Total Chromatic Number and Chromatic Index of Dually Chordal Graphs

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Abstract

A graph is dually chordal if its clique hypergraph forms a hypertree. Alternatively, a graph is dually chordal if it admits a maximum neighbourhood order. This dual class to chordal graphs is not a class of perfect graphs. This class generalizes known subclasses of chordal graphs such as doubly chordal graphs, strongly chordal graphs and interval graphs. We prove that the total-colour conjecture holds for dually chordal graphs. We describe a new heuristic that yields an exact total-colouring algorithm for even maximum degree dually chordal graphs and an exact edge-colouring algorithm for odd maximum degree dually chordal graphs.

Key words. graph algorithms, chordal graphs, total-colour, edge-colour, clique graphs
AMS subject classification. 05C85, 05C15, 68R10, 90C27

1 Introduction

We consider the problem of total-colouring and edge-colouring dually chordal graphs. The total-colouring of a graph is a colouring of vertices and edges in such a way that no adjacent vertices, no adjacent edges, no incident vertices and edges obtain the same colour. Edge-colouring is a partial case of total-colouring when only the edges are coloured. The minimum number of colours needed is called the total-chromatic number and chromatic index repectively. Vizing's total-colour conjecture says that every graph admits a total-colouring with maximum degree plus two colours and Vizing's celebrated theorem says that the chromatic index is at most the maximum degree plus one [15]. A constructive proof of Vizing's theorem appeared in [12].

Graphs whose chromatic index equals the maximum degree are said to be *Class* 1; graphs whose chromatic index exceeds the maximum degree by one are said to be *Class* 2. It is

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well known that the set of graphs in Class 1 is NP-complete [11]. In [7, 8] we considered classes of graphs that are defined by special vertex *perfect elimination orders.* Our method identified subclasses of chordal graphs for which a simple constructive proof of Vizing's theorem is presented: doubly chordal graphs, strongly chordal graphs, interval graphs and indifference graphs.

The total-colour conjecture has been established for several classes of graphs and a recent survey book has been devoted to the subject [16]. The total-colour conjecture was proved recently for a subclass of chordal graphs: the class of split graphs [5]. Graphs whose total-chromatic number equals the maximum degree plus one are said to be *Type* 1; graphs whose total chromatic number equals the maximum degree plus two are said to be *Type* 2.

In this paper, we consider colourings of dually chordal graphs. We extend in two ways the results of [7] where we considered edge-colouring doubly chordal graphs: we now consider both total-colourings and edge-colourings and we now consider a larger class of graphs. Dually chordal graphs have attracted much attention lately as the class of clique graphs of chordal graphs and as a natural generalization of strongly chordal graphs [4, 2, 3, 14]. We prove that the total-colour conjecture holds for dually chordal graphs. We describe a new heuristic that yields an exact total-colouring algorithm for even maximum degree dually chordal graphs and an exact edge-colouring algorithm for odd maximum degree dually chordal graphs. In Section 2 we define the classes of graphs considered in this paper. In Section 3 we define pull back and we use this graph homomorphism to obtain a sufficient local condition for a class of graphs to satisfy Vizing's total-colour conjecture. In Section 4 we apply our pull back method to the class of dually chordal graphs. Our conclusions in Section 5 consider two possible natural extensions of our results to chordal graphs and to neighbourhood-Helly graphs.

2 Definitions and Notations

General terms

In this paper, G denotes a simple, undirected, finite, connected graph. $V(G)$ and $E(G)$ are the vertex and edge sets of G. A *clique* is a set of vertices pairwise adjacent in G. A *maximal clique* of G is a clique not properly contained in any other clique. A *subgraph* of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, we denote by $G[X]$ the *subgraph induced by* X, that is, $V(G[X]) = X$ and $E(G[X])$ consists of those edges of $E(G)$ having both ends in X.

For each vertex v of a graph G, $Adj(v)$ denotes the set of vertices which are adjacent to v. The *degree* of a vertex v is $\deg(v) = |Adj(v)|$. The maximum degree of a graph G is then $\Delta(G) = \max_{v \in V(G)} \deg(v)$. A vertex u is *universal* if $\deg(u) = |V(G)| - 1$. $N(v)$ denotes the *neighbourhood* of v, that is, $N(v) = Adj(v) \cup \{v\}$. A subgraph which is induced by the neighbourhood of a vertex is simply called a *neighbourhood.* We denote by $N^2(v)$

the family of sets $\{N(u) : u \in N(v)\}$. The *distance* between two vertices u and v is the minimum length (i.e., number of edges) of a path between u and v. Given a graph G we denote by G^2 the graph having $V(G^2) = V(G)$ and satisfying $xy \in E(G^2)$ if and only if x and y are distinct and their distance in G is at most 2. A vertex v is *simple* if $N^2(v)$ is linearly ordered by set inclusion. A vertex $u \in N(v)$ is a *maximum neighbour of* v if and only if for all $w \in N(v)$, $N(w) \subseteq N(u)$ holds. A *maximum neighbourhood elimination order* of a graph G is a linear order on its vertex set $v_1, v_2, \ldots, v_n$ such that for each i, $1 \leq i \leq n$, there is a maximum neighbour u_i of v_i in $G[v_1, v_2, \ldots, v_i]$.

A vertex v is *simplicial* if $N(v)$ is complete. A *perfect elimination order* of a graph G is a linear order on its vertex set $v_1, v_2, \ldots, v_n$ such that for each i, $1 \leq i \leq n$, v_i is simplicial in $G[v_1, v_2, \ldots, v_i]$. A graph is *chordal* if it admits a perfect elimination order.

A *simple perfect elimination order* of a graph G is a linear order on its vertex set $v_1, v_2, \ldots, v_n$ such that for each i, $1 \leq i \leq n$, v_i is simple in $G[v_1, v_2, \ldots, v_i]$. A vertex is *doubly simplicial* if it is simplicial and has a maximum neighbour. A *doubly perfect elimination order* of a graph G is a linear order on its vertex set $v_1, v_2, \ldots, v_n$ such that for each i, $1 \leq i \leq n$, v_i is doubly simplicial in $G[v_1, v_2, \ldots, v_i]$. A graph is *strongly chordal* if it admits a simple perfect elimination order. A graph is *doubly chordal* if it admits a doubly perfect elimination order [13]. We note that, by definition, every strongly chordal graph is doubly chordal.

A graph is *dually chordal* if it admits a maximum neighbourhood elimination order. The reason for this name is the following characterization of graphs with maximum neighbourhood order: a graph G has a maximum neighbourhood order if and only if its clique hypergraph $\mathcal{C}(\mathcal{G})$ forms a hypertree [4, 14, 10]. This means duality to chordal graphs. Recognition of dually chordal graphs can be done in $O(n^2m)$ time [14] and in $O(m)$ time [2]. Maximum neighbourhood orders are algorithmically useful, especially for domination-like problems and distance problems on dually chordal graphs [3]. For a dually chordal graph a maximum neighbourhood order can be computed in linear time [2]. Note that unlike the chordal graphs, dually chordal graphs are not perfect: every graph containing a universal vertex is dually chordal. A graph is doubly chordal if and only if it is chordal and dually chordal. A graph is strongly chordal if and only if all induced subgraphs are dually chordal [1].

For us, K_n denotes the complete graph on $n \geq 1$ vertices.

Colouring

An *assignment of colours* to the vertices of G is a function $\lambda\colon V(G) \to S$. The elements of the set S are called *colours*. A *conflict* in an assignment of colours is the existence of two adjacent vertices with the same colour. A *vertex-colouring* of a graph is an assignment of colours such that there are no conflicts. The *chromatic number* of a graph G is the minimum number of colours used among all vertex-colourings of G and is denoted by $\chi(G)$.

An *assignment of colours* to the edges of G is a function $\kappa\colon E(G) \to S$. Again, the elements of the set S are called *colours*. A *conflict* in an assignment of colours is the existence of two edges with the same colour incident to a common vertex. A vertex u is said to be *satisfied* when $\kappa(uv) = \kappa(uw)$ implies $v = w$, for all neighbours v, w of u. An *edge-colouring* of a graph is an assignment of colours such that every vertex is satisfied or, equivalently, such that there are no conflicts. The *chromatic index* of a graph G is the minimum number of colours used among all edge-colourings of G and is denoted by $\chi'(G)$.

An *assignment of colours* to the vertices and edges of G is a function $\tau: V(G) \cup E(G) \to S$. The elements of the set S are called *colours*. A *conflict* in an assignment of colours is the existence of two adjacent vertices or edges with the same colour. A *total-colouring* of a graph is an assignment of colours to vertices and edges of the graph such that there are no conflicts. The *total-chromatic number* of a graph G is the minimum number of colours used among all total-colourings of G and is denoted by $\chi''(G)$.

The *greedy method* for vertex-colouring considers the vertex set of a graph according to a linear order and assign to the current vertex the smallest available colour that does not create conflicts. A *perfect order* is a linear order on the vertex set of a graph such that the greedy method vertex-colours optimally all its induced subgraphs [6]. Every chordal graph admits a perfect order as every perfect elimination order is a perfect order.

In this paper, we use a maximum neighbourhood order of a dually chordal graph G to show that G^2 admits a perfect order. A $\Delta(G) + 1$ vertex-colouring of G^2 is used in turn to give a simple constructive proof of Vizing's total-colour conjecture and of Vizing's theorem for the class of dually chordal graphs.

3 Pull back of complete graphs

Given two graphs G and G', a *pull back* is a function $f: V(G) \to V(G')$, such that

- f is a homomorphism, i.e., if $xy \in E(G)$, then $f(x)f(y) \in E(G')$;
- f is injective when restricted to $N(x)$, for all $x \in V(G)$.

Lemma 1 *If $f: V(G) \to V(G')$ is a pull back and τ' is a total-colouring of G', then the colour assignment τ defined by*

$$\tau(x) = \tau'(f(x))$$

$$\tau(xy) = \tau'(f(x)f(y))$$

is a total-colouring of G.

Proof: Each vertex and each edge of G has a colour defined by τ. This is because $f: V(G) \to V(G')$ is a homomorphism.

Moreover, the colour assignment τ has no conflits. In fact, suppose we have two distinct edges xy and xz. Assume for a moment that $\tau(xy) = \tau(xz)$. Hence, $\tau'(f(x)f(y)) =$

$\tau'(f(x)f(z))$. Since f is injective on $N(x)$ and $y \neq z \in N(x)$, we have $f(y) \neq f(z) \in N(f(x))$. Thus, τ' is not a total-colouring of G', a contradiction. Now suppose we have two distinct adjacent vertices x and y. Because $f\colon V(G) \to V(G')$ is a homomorphism, we have that $f(x)$ and $f(y)$ are two distinct adjacent vertices of G'. Assume for a moment that $\tau(x) = \tau(y)$. Hence, $\tau'(f(x)) = \tau'(f(y))$. Thus, τ' is not a total-colouring of G', a contradiction. Now suppose we have a vertex x and an edge xy. Because $f\colon V(G) \to V(G')$ is a homomorphism, we have that $f(x)f(y)$ is an edge of G'. Assume for a moment that $\tau(x) = \tau(xy)$. Hence, $\tau'(f(x)) = \tau'(f(x)f(y))$. Thus, τ' is not a total-colouring of G', a contradiction. ■

Corollary 1 *If $f\colon V(G) \to V(G')$ is a pull back and κ' is an edge-colouring of G', then the colour assignment κ defined by*

$$\kappa(xy) = \kappa'(f(x)f(y))$$

is an edge-colouring of G.

Proof: Analogous to proof of Lemma 1. ■

A total-colouring τ for G is said to be a *pull back* from a total-colouring τ' for G', if τ is defined from τ' as in Lemma 1. Analogously we define an edge-colouring κ for G to be a *pull back* from an edge-colouring κ' for G'.

For the convenience of the reader, we recall next a result proved in [7]: if G^2 admits a vertex-colouring with l colours, then there is a pull back $f\colon V(G) \to V(K_l)$. We note that, by definition, any vertex-colouring of G^2 needs $l \geq \Delta(G) + 1$ colours. Hence, this result actually says that, if we have a vertex-colouring of G^2 with $\Delta(G) + 1$ colours, then there is a natural way of proving that we can total-colour G with $\Delta(G) + 2$ colours and that we can edge-colour G with $\Delta(G) + 1$ colours.

Theorem 1 [**7**] *There is a pull back $f\colon V(G) \to V(K_l)$ if and only if $\chi(G^2) \leq l$.*

Proof: For the "if" part, let $\lambda\colon V(G^2) \to S$ be a vertex-colouring of G^2 with $|S| = l$.

Consider a bijection $g\colon S \to V(K_l)$. We shall show that the composition $f = g \circ \lambda$ is a pull back $f\colon V(G) \to V(K_l)$ (remember that $V(G) = V(G^2)$).

It is clear that the first condition of a pull back function is satisfied by f. Now, notice that λ is also a vertex-colouring of G with l colours, and that all vertices which have distance in G at most two, have distinct colours. Hence, f is injective when restricted to $N(v)$, for all $v \in V(G)$.

For the "only if" part, let $f\colon V(G) \to V(K_l)$ be a pull back. This can be viewed as a colour assignment with $S = V(K_l)$. We are using $|S| \leq l$ colours.

If $uv \in E(G)$, then $f(u)f(v) \in E(K_l)$, that is, $f(u) \neq f(v)$. If u and v are at distance 2 in G then there is $w \in V(G)$ with $u, v \in N(w)$. Since f is injective in $N(w)$, $f(u)$ is again distinct from $f(v)$. Hence, there are no conflicts and f is a vertex-colouring of G^2. It follows that $\chi(G^2) \leq l$. ■

The total-chromatic numbers of complete graphs are well-known [9]: K_l is Type 2, if $l > 0$ is even and Type 1, if l is odd. In particular, we get sufficient conditions for a graph G to be Type 1: G has even maximum degree and G^2 admits a vertex-colouring with $\Delta(G)+1$ colours.

Corollary 2 *Suppose* $\chi(G^2) \leq l$. *Then*

- $\chi''(G) \leq l$, *if* l *is odd;*
- $\chi''(G) \leq l+1$, *if* l *is even and* $l > 0$.

Proof: It follows immediately from Lemma 1, Theorem 1, and the above remarks. ■

Another useful feature of Corollary 2 is that it gives a sufficient condition for a class of graphs to satisfy Vizing's total-colour conjecture: the square of each of its graphs have chromatic number $\Delta + 1$.

4 Dually chordal graphs

In this section we shall prove that Vizing's total-colour conjecture holds for the class of dually chordal graphs. We denote by $G_i = G[v_1, \ldots, v_i]$ the subgraph induced by $\{v_1, \ldots, v_i\}$ and by $N_i(v)$ the neighbourhood of v in G_i.

Lemma 2 *If* G *is dually chordal, then* $\chi(G^2) = \Delta(G) + 1$.

Proof: Let $v_1, \ldots, v_n$ be a maximum neighbourhood elimination order of G.

For each i, $1 \leq i \leq n$, let u_i be a maximum neighbour of v_i in G_i. By definition, for all $w \in N_i(v_i)$, $N_i(w) \subseteq N_i(u_i)$. In other words, $N_i(w) \in N_i^2(v_i)$ implies that $N_i(w) \subseteq N_i(u_i)$. Hence, there are at most $\Delta(G) + 1$ vertices in the family of sets $N_i^2(v_i)$, for each i.

Thus, given a maximum neighbourhood order of G, the greedy method uses at most $\Delta(G) + 1$ colours to colour the vertices of G^2. ■

It follows from Theorem 1 and Lemma 2 that we can pull back total-colourings and edge-colourings from complete graphs to colour dually chordal graphs.

Corollary 3 *All dually chordal graphs can be total-coloured with* $\Delta(G) + 2$ *colours. All dually chordal graphs with even maximum degree are Type* 1. *All dually chordal graphs with odd maximum degree are Class* 1.

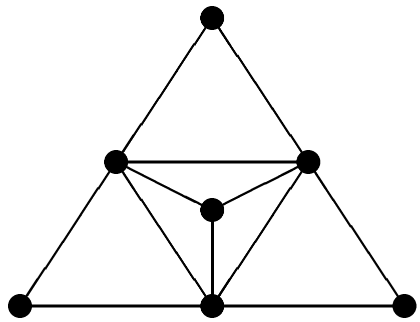

Figure 1: A chordal graph with $\chi(G^2) > \Delta(G) + 1$.

We remark that the results of Corollary 3 hold for subclasses of dually chordal graphs such as doubly chordal graphs, strongly chordal graphs and interval graphs.

5 Conclusions

We consider the extension of our results to two classes graphs: neighbourhood-Helly graphs and chordal graphs.

A graph is *neighbourhood-Helly* when the set $\{N(v) : v \in V\}$ satisfies the Helly property. A characterization of dually chordal graphs says that G is dually chordal if and only if for all $k \geq 0, G^{2k}$ is chordal and $\{N^k[v] : v \in V\}$ has the Helly property [1].

Consider the class of graphs G that satisfy the following two properties:

1. the cardinality of a maximum clique of G^2 equals the chromatic number of G^2;

2. for every maximum clique C of G^2 there exists a vertex v of G such that $N_G(v) = C$.

These two properties are enough to imply that $\chi(G^2) = \Delta + 1$. Note that property 1 is satified by graphs G having G^2 perfect. Property 2 is satisfied by neighbourhood-Helly graphs G.

Consider now the class of chordal graphs. For consider the chordal graph G depicted in Figure 1. This graph has diameter equals to 2, i.e., every pair of vertices is at distance at most 2, or more precisely $G^2 = K_7$. In this case, we need 7 colours for any vertex-colouring of G^2. On the other hand, G has no universal vertex, $\Delta(G) = 5$. Therefore, Lemma 2 does not generalize to chordal graphs.

On the other hand, we were unable to find any evidence that Corollary 3 does not hold for chordal graphs (note that graph in Figure 1 has chromatic index equals 5). We conjecture that all chordal graphs with odd maximum degree are in fact Class 1. A more general question would be to determine the largest graph class for which all its odd maximum degree graphs are Class 1 and for which all its even maximum degree graphs are Type 1.

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