On Clique-Complete Graphs

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Abstract

A graph is *clique-complete* if no two of its maximal cliques are disjoint. A vertex is *universal* if it is adjacent to all other vertices in the graph. We prove that every clique-complete graph either contains a universal vertex or an induced subgraph in an indexed family $\mathcal{Q} := \{Q_{2n+1} : n \geq 1\}$, defined in the text. We show that this is precisely the family of minimal graphs which are clique-complete but have no universal vertices. The minimality used here refers to induced subgraphs.

For $n \geq 2$, we show that Q_{2n+1} is neither perfect nor planar. It follows that every planar clique-complete graph without a universal vertex contains an induced subgraph isomorphic to Q_3. A similar result holds for perfect clique-complete graphs without universal vertices. We also specialize the latter result for certain classes of perfect graphs.

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1 Clique-complete Graphs

We present in this paper a proof of a revised version of a conjecture due to the second-named author, first presented in her Ph. D. thesis [1], written under the supervision of the third-named author.

Theorem 1 *A clique-complete graph free of universal vertices contains an induced subgraph isomorphic to Q_{2n+1}, for some positive integer n.*

A *graph* G here is a simple graph, that is, a graph without loops and multiple edges. We denote by VG the vertex set of G. A *clique* K in a graph G is a set of vertices pairwise adjacent in G; clique K is *maximal* if no proper superset of K is a clique, and *maximum* if no larger set of vertices is a clique.

For each vertex v of graph G, we denote by $N(v)$ the *neighborhood* of v, that is, the set consisting of v plus each vertex to which v is adjacent. Vertex v is *universal in* G if it is adjacent to each vertex of $VG - v$, that is, if $N(v) = VG$. We extend the domain of N to subsets X of VG by making $N(X) := \cup_{v \in X} N(v)$.

Graph G is *clique-complete* if every two of its maximal cliques have nonnull intersection. Every nonnull complete graph is clique-complete. In fact, every graph containing a universal vertex is clique-complete. A more interesting example is shown in Figure 1.

For X a set of vertices of G, we denote by $G[X]$ the *subgraph of* G *induced by* X, that is, the vertex set of $G[X]$ is X and the edge set of $G[X]$ consists of those edges of G having both ends in X.

We now define graph Q_n, for each integer $n \geq 3$. A *circuit* C_n is a connected graph with $n \geq 3$ vertices, each of which has degree 2:

- $VQ_n := \{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\}$ is a set of $2n$ vertices.
- $Q_n[\{v_1, \cdots, v_n\}] \simeq \overline{C_n}$.
- For each i, $(1 \leq i \leq n)$, $N(u_i) = VQ_n - v_i$.

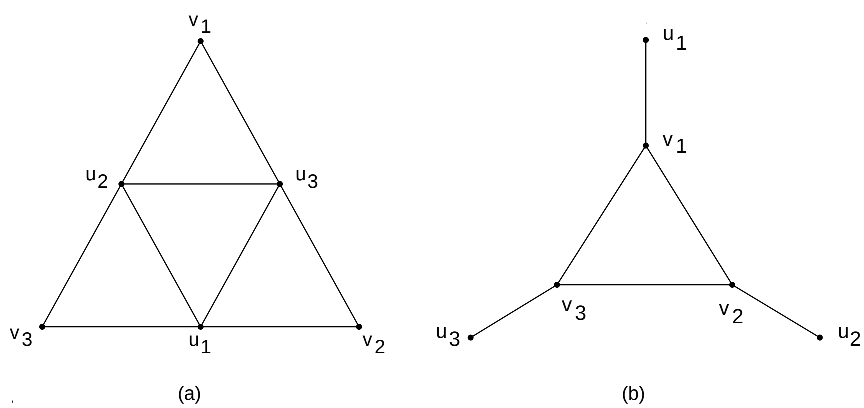

Figure 1: (a) Graph Q_3, the smallest clique-complete graph free of universal vertices. (b) The complement of Q_3.

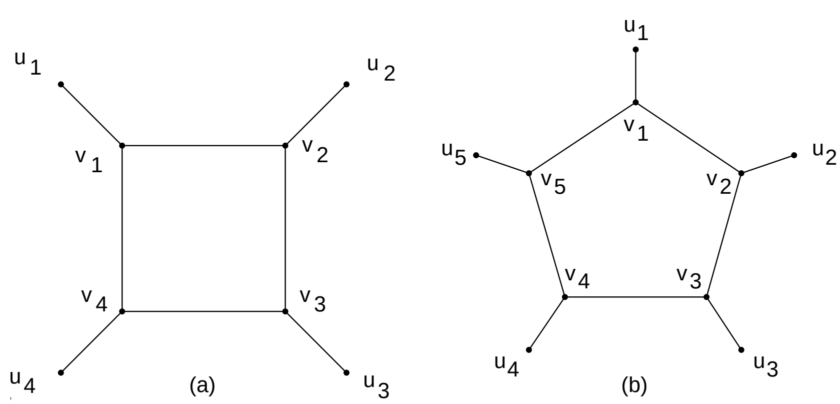

Figure 2: (a) Graph $\overline{Q_4}$. (b) Graph $\overline{Q_5}$.

Figure 2 shows graphs $\overline{Q_4}$ and $\overline{Q_5}$.

Proposition 2 *For each odd integer n, $(n \geq 3)$, graph Q_n is clique-complete.*

Proof. Let A be any maximal clique of Q_n. Since $Q_n[\{v_1, \cdots, v_n\}] = \overline{C_n}$, we have

$$|\, A \cap \{v_1, \cdots, v_n\}\,| \leq (n-1)/2.$$

On the other hand, for each i $(1 \leq i \leq n)$, precisely one of u_i and v_i lies in A, since u_i is universal in $G - v_i$. Consequently, $|A| = n$, whence

$$|\, A \cap \{u_1, \cdots, u_n\}\,| \geq (n+1)/2.$$

Since this inequality holds for every maximal clique A of Q_n, this graph is clique-complete. □

Proposition 3 *Graph Q_n is free of universal vertices and also free of induced subgraphs isomorphic to Q_p, for every integer p such that $3 \leq p \neq n$.*

Proof. Graph $\overline{Q_n}$ is free of isolated vertices, whence Q_n is free of universal vertices. For each integer $k \geq 3$, $\overline{Q_k}$ contains precisely one circuit, which consists of k vertices. We conclude that if $3 \leq p \neq n$ then no subgraph of Q_n is isomorphic to Q_p. □

2 Proof of Theorem 1

Graph G is *critical* if for each induced proper subgraph H of G, either H contains a universal vertex or H is not clique-complete.

Proof of Theorem 1. Let G be a clique-complete graph free of universal vertices. We show, by induction on $|VG|$, that G contains an induced subgraph isomorphic to Q_{2n+1}, for some positive integer n.

If G is not critical, then it contains an induced proper subgraph, H, that is clique-complete and free of universal vertices. By induction hypothesis, H contains an induced subgraph isomorphic to Q_{2n+1}. If G is critical, then, by Theorem 4, asserted below, $G \simeq Q_{2n+1}$. In both cases the assertion holds.

Theorem 4 *Every graph G free of universal vertices, clique-complete and critical, is isomorphic to Q_{2n+1}, for some positive integer n.*

Proof. We derive first some properties of G.

Proposition 5 *The complement $\overline{G}$ of G is connected.*

Proof. Assume the contrary. Let X be the vertex set of a connected component of $\overline{G}$. Thus, in G,

$$VG \setminus X \subseteq N(v)\, (\forall v \in X).$$

Let $H := G[X]$, $K := G[VG \setminus X]$. Since G is free of universal vertices, so too are H and K. Since G is critical, neither H nor K are clique-complete. Let A_H and B_H be disjoint maximal cliques of H; likewise, denote by A_K and B_K disjoint maximal cliques of K.

Sets $A_H \cup A_K$ and $B_H \cup B_K$ are disjoint maximal cliques of G, a contradiction. □

Proposition 6 *The complement $\overline{G}$ of G is not bipartite.*

Proof. Assume the contrary, let $\{A, B\}$ be a bipartition of $\overline{G}$.

Consider first the case in which A and B are both nonnull. By Proposition 5, $\overline{G}$ is connected, thus each vertex of A (respectively, B), is adjacent in $\overline{G}$ to at least one vertex of B (respectively, A). We conclude in this case that A and B are (disjoint) maximal cliques of G, a contradiction.

We may thus assume that at least one of A and B, say, A, is null. By Proposition 5, $\overline{G}$ is connected. It follows that G consists of at most one vertex. Since G is clique-complete, it consists of precisely one vertex, a universal vertex.

In both cases, a contradiction is obtained, which proves that $\overline{G}$ is not bipartite. □

Vertex v of G is *quasi-universal* if it is adjacent to all but one vertex of $VG - v$; that is, $VG \setminus N(v)$ is a singleton. The unique element of $VG \setminus N(v)$ is *antipodal* to v. It should be noticed that each vertex u_i of Q_n is quasi-universal, v_i its antipodal.

Proposition 7 *Graph G contains a quasi-universal vertex.*

Proof. Let u be a vertex of maximum degree in G. Since G is free of universal vertices, $VG \setminus N(u)$ is nonnull, let v be one of its vertices, let $H := G[N(u) + v]$. Since u has maximum degree in G, H is free of universal vertices. Since G is critical, either $H = G$ or H is not clique-complete.

It thus suffices to show that H is clique-complete. For this, assume that there exist in H two disjoint maximal cliques, A and B. By definition of H, vertex u is quasi-universal in H, v its antipodal vertex. It follows that one of A and B contains u, the other contains v. Say, $u \in A, v \in B$.

Clique A is maximal in G, for A is maximal in H, u lies in A and no vertex of $VG \setminus VH$ is adjacent to u, by definition of H.

Set $B \cup (VG \setminus VH)$ includes some maximal clique C of G, for B is maximal in H. Thus A and C are disjoint maximal cliques in G, a contradiction.

Indeed, H is clique-complete and free of universal vertices. By the criticality of G, $G = H$, whence u is quasi-universal in G. □

Let $RG := \{v \in VG : G - v \text{ is clique-complete}\}$. Clearly, the antipodal of every quasi-universal vertex of G lies in RG. The following assertion establishes the converse.

Proposition 8 *Each vertex v of RG is the antipodal of some quasi-universal vertex, denoted $u(v)$, in G.*

Proof. By hypothesis, $G - v$ is clique-complete and G is critical. Thus, $G - v$ contains a universal vertex, $u(v)$. But G is free of universal vertices, whence $u(v)$ is quasi-universal in G, v its antipodal vertex. □

Proposition 9 *For each vertex v of RG, $u(v) \in VG \setminus RG$.*

Proof. Assume the contrary. By Proposition 8, $u(v)$ is the antipodal vertex of some quasi-universal vertex w in G. Clearly, $w = v$. This implies that $\{v, u(v)\}$ is the vertex set of a connected component of $\overline{G}$. By Proposition 5, $\overline{G}$ is a complete graph with just two vertices. Thus G consists of two isolated vertices, therefore it is not clique-complete, a contradiction. □

We have thus established that RG is the set of vertices that are antipodal to quasi-universal vertices of G.

Proposition 10 *For each vertex v of RG, each of its non-neighbors, except $u(v)$, lies in RG.*

Proof. Let w be a vertex in $VG \setminus N(v)$, distinct from $u(v)$. Assume, to the contrary, that $G - w$ is not clique-complete. Let A and B be disjoint maximal cliques of $G - w$. Since G is clique-complete, $A + w$ and $B + w$ are (maximal) cliques in G. Since $w \in VG \setminus N(v)$, vertex v does not lie in $A \cup B$. By the maximality of A and B, and since $w \neq u(v)$, it follows that $u(v) \in A \cap B$, a contradiction. □

We are now in position to show that $G \simeq Q_{2n+1}$, for some positive integer n. By Propositions 8 and 9, $u : RG \to VG \setminus RG$. Clearly, u is injective.

We now show that u is surjective, that is, $\{RG, u(RG)\}$ is a partition of VG. For this, let $S := RG \cup u(RG)$.

By Proposition 8, each vertex of $u(RG)$ is adjacent to each vertex of $VG \setminus RG$. On the other hand, by Proposition 10, each vertex of RG is adjacent to each vertex of $VG \setminus S$. We conclude that each vertex of S is adjacent to each vertex of $VG \setminus S$.

By Proposition 5, $\overline{G}$ is connected, whence one of S and $VG \setminus S$ is null. By Proposition 7, G contains a quasi-universal vertex, whence its antipodal vertex lies in RG. We conclude that $VG = S$ and u is bijective.

By Proposition 6, $\overline{G}$ is not bipartite. Since each vertex of $u(RG)$ has degree one in $\overline{G}$, it follows that $\overline{G}[RG]$ is not bipartite.

Let X be a minimal subset of RG such that $\overline{G}[X]$ is not bipartite. Clearly, $\overline{G}[X]$ is a circuit, say, C_{2n+1}. Consequently, $G[X \cup u(X)] \simeq Q_{2n+1}$.

By Propositions 2 and 3, Q_{2n+1} is clique-complete and free of universal vertices. Since G is critical, we conclude that $G \simeq Q_{2n+1}$.

The proof of Theorem 4 completes the proof of Theorem 1. □□

From Theorems 1 and 4 we deduce that family $\mathcal{Q} := \{Q_{2n+1} : n \geq 1\}$ is the family of minimal clique-complete graphs free of universal vertices.

Corollary 11 *A graph free of universal vertices is clique-complete and critical if and only if it is isomorphic to Q_{2n+1}, for some positive integer n.*

Proof. Theorem 4 asserts that every clique-complete critical graph free of universal vertices is isomorphic to Q_{2n+1}, for some positive integer n. To prove the converse, let n be a positive integer, let H be a clique-complete induced proper subgraph of Q_{2n+1}. By Theorem 1, either H contains a universal vertex or it contains an induced subgraph isomorphic to Q_{2p+1}, for some positive integer p. In the latter case, Q_{2n+1} would contain a proper induced subgraph isomorphic to Q_{2p+1}, in contradiction to Proposition 3. Therefore, H contains a universal vertex. Since this conclusion holds for every clique-complete proper induced subgraph of Q_{2n+1}, this graph is critical. □

We conclude this section by giving a finite family of graphs that occur as induced subgraphs of each clique-complete graph. This family consists

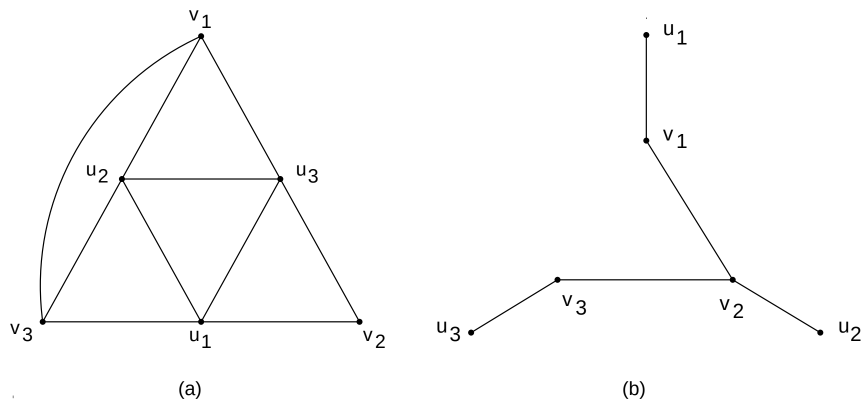

Figure 3: (a) Graph Q'_3. (b) The complement of Q'_3.

of just two graphs, namely, Q_3 and Q'_3, the latter obtained from Q_3 by adding one edge joining v_1 and v_3 (Figure 3).

Corollary 12 *A graph free of universal vertices and clique-complete contains one of Q_3 and Q'_3 as an induced subgraph.*

Proof. Graph $\overline{Q'_3}$ is an induced subgraph of $\overline{Q_n}$ for each $n \geq 4$ (see Figure 3). Thus Q'_3 is an induced subgraph of Q_n, for each $n \geq 4$. The assertion follows from Theorem 1. □

3 Conclusions

Graph G is *perfect* if, for each induced subgraph H of G, its chromatic number equals the size of its maximum clique [2].

Every perfect graph is free of induced circuits C_{2n+1} and their complements, for any integer $n \geq 2$. Thus, for each $n \geq 2$, Q_{2n+1} is not perfect. On the other hand, Q_3 is perfect.

Corollary 13 *Every clique-complete perfect graph free of universal vertices contains Q_3 as induced subgraph.* □

We observe that graph Q_3 is neither a comparability nor a co-comparability graph [2].

Corollary 14 *Every clique-complete (co-)comparability graph contains a universal vertex.* □

Corollary 15 *Every clique-complete interval graph contains a universal vertex.*

Finally, it follows that every clique-complete graph free of universal vertices and not containing Q_3 as an induced subgraphs necessarily contains the complete graph K_{2n+1} for $n \geq 2$.

Corollary 16 *Every clique-complete planar graph free of universal vertices contains Q_3 as an induced subgraph.*

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