraphs of G. Then, it is easy to verify that $\mathcal{P}'$ is a canonical ear decomposition of G with d double ears.

Case 2 $|\mathbf{S}| > 3$: Firstly, let us suppose that $G\{\overline{S}, s\}$ is a brace, and let e be any edge of $G\{\overline{S}, s\}$ which is not incident with s, and hence not belonging to the cut C. Then, by (3.2), e is removable in $G\{\overline{S}, s\}$. Since e is not in S, e is removable in G as well. Let us write $G' = G$. Then, by hypothesis, $G'\{S, s\} = G\{S, s\}$ has a canonical ear decomposition, and $G'\{\overline{S}, s\}$ is bipartite. By induction, G' has a canonical ear decomposition. It can be extended to a canonical ear decomposition of G by adding e at the end as a single ear.

If not, $G\{\overline{S}, s\}$ is not a brace, then it has a non-trivial tight cut $D = \nabla(T)$ where T is a proper subset of S. But D is also a tight cut of G. The proof can now be completed by two applications of the induction hypothesis. ■

A matching covered graph G is **near-bipartite** if there are two edges e and f of G such that (i) $G - \{e, f\}$ is a bipartite graph with bipartition (A, B), and (ii) e has both its ends in A, and f has both its ends in B. We shall refer to H as the **underlying bipartite graph**, and the edges e and f as **special edges**. In any ear decomposition of a bipartite matching covered graph, the first non-bipartite member of the sequence is a near-bipartite graph. Thus, by studying ear decompositions of near-bipartite matching covered graphs, we are able to deduce theorems concerning ear decompositions of non-bipartite matching covered graphs.

Lemma 5.2 *Every near-bipartite matching covered graph has a canonical ear decomposition which uses exactly one double ear.*

Proof: By induction on $|V| + |E|$. We may assume that $|V| \geq 6$.

If G has a non-trivial tight cut C, then it is also a tight cut of H. Furthermore, one of the C-contractions of G is bipartite and the other is a near-bipartite graph. The proof follows from induction and Lemma (5.1).

So, we may assume that G is a brick. If G is either K_4 or $\overline{C}_6$, then there is nothing more to be proved. If not, by 1.5, G has a removable edge, say h. Clearly, h cannot be either e or f. By induction, $G - h$ has a canonical ear decomposition which uses exactly one double ear. We can add h as a single ear at the end to obtain an ear decomposition of G which uses exactly one double ear. ■

Proof of Theorem 1.7: Let G be any graph which has an ear decomposition

$$\mathcal{P}: \quad G_1, G_2, ..., G_r = G$$

which uses d double ears. Then, we wish to prove that G has a canonical ear decomposition $\mathcal{P}^*$ which also uses exactly d double ears. In order to prove this, it suffices to show that the first non-bipartite graph in $\mathcal{P}$ has a canonical ear decomposition which uses exactly one double ear. Let G_i be the first non-bipartite graph in $\mathcal{P}$. Then, G_{i-1} is bipartite and G_i is obtained from G_{i-1} by adding a double ear (P_1, P_2), where P_1 is an odd path with ends in one part of the bipartition of G_{i-1}, and P_2 is an odd path with ends in the other part of the bipartition of G_{i-1}. If we replace P_1 and P_2 by edges e and f, we obtain a near-bipartite matching covered graph, say G'. To prove the theorem, it clearly suffices to show that G' has a canonical ear decomposition which uses exactly one ear. But this is valid by (5.2). ■

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