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An (l, u)-Transversal Theorem for Bipartite Graphs

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Abstract

Continuing the work begun by Philip Hall in 1935, we here give necessary and sufficient conditions for the existence, in a bipartite graph, of a set of edges satisfying specified lower and upper bounds. Here the graph is directed bipartite; lower and upper bounds are specified by integer-valued functions, l and u, on the collection of all directed sets of vertices, or perhaps on some subcollection, such as the collection of singletons. We require these functions to be super- and sub-modular, respectively. An (l, u)-transversal is a set t of edges that satisfies these bounds. A second restriction, $q \subseteq t \subseteq r$, for edge sets q and r, is also permitted.

One might hope to give necessary and sufficient conditions for the existence of a general (l, u)-transversal. In this paper, this is done for the special case in which the domain of one of the functions, say u, is restricted to singletons. Graph G contains an (l, u)-transversal t such that $q \subseteq t \subseteq r$ if and only if for each Xin Dom l and each subset N of VG, $lX \leq uN + [q, r](X \oplus N)$. This function [q, r], when applied to a set Y of vertices of G, is the number of edges of r directed away from Y minus the number of edges of q directed toward Y.

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This work is motivated by the Woodall Conjecture, which states: in any directed graph, a largest packing of transversals of directed coboundaries is equal in size to a smallest directed cut. We observe that the domain of this Conjecture can be reduced to directed bipartite graphs. For such graphs, the partial (l, u)-theory developed here is used to show that the edge set of any directed bipartite graph can be partitioned into two subsets, one a transversal of directed coboundaries, the other a (k - 1)-transversal of the vertex coboundaries. In this application we require the supermodularity of the size of a maximum partition of a directed coboundary into directed cuts.

1 Woodall's Conjecture and Directed Bipartite Graphs

Let G be a graph with vertex set VG and edge set eG. For any set X of vertices in G, \overline{X} denotes set $VG \setminus X$. The *coboundary* of a set X of vertices in G is the set of edges that each have one end in X and one end in \overline{X} . A set d of edges is a coboundary if there is a set X of vertices such that $d = \delta X$. A *cut* is a minimal nonnull coboundary.

For directed graph G, each edge α of G leaves its positive end $p\alpha$ and enters its negative end $n\alpha$. The constituents of a coboundary δX are $\delta^+ X := \{\alpha \in \delta X : p\alpha \in X\}$ and $\delta^- X := \{\alpha \in \delta X : n\alpha \in X\}$. Vertex set X is outdirected if $\delta^- X = \emptyset$, indirected if $\delta^+ X = \emptyset$; in either case X is directed. For X a directed set of vertices, δX is a directed coboundary. A directed cut is a cut that is a directed coboundary.

A transversal of directed cuts in G is a set t of edges that intersects (has a nonnull intersection with) each directed cut.

Woodall's Conjecture In directed graph G, let T^* be a maximum packing of directed cut tranversals; let d_* be a minimum directed cut: then $|T^*| = |d_*|$.

Woodall [11] described this Conjecture as the Menger dual of the

Lucchesi-Younger Theorem A minimum directed cut transversal has size equal to that of a maximum packing of directed cuts: $|t_*| = |D^*|$. [7, 8, 6]

A graph G is directed bipartite if its edge set eG is a directed coboundary in G. Equivalently, G has a directed bipartition, a bipartition (V^+, V^-) such that each edge α has $p\alpha$ in V^+ and $n\alpha$ in V^- .

The Woodall Conjecture can be reduced, without loss of generality, to directed bipartite graphs. Let G be any directed graph, with maximum packing T^* and minimum directed cut d_* . Let $k := |d_*|$. For each vertex v in G that is not a source or sink, replace v in G by two vertices, a source v^+ and sink v^- , whose incident edges are those in G with positive end vand negative end v, respectively. Add k edges directed from v^+ to v^- . The sources and sinks are left unchanged. Let G' denote the directed bipartite graph thus obtained. Now, each directed cut of G is a directed cut of G'. And each directed cut of G' that is not in G has at least kedges. Moreover, for every packing of transversals of G', the images of these transversals in G make up a packing of directed cut transversals of G. So the minimax equality holds in G if and only if it holds in directed bipartite graph G'.

2 *f*-Coverings and Supermodular Functions

In this section, we characterize when an f-covering in a directed graph exists in terms of the existence of a supermodular function satisfying a certain inequality.

For directed graph G, let f be a real-valued function defined on some subsets of VG. An f-covering is a set t of edges of G such that for each X in the domain Dom f of f, $fX \leq |t \cap \delta^+ X| - |t \cap \delta^- X|$. Given two real-valued functions f and g, we say that $f \leq g$ if $Dom f \subseteq Dom g$ and, for all X in Dom f, $fX \leq gX$. Note that \leq is transitive.

Whenever the domain of f is closed under intersection and union, f is supermodular if $f(X \cap Y) + f(X \cup Y) \ge fX + fY$ for each X, Y in Dom f; f is submodular if the reverse inequality holds, and modular if equality holds.

For q and r subsets of eG, let $[q, r] := [r]^+ - [q]^-$, where $[r]^+X := |\delta^+ X \cap r|$ and $[q]^- X := |\delta^- X \cap q|$, for all subsets X of VG. Let [t] abbreviate [t, t]. Then t is an f-covering if and only if $f \leq [t]$.

Proposition 1 For edge sets q, r and vertex sets X, Y of G,

Proof. The asserted equation follows from

$$[r]^{+}X + [r]^{+}Y = [r]^{+}(X \cap Y) + [r]^{+}(X \cup Y) + |\delta^{+}X \cap \delta^{-}Y \cap r| + |\delta^{-}X \cap \delta^{+}Y \cap r|,$$

and the similar relation for $q.\Box$

Corollary [q,r] is submodular if $q \subseteq r$ and modular if $q = r.\Box$

The main result of this section is

Theorem 2 For directed graph G, let f be a real-valued function on subsets of the vertex set of G; let q and r be subsets of eG such that $q \subseteq r$. There is an f-covering t satisfying $q \subseteq t \subseteq r$ if and only if there exists a supermodular integer-valued function h such that $f \leq h \leq [q, r]$.

Proof. For necessity, let t be an f-covering satisfying $q \subseteq t \subseteq r$. Then $f \leq [t] = [t, t] \leq [q, r]$. The inequality is satisfied with [t] in the role of h. By the above Corollary, [t] is modular.

For sufficiency, let h be an integer-valued supermodular function such that $f \leq h \leq [q, r]$. We use induction on $r \setminus q$. If this difference is null, i.e., q = r, then [q, r] = [q, q] = [q], whereupon $f \leq [q]$, i.e., q is an f-covering.

Assume then that $r \setminus q$ is nonnull.

Lemma For each edge α in $r \setminus q$, either $h \leq [q, r \setminus \{\alpha\}]$ or $h \leq [q \cup \{\alpha\}, r]$.

Proof. Since $h \leq [q, r]$, either

- (a) $h \leq [q, r \setminus \{\alpha\}]$, or
- (b) $\exists X \subseteq VG$ such that hX = [q, r]X and $\alpha \in \delta^+ X$.

Likewise, either

(a) h ≤ [q ∪ {α} , r], or
(b) ∃Y ⊆ VG such that hY = [q, r]Y and α ∈ δ⁻Y.

Suppose that in each case, alternative (b) holds. Then $\alpha \in \delta^+ X \cap \delta^- Y$, whence by Proposition 1,

 $[q, r]X + [q, r]Y > [q, r](X \cap Y) + [q, r](X \cup Y).$

The left side is equal to hX + hY; the right side is at least as large as $h(X \cap Y) + h(X \cup Y)$. This contradicts the supermodularity of h. So at least one of alternatives (a) holds. \Box

Under each of the alternatives of the Lemma, there is by induction hypothesis an f-covering t such that $q \subseteq t \subseteq r$. The Theorem follows by induction. \Box

3 (l, u)-Transversals for Directed Bipartite Graphs

Consider a graph G with directed bipartition (V^+, V^-) . We seek necessary and sufficient conditions for the existence of a set of edges in G satisfying lower and upper bounds, l and u, on directed coboundaries.

The first such Theorem we take to be Hall's [5]: the lower bound is 1 on each vertex in V^+ ; the upper bound is 1 on each vertex in VG. There exists such an (l, u)-transversal in G if and only if, for each subset X of V^+ , $|X| \leq |N|$, where N is the neighbor set of X. Hall's Theorem has been generalized to arbitrary integer-valued functions l and u on the vertices of G. As a notational device, the domains of l and u are extended to subsets of VG by $lX := \sum \{lv : v \in X\}$ and likewise for u. There exists an (l, u)-transversal in G if and only if, for each X and N such that one of X and N is a subset of V^+ and the other a subset of V^- , the following inequality holds:

$$lX \le uN + |\delta X \setminus \delta N| \,. \tag{1}$$

There have been results which extend this Theorem further, so that the domain of one of l and u includes directed subsets of VG other than singletons. Theorems of this type have been found by McWhirter-Younger [9], Rolle [10], Edmonds-Giles [1] and Feofiloff [2, 3]. The generalization described here is easy to relate to special cases, even to Hall's Theorem.

We begin with a general definition of (l, u)-transversal. Let \mathcal{V}^+ and \mathcal{V}^- be the collections of all outdirected and indirected subsets of VG, respectively. Let l be an integer-valued function on some subcollection Dom l of \mathcal{V}^+ that is closed under intersection and union; for X in $\mathcal{V}^+ \cap \mathcal{V}^-$, we take lX = 0; finally, l is supermodular. Let u have the same defining properties over \mathcal{V}^- , except that u is submodular rather than supermodular. Subset t of eG is an (l, u)-transversal if $lX \leq |\delta^+ X \cap t|$ for each X in Dom l, and $|\delta^- X \cap t| \leq uX$ for each X in Dom u. More compactly, set t of edges is an (l, u)-transversal if $l \leq [t]$ and $-u \leq [t]$. Note that an (l, u)-transversal is a directed cut transversal if $lX \geq 1$ for each X in $\mathcal{V}^+ \setminus \mathcal{V}^-$.

Let the join of l and -u be a function $\langle l, -u \rangle$ whose domain is $Dom l \cup Dom u$ and whose value is lX if $X \in Dom l$, and -uX if $X \in Dom u$. Set t of edges is an (l, u)-transversal if and only if $\langle l, -u \rangle \leq [t]$. Consequently, an (l, u)-transversal is an $\langle l, -u \rangle$ -covering, and conversely.

In the following Theorem, function u is restricted to singletons and co-singletons: $Dom u := \{\{v\} : v \in V^-\} \cup \{\overline{\{v\}} : v \in V^+\}$. Under these restrictions, we adopt the following notational conventions: $uv := u\{v\}$

for $v \in V^-$ and $uv := u\overline{\{v\}}$ for $v \in V^+$. Moreover, for N a subset of VG, $uN := \sum \{uv : v \in N\}$. We also assume that $u \ge 0$.

Theorem 3 Let G be a graph with directed bipartition (V^+, V^-) . Let q and r be subsets of eG such that $q \subseteq r$. There exists an (l, u)-transversal t such that $q \subseteq t \subseteq r$ if and only if for each X in Dom l and subset N of VG,

$$lX \le uN + [q, r](X \oplus N).$$

Remark To see that this inequality generalizes (1), observe that

if
$$X \subseteq V^+, N \subseteq V^-$$
, then $\delta X \setminus \delta N = \delta^+ (X \cup N) = \delta^+ (X \oplus N)$;
if $X \subseteq V^-, N \subseteq V^+$, then $\delta X \setminus \delta N = \delta^+ (\overline{X} \setminus N) = \delta^+ (\overline{X} \oplus N)$.

Proof of Theorem 3. To prove necessity, consider any X in Dom l and subset N of VG. Let $N_1 := N \cap X$ and $N_2 := N \setminus X$. Let t be an (l, u)-transversal. By hypothesis, $l \leq [t]$; by the Corollary of Proposition 1, [t] is modular. Thus,

$$lX \leq [t]X = [t](X \oplus N) + [t]N_1 - [t]N_2.$$

By hypothesis, $[t] \leq u$ and $0 \leq u$, whence

$$\begin{array}{rcl} [t]N_1 &\leq & [t]^+N_1 &\leq & u(V^+ \cap N_1) &\leq & uN_1 \\ -[t]N_2 &\leq & [t]^-N_2 &\leq & u(V^- \cap N_2) &\leq & uN_2. \end{array}$$

Using these inequalities, we conclude that

$$lX \leq uN_1 + uN_2 + [t](X \oplus N)$$

$$\leq uN + [t](X \oplus N).$$

By hypothesis, $q \subseteq t \subseteq r$, whence $[t] \leq [q, r]$: the cited inequality holds.

To prove sufficiency, assume that the inequality stated in the Theorem holds. Define function $h: 2^{VG} \to Z$ by

$$hY := max\{lX - uN : Y = X \oplus N, X \in Dom \, l, N \subseteq VG\}.$$

Function h is well-defined since for every subset Y of VG, it is the case that $Y = X \oplus Y$ for $X = \emptyset \in Dom l$. We claim that this function h satisfies $\langle l, -u \rangle \leq h \leq [q, r]$ and is supermodular. Assuming this, since h is integer-valued, there is, by Theorem 2, an $\langle l, -u \rangle$ -covering t such that $q \subseteq t \subseteq r$. An $\langle l, -u \rangle$ -covering is an (l, u)-transversal. So the proof is completed by verifying this claim.

(i) $\langle l, -u \rangle \le h \le [q, r].$

For X in Dom l, $X = X \oplus \emptyset$, whence $hX \ge lX - u\emptyset = lX$. For $Y \subseteq VG$, $Y = \emptyset \oplus Y$, whence $hY \ge l\emptyset - uY = -uY$. Thus $\langle l, -u \rangle \le h$.

For Y a subset of VG, there exist X in Dom l and subset N of VG, where $Y = X \oplus N$, such that hY = lX - uN. Since $lX - uN \leq [q, r](X \oplus N) = [q, r]Y$, there follows $h \leq [q, r]$.

(ii) h is supermodular.

Let Y, Y' be subsets of VG. There exist X, X' in Dom l and subsets N, N' of VG, where $Y = X \oplus N$, $Y' = X' \oplus N'$, for which

$$hY = lX - uN$$

$$hY' = lX' - uN'.$$

Let $X_I := X \cap X', X_U := X \cup X'$. By the supermodularity of l,

$$lX + lX' \le lX_I + lX_U.$$

Let $Y_I := Y \cap Y'$, $Y_U := Y \cup Y'$. There is just one pair N_I, N_U of subsets of VG such that $Y_I = X_I \oplus N_I, Y_U = X_U \oplus N_U$, namely, $N_I :=$ $X_I \oplus Y_I, N_U := X_U \oplus Y_U$. A look at the Venn Diagrams (Figure 1) verifies that $N \cap N' \supseteq N_I \cap N_U$ and $N \cup N' \supseteq N_I \cup N_U$; since u is nonnegative,

$$uN + uN' \ge uN_I + uN_U.$$

Therefore

$$hY + hY' \le lX_I - uN_I + lX_U - uN_U.$$

The right side of this inequality is at most $hY_I + hY_U$. We conclude that h is supermodular. \Box



Figure 1: A comparison of N and N' with N_I and N_U .

4 An Application of the (l, u)-Transversal Theorem

Proposition 4 For directed bipartite graph G, let k be the size of a minimum directed cut. There is a transversal t of directed cuts such that $eG \setminus t$ is a (k-1)-transversal of vertex coboundaries.

A subset t of eG is a (k-1)-transversal of vertex coboundaries if each nonisolated vertex of G has at least k-1 edges of t incident.

Proof of Proposition 4. For each set of vertices X in \mathcal{V}^+ , denote by μX the cardinality of a maximum partition of δX into nonnull directed coboundaries. Frank, Sebö and Tardos [4] showed that function μ is supermodular. Let l be this function μ . Let u be the function on vertices which assigns each vertex v the value $|\delta\{v\}| - (k-1)$ if v is nonisolated, and 0 otherwise.

Consider the conditions of the (l, u)-transversal Theorem, with $q := \emptyset$ and r := eG. If $X \in \mathcal{V}^+$ and $N \subseteq VG$, let $N^+ := N \cap V^+$ and $N^- := N \cap V^-$. By the hypotheses on k and r, and the definition of l and u,

$$\begin{aligned} k \cdot lX &\leq |\delta X|, \\ |\delta N^+| &\leq k \cdot uN^+, \\ |\delta N^-| &\leq k \cdot uN^-. \end{aligned}$$

Every edge of δX either has one end in N or lies in $\delta^+(X \oplus N)$, whence

$$\left|\delta X\right| \le \left|\delta N^{+}\right| + \left|\delta N^{-}\right| + \left|\delta^{+}(X \oplus N)\right|.$$

From these inequalities we conclude that

$$k \cdot lX \le k \left[uN^+ + uN^- + \left| \delta^+ (X \oplus N) \right| / k \right],$$

whence $lX \leq uN + [\emptyset, eG](X \oplus N)$, i.e., the conditions of Theorem 3 are satisfied. By that Theorem, G has an (l, u)-transversal t. For these values of l and u, set t is a transversal of directed cuts and $eG \setminus t$ is a (k-1)-transversal of vertex coboundaries. \Box

While this result is modest, it suggests that a strengthened (l, u)-transversal Theorem could have useful implications for the Woodall Conjecture.

References

- [1] J. Edmonds and R. Giles, A min-max relation for submodular functions on graphs. Annals of Discrete Math. 1 (1977), 185-204.
- [2] P. Feofiloff, Disjoint Transversals of Directed Coboundaries, Ph.D. Dissertation, Dept. of Combinatorics and Optimization, University of Waterloo, 1983.
- [3] P. Feofiloff, Transversais de cortes orientados em grafos bipartidos, Technical report, IME-University of S. Paulo, 1985. (In Portuguese.)
- [4] A. Frank, É. Tardos, and A. Sebö, Covering directed and odd cuts, Mathematical Programming Study 22 (1984), 99-112.
- [5] P. Hall, On representatives of subsets, J. London Math. Soc. 10 (1935), 26-30.
- [6] L. Lovász, On two minimax theorems in graph theory, J. Combinatorial Theory, Ser B 21 (1976), 96-103.
- [7] C. L. Lucchesi, A Minimax Equality for Directed Graphs, Ph.D. Dissertation, Dept. of Computer Science, University of Waterloo, 1976.
- [8] C. L. Lucchesi and D. H. Younger, A minimax theorem for directed graphs, J. London Math. Soc. 17 (1978), 369-374.
- [9] I. P. McWhirter and D. H. Younger, Strong covering of a bipartite graph, J. London Math. Soc. 3 (1971), 86-90.
- [10] M. Rolle, Matching and strong covering of directed bipartite graphs, (unpublished), 1977.
- [11] D. R. Woodall, Menger and König Systems, Lecture Notes in Math.
 642 (1978), 620-635, Springer-Verlag.

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