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Representing conics using the oriented projective plane

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Abstract

We present a geometric definition of conic sections in the oriented projective plane and describe some of their nice properties. The three main classes of affine conics are unified by a generalized distance notion on that space. This definition leads to a very simple representation of conic arcs suitable for implementations of geometric solutions to problems involving the concept of distance, in particular, the construction of various generalizations of Voronoi diagrams.

Keywords: Conic sections; Oriented projective plane; Representation of curves

1 Motivation and basics

Representation of conics and conic arcs can be done in many different ways and the choice depends on the application. For instance, in computer graphics, Herman (1991) presents a representation suitable for viewing pipelines, where conics are described by a set of characteristic points and the goal is to achieve first affine and then projective invariance in order to reduce the cost of the pipeline process.

For CAGD applications, in (Farin, 1988), we can find how to represent conic arcs as quadratic NURBS curves in an affine invariant form.

We concentrate here on representing conic arcs which appear naturally in computational geometry problems such as generalizations of Voronoi diagrams (Yap, 1987), shortest path maps and visibility diagrams. These problems are defined based on concepts of distance and, usually, coordinates of foci of conics are present in the input.

In this computational geometry context, the work of Held (1991) is, as far as we were able to determine, the only one involving conics that discusses how to represent them. In the problem of constructing offsets of boundaries formed by line segments and circular arcs, conic arcs are used as a step in the construction. Held uses a representation that is in itself a parameterization of conic arcs, whose parameter is the distance from the boundary. This representation is convenient for building offsets but is too specific to be used as a protocol in other applications.

Usually, text books on conic sections, such as (Salmon, c. 1913), give more emphasis on the algebraic characterization than on the geometric one. For instance, an ellipse is defined

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as a second degree equation whose coefficients satisfy certain constraints, instead of the set of all points whose sum of distances from two given foci is constant. However, as it will be shown in section 3, this geometric definition, along with a generalized notion of distance on the oriented projective plane (Stolfi, 1991), leads to a very simple representation unifying all classes of affine conics. This representation is adequate for most applications and for Euclidean ones through a straightforward embedding of the Euclidean plane in the oriented projective plane.

1.1 Generalized distance notions

While the classical projective plane $\mathbb{P}^2$, can be viewed as the Euclidean plane $\mathbb{E}^2$ plus a line at infinity, the oriented projective plane $\mathbb{T}^2$ is composed of *two* copies of $\mathbb{E}^2$ plus a line at infinity. This follows from the fact that in $\mathbb{T}^2$ we use *signed* homogeneous coordinates: $p = [x, y, w]$ is *not* identified to $[-x, -y, -w]$, which is denoted $\neg p$, and these are called antipodal points. (We adopt throughout this paper the notations used by (Stolfi, 1991)).

The set of points with $w > 0$ is called the front range (or side) of the plane, and those with $w < 0$ form the back range. The set of points with $w = 0$ (except for the invalid triplet $[0, 0, 0]$), which are referred to as improper points, is the line at infinity, Ω. There are (at least) two geometric models for $\mathbb{T}^2$: the flat model, with the usual mapping to Cartesian coordinates $[x, y, w] \mapsto (x/w, y/w)$, and the spherical model with the mapping $[x, y, w] \mapsto (x, y, w)/\sqrt{x^2 + y^2 + w^2}$. It can easily be seen that these two models are related by central projection and we will take advantage of them in the illustrations, by using for each situation the one that best serves to convey a given idea.

The extension of Euclidean concepts, such as perpendicularity, distance and angular measure to the oriented projective plane defines the so called two-sided Euclidean Plane. The distance between two proper points ($w \neq 0$) is defined by the expression:

$$\mathrm{dist}(a, b) = \frac{\sqrt{(x_a w_b - x_b w_a)^2 + (y_a w_b - y_b w_a)^2}}{w_a w_b}.$$

This formula yields positive values for points on the same side of the plane, and negative values for points on different sides. Note that $\mathrm{dist}(a, b) = \mathrm{dist}(b, a)$, and that $\mathrm{dist}(\neg a, b) = \mathrm{dist}(a, \neg b) = -\mathrm{dist}(a, b)$.

On $\mathbb{T}^2$ we can define without ambiguity *the* segment between a proper and an improper point, which corresponds to a ray on $\mathbb{E}^2$. This allows the extension, by projective tools, of a very intuitive concept of $\mathbb{E}^2$: if we sweep the plane from infinity with a straight line, in a given direction, and the line encounters point b after point a, we can say that b is *closer to infinity in that direction* than a is.

Definition 1 *Let a and b be two proper points and c be an improper point* (Fig. 1(a)). *Let $r_a = a \vee c$ and $r_a^{\perp} = a \vee \mathrm{norm}(r_a)$. We say that:*

$$if\ b \diamond r_a^{\perp} = \left\{ \begin{array}{c} +1 \\ -1 \end{array} \right\} then \left\{ \begin{array}{c} a \\ b \end{array} \right\} is\ closer\ to\ c\ than \left\{ \begin{array}{c} b \\ a \end{array} \right\} is$$

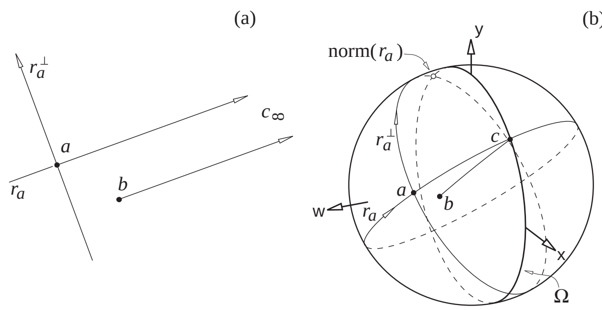

Figure 1: Relative distance from improper points: the flat and the spherical models

and that a is as close to c as b is if $b \diamond r_a^\perp = 0$, where $p \vee q$ denotes the oriented line defined by points p and q, and $p \diamond r$ is the predicate that decides the sidedness of point p with respect to line r.

We can also determine, in the context of definition 1, how much closer say, b is to c than a is to c, by simply computing $\delta_r(b, r_a^\perp)$, where δ_r is the usual Euclidean distance from the point b to the line $r_a^\perp$. This concept is called *relative distance between proper and improper points*. Even though it may seem strange to compare $\mathrm{dist}(a, c)$ and $\mathrm{dist}(b, c)$ when both are infinite (and thus, might be regarded as equal), the comparison leads to no inconsistencies when we consider computation involving improper points in algorithms.

2 Conics in the Two-Sided Euclidean Plane

Due to the usual relation between $\mathbb{E}^2$ and $\mathbb{T}^2$ which consists of mapping Cartesian to signed homogeneous coordinates, converting formulas and concepts from $\mathbb{E}^2$ to $\mathbb{T}^2$ is usually very simple: all it takes is to convert the coordinates.

When dealing with polynomial equations defined on $\mathbb{E}^2$, and converted to $\mathbb{T}^2$, it is easy to see that if a point satisfies the equation, so does its antipode. Consider the parabola $y = x^2$, which in homogeneous coordinates, becomes $yw = x^2$. The set of points of $\mathbb{T}^2$ which satisfy this equation has two connected components (Fig. 2(a)). This fact raises the issue that a good representation of conic arcs must be able to distinguish these two components.

In spherical geometry, conics can be geometrically defined as the set of points whose sum of distances from two given foci is constant (Salmon, 1914, Chapter X). This is the definition of ellipses on $\mathbb{E}^2$, and in fact, sphero-conics resemble ellipses. It is easy to see that the central projection of affine conics in the spherical model of $\mathbb{T}^2$ (see Fig. 2) are, indeed, sphero-conics. Nonetheless, the foci of an affine conic are *not* the foci of the projected sphero-conic. Despite this, we can apply the same definition with respect to the affine foci on the plane using the generalized distance notions from the previous section.

It is known that a parabola may be considered, in every aspect, as an ellipse with one of its foci shifted to infinity (Salmon, c. 1913, p.202). On the two-sided plane we can drive

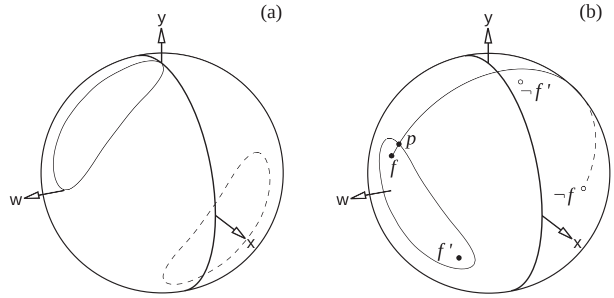

Figure 2: Algebraic conics in the two-sided plane

this focus beyond infinity and consider the resulting hyperbola as an ellipse with foci on different sides of the plane.

Definition 2 *On the two-sided Euclidean plane, a* conic *is the set of all points whose sum of distances from two given foci is constant. This yields an ellipse if the foci are in the same range, a parabola if one focus is at infinity and a hyperbola if they are in different ranges.*

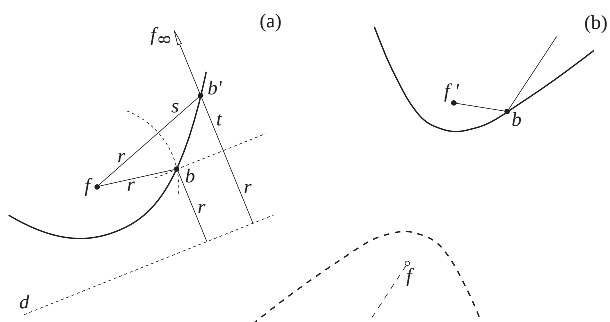

Figure 3: The definition 2 as seen from the flat model

It is interesting to show how the definition yields affine parabolas and hyperbolas. For the parabola, we use the relative distance from improper points, as follows. Consider the parabola in Fig. 3(a): as we move from b to b' the relative distance from the improper focus decreases by t. From the definition of affine parabola we know that $r + t = r + s$, so that $s = t$ and hence the distance from the proper focus increases by t. For the hyperbola, note that one of the distances is always negative. In Fig. 3(b) we have $\mathrm{dist}(b,f) = -\mathrm{dist}(b,\neg f)$. Therefore, if the foci are on different sides of the plane, the effect of adding the two distances is to subtract their absolute values, which is the usual definition of a hyperbola.

This definition also distinguishes the connected components of the conic defined by a polynomial equation. We consider them as antipodal conics, each one with two possible orientations. Figure 2(b) shows an ellipse defined by foci f and f' and some constant sum c, which is the same set of points of the ellipse defined by foci $\neg f$ and $\neg f'$ and constant sum $-c$. The former is counterclockwise oriented and the latter is clockwise oriented. Note that if we move a point p counterclockwise on the ellipse (as seen from the outside of the sphere) the segment fp rotates counterclockwise around f, but $\neg fp$ rotates clockwise around $\neg f$. This orientation distinction may also be understood by considering the equation $\text{dist}(p,f) + \text{dist}(p,f') = c$. To get the oppositely oriented ellipse, we multiply the equation by -1: $-\text{dist}(p,f) - \text{dist}(p,f') = -c$, which is equivalent to $\text{dist}(p,\neg f) + \text{dist}(p,\neg f') = -c$.

To get the antipodal ellipse, we change the sign of the constant sum: $\text{dist}(p,\neg f) + \text{dist}(p,\neg f') = c$. To change the orientation of this ellipse, we multiply the equation by -1, obtaining $\text{dist}(p,f) + \text{dist}(p,f') = -c$. Note that if we try to extract the radicals, by appropriately squaring these equations twice, we get the polynomial equation of the algebraic conic which does not distinguish antipodal points. Before proposing the computational representation, we list three properties of the conic introduced in definition 2:

1. If we remove a conic from $\mathbb{P}^2$ what remains are two subspaces, one of which is topologically equivalent to an open disc and the other to a Möbius strip. If we remove any conic from $\mathbb{T}^2$ what remains are two subspaces, both equivalent to an open disc.

2. Let us call the interior of a conic *the* open subspace which contains at least one of the foci. The segments joining *any* point on a conic to its foci are entirely contained in the interior of the conic, as opposed to what happens with hyperbolas on $\mathbb{E}^2$.

3. From the preceding property we see that the interior of a conic is always star-shaped with respect to the foci. Furthermore, if the interior is convex the conic is oriented counterclockwise (as defined before).

3 Representation of Conics and Conic arcs

We represent a conic by the homogeneous coordinates of three points in the two-sided Euclidean plane. The points are the two foci f_1 and f_2, and one point d on the conic, which are sufficient to uniquely define the conic. The affine class of the conic is implicitly given by the relative location of the foci. For all classes, to change the orientation of a conic, we simply exchange the foci for their antipodes, which means multiplying their coordinates by -1, and to get the antipodal conic, we exchange the point d for its antipode.

Representation of Arcs. The coordinates of the endpoints of an arc may, themselves, be used to determine the shape of the conic. This is the case in NURBS representations. However, the coordinates of the foci of the conic are usually present in the input to algorithms. This means that we should constrain the representation of the conic to use these coordinates as its foci directly, in order to minimize errors. Thus, we represent an arc of a conic with the same three points (f_1, f_2 and d that represent the conic itself) and two more points a_1 and a_2 used to determine the endpoints of the arc.

Due to the fact that the conic is star-shaped with respect to the foci, all lines passing through one focus intersect the conic in two points. Let $r_1 = f_1 \vee a_1$ and $r_2 = f_1 \vee a_2$. We define the first endpoint of the arc as the intersection between the conic and r_1 when it leaves the interior of the conic (see Fig. 4 for an arc of a hyperbola). The second endpoint is, as expected, the intersection between the conic and r_2 when it leaves the interior of the

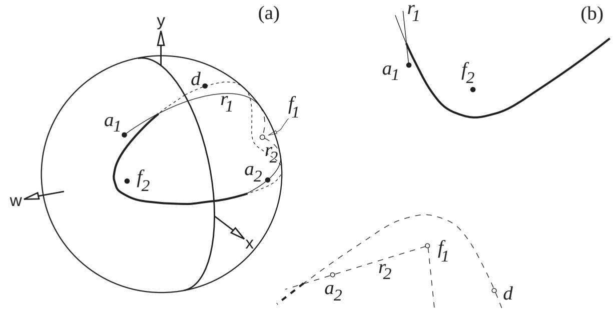

Figure 4: Arc of a hyperbola on the two-sided Euclidean plane

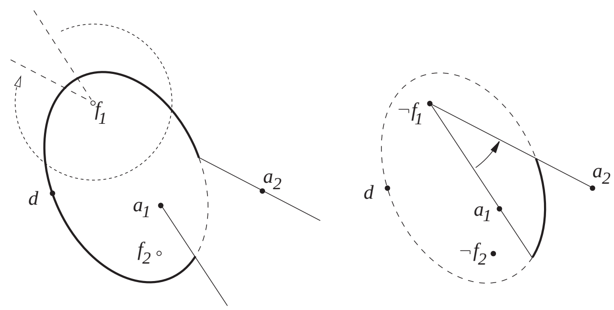

Figure 5: Complementary arc changing the orientation of the conic

conic. Note that there are two complementary arcs satisfying this representation. We use the orientation of the conic to decide which arc is represented. The arc is the one traced by the intersection of r_1 with the conic when we rotate r_1 counterclockwise around f_1 until r_1 coincides with r_2. For all classes, to exchange an arc for its complement, either change the orientation of the conic (see Fig. 5) or exchange a_1 for a_2. To get the antipodal arc, exchange d, a_1 and a_2 for their antipodes.

Degenerate Cases. This representation can describe with no redundancy (and therefore without risk of inconsistency) any arc of non-degenerate conics. However, one may set the coordinates of the points f_1, f_2, d, a_1 and a_2 so that the resulting data does not properly define a conic. We identify these degenerate situations:

1. when a_1 (a_2) and f_1 are coincident or antipodal, r_1 (r_2) is not defined;

2. when both f_1 and f_2 are at infinity;

3. when f_1 and f_2 are antipodal points;

4. when d is at infinity and f_1 and f_2 are on a single range. In this case, we may regard the conic as being equal to Ω.

   Note that if d is at infinity and f_1 and f_2 are on different ranges, we still have a properly defined hyperbola, but still this case requires special computational treatment.

Concluding Remarks

The representation of conics and conic arcs that we offered portrays a concise and unifying manner of dealing with affine conics which is both immune to inconsistencies and simple to deal with. It serves well the purpose of computing with conics on the oriented projective plane as well as on the regular Euclidean plane (by viewing it as a single range of the two-sided plane).

We have successfully employed this representation to implement the construction of nearest and furthest neighbor Voronoi diagrams with additive weights, which contains regions whose boundaries often include conic arcs (Pinto and de Rezende, 1998).

On the other hand, we should note that this representation is *not* affine invariant, as the foci are not necessarily preserved by affine transformations. However, it is certainly similarity invariant and fits well most situations where conics appear as input (or output) data of geometric problems, specially those dealing with generalized proximity concepts.

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