On Edge-Colouring Indifference Graphs

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Abstract. Vizing's theorem states that the chromatic index $\chi'(G)$ of a graph G is either the maximum degree $\Delta(G)$ or $\Delta(G)+1$. A graph G is called *overfull* if $|E(G)| > \Delta(G)\lfloor |V(G)|/2 \rfloor$. A sufficient condition for $\chi'(G) = \Delta(G)+1$ is that G contains an overfull subgraph H with $\Delta(H) = \Delta(G)$. Plantholt proved that this condition is necessary for graphs with a universal vertex. In this paper, we conjecture that, for indifference graphs, this is also true. As supporting evidence, we prove this conjecture for general graphs with three maximal cliques and with no universal vertex, and for indifference graphs with odd maximum degree. For the latter subclass, we prove that $\chi' = \Delta$.

1 Introduction

An *edge-colouring* of a graph is an assignment of colours to its edges such that no adjacent edges have the same colour. The *chromatic index* of a graph is the minimum number of colours required to produce an edge-colouring for that graph. In this paper, we address the computation of the chromatic index for indifference graphs.

Indifference graphs are graphs whose vertices can be linearly ordered so that the vertices contained in the same maximal clique are consecutive in this order [12]. We shall call such an order an *indifference order*. Indifference graphs form an important subclass of interval graphs: they are also called unitary interval graphs.

An easy lower bound for the chromatic index is the maximum vertex degree. A celebrated theorem by Vizing states that these two quantities differ by at most one [13]. Graphs whose chromatic index equals the maximum degree are said to be *Class* 1; graphs whose chromatic index exceeds the maximum degree by one are said to be *Class* 2.

Despite the powerful restriction imposed by Vizing's result, it is very hard to compute the chromatic index in general. The set of graphs in Class 1 is NP-complete, whereas Class 2 is co-NP-complete [7]. This holds for various subclasses, such as perfect graphs and 3-regular graphs [1, 7]. Very little is known about the complexity of computing the chromatic index and the goal of this paper is to investigate this question. We focus our attention on the class of *indifference* graphs. In particular, we are interested in the role played by *overfullness* in determining the chromatic index.

By definition of edge-colouring, each colour determines a matching and can cover at most $\lfloor n/2 \rfloor$ edges, where n is the number of nodes. Therefore, if the total number of edges is greater than the product of the maximum degree by $\lfloor n/2 \rfloor$, then the graph is necessarily Class 2. Graphs to which this argument can be applied are called *overfull*. More generally, if a graph has an overfull subgraph with the same maximum degree, then the same counting argument shows that the supergraph is Class 2. We call such graphs *subgraph-overfull*. We consider classes of graphs for which being Class 2 is equivalent to being subgraph-overfull. For such graphs the chromatic index problem is in P: a decomposition algorithm due to Padberg and Rao [9] checks in polynomial time whether a graph is subgraph-overfull. We note that historical results of König on bipartite graphs and of Tait on 3-regular planar graphs show that these two classes of graphs are trivial examples: any set of Class 1 graphs satisfies vacuously the above equivalence [4].

Hilton conjectured that a lower bound on the maximum degree would ensure the equivalence between being Class 2 and being subgraph-overfull [5]. Every Class 2 graph with either a universal vertex or a quasi-universal vertex is subgraph-overfull [10, 11]. As noted by Ortiz [8], this result can be used to compute efficiently the chromatic index for the following two subclasses of indifference graphs: clique-complete indifference graphs and split-indifference graphs. Another evidence for Hilton's conjectured lower bound was given recently: every Class 2 complete multipartite graph is subgraph overfull [6].

We consider a new version of overfullness that is not as powerful as subgraph-overfullness but is trivially checkable: a graph is said to be *neighbourhood-overfull* when it has a maximum degree vertex whose neighbourhood induces an overfull subgraph. Every neighbourhood-overfull graph is subgraph-overfull, but the converse is not valid (see Figure 1). We consider classes of graphs for which being Class 2 is equivalent to being neighbourhood-overfull.

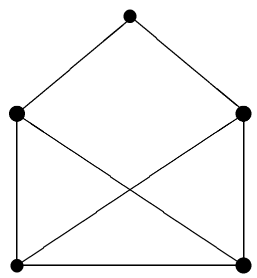

Fig. 1. A subgraph-overfull graph which is not neighbourhood-overfull.

We prove that every indifference graph with odd maximum degree is Class 1. Since graphs with an even number of vertices cannot be overfull, graphs with odd maximum degree cannot be neighbourhood-overfull. Hence, being Class 2 and being neighbourhood-overfull are vacuously equivalent for indifference graphs with odd maximum degree.

We also prove that Class 2 and neighbourhood-overfullness are equivalent for indifference graphs with three maximal cliques and no universal vertex. Since every non indifference graph with three maximal cliques must contain a universal vertex, our result actually says that Class 2 and neighbourhood-overfullness are equivalent for general graphs with at most three maximal cliques.

In this paper, G denotes a simple undirected, finite, connected graph. $V(G)$ and $E(G)$ are the vertex and edge sets of G. A *clique* is a set of vertices pairwise adjacent in G. A *maximal clique* of G is a clique not properly contained in any other clique. A *subgraph* of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, we denote by $G[X]$ the *subgraph induced by* X, that is, $V(G[X]) = X$ and $E(G[X])$ consists of those edges of $E(G)$ having both ends in X.

For each vertex v of a graph G, $Adj(v)$ denotes the set of vertices which are adjacent to v. In addition, $N(v)$ denotes the *neighbourhood* of v, that is, $N(v) = Adj(v) \cup \{v\}$. A subgraph which is induced by the neighbourhood of a vertex is simply called a *neighbourhood*. Two vertices v, w are *twins* when $N(v) = N(w)$. The *twin set* of a vertex v of G is the subset of $V(G)$ formed by v and all its twins. The *clique partition* of G is the partition of the vertex set $V(G)$ into twin sets. Thus, each set of the clique partition is a subset of $V(G)$ formed by the vertices belonging to the same maximal cliques.

Figure 2 illustrates a graph with two maximal cliques $C_1 = \{a, b, e\}$ and $C_2 = \{c, d, e\}$. The sets of the clique partition are: $\{a, b\}$, $\{e\}$ and $\{c, d\}$.

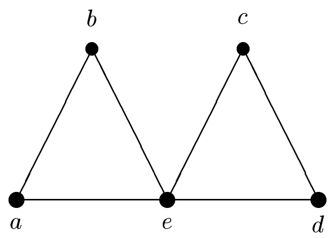

Fig. 2. A graph with two maximal cliques.

The *degree* of a vertex v is $\deg(v) = |\mathrm{Adj}(v)|$. The maximum degree is then $\Delta(G) = \max_{v \in V(G)} \deg(v)$. A vertex u is *universal* if $\deg(u) = |V(G)| - 1$.

The remainder of the paper is organized as follows. In Sections 2 and 3 we prove the results cited above. Finally, our concluding remarks and plans for future work appear in Section 4.

2 Indifference graphs with odd maximum degree

In this section we prove the following theorem:

Theorem 1. *All indifference graphs with odd maximum degree are Class* 1.

Proof. Let G be an indifference graph with $\Delta = \Delta(G)$ odd. We shall explicitly construct an edge-colouring of G using Δ colours.

numbering	0	1	2	3	4	5	6	7	8	9	10	11	12
labels	0	1	2	3	4	$\mathcal{D}$	0	1	2	3	4	$\mathcal{D}$	0

Fig. 3. Numbers and labels for an indifference graph with $\Delta = 5$.

We arrange the vertices of G into an indifference order, and number them with consecutive integers starting from 0 (see Figure 3). We label each vertex with an element of the set $\{0, 1, 2, \dots, \Delta - 1, \mathcal{D}\}$, where $\mathcal{D}$ is a special symbol, not a number. The labeling is as follows:

$$\lambda(x) = \begin{cases} \mathcal{D}, & \text{if } x \equiv \Delta \pmod{\Delta + 1}; \\ x \bmod (\Delta + 1), & \text{otherwise.} \end{cases}$$

The colour $\kappa(xy)$ of edge xy will be a symmetric function of the labels of x and y. The colours used will be elements of the set $\{0, 1, 2, \dots, \Delta - 1\}$, assigned as follows:

$$\kappa(xy) = \begin{cases} (\lambda(x) + \lambda(y)) \bmod \Delta, & \text{if both } \lambda(x) \text{ and } \lambda(y) \text{ are numeric;} \\ (2\,\lambda(x)) \bmod \Delta, & \text{if } \lambda(x) \text{ is numeric and } \lambda(y) = \mathcal{D}; \\ (2\,\lambda(y)) \bmod \Delta, & \text{if } \lambda(x) = \mathcal{D} \text{ and } \lambda(y) \text{ is numeric.} \end{cases}$$

No rule is needed for colouring edges connecting two vertices labeled $\mathcal{D}$, because such edges do not exist. In fact, if a vertex x with label $\mathcal{D}$ were adjacent to the next one labeled $\mathcal{D}$, it would have to meet all the vertices in between due to the properties of an indifference order. But there are Δ vertices in between. Therefore, we would have $\deg(x) \geq \Delta + 1$, contradicting the fact that Δ is the maximum degree. The same argument can be used to show the following facts:

- there are no edges between vertices with the same label;
- no vertex has two neighbours with the same label.

In both cases, a neighbourhood would contain all vertices between those two vertices with the same label, totaling more than $\Delta + 1$ vertices.

We will now show that κ is in fact an edge-colouring.

Suppose we have two distinct edges xy and xz. By the previous argument, we know that the labels of x, y, and z are also distinct. Assume for a moment that $\kappa(xy) = \kappa(xz)$. There are two cases to consider.

Case 1: $\lambda(x) = \mathcal{D}$.

In this case, the labels of y and z are necessarily numeric and

$$2\,\lambda(y) \equiv 2\,\lambda(z) \pmod{\Delta}.$$

Since Δ is odd, we may cancel the factor 2 in both sides to get

$$\lambda(y) \equiv \lambda(z) \pmod{\Delta}.$$

This implies $\lambda(y) = \lambda(z)$, a contradiction. Thus, this case is not possible.

Case 2: $\lambda(x)$ is numeric.

In this case, $\lambda(y)$ and $\lambda(z)$ cannot both be $\mathcal{D}$. If one of them is $\mathcal{D}$, say, $\lambda(y) = \mathcal{D}$, then

$$2\,\lambda(x) \equiv \lambda(x) + \lambda(z) \pmod{\Delta},$$

which implies $\lambda(x) = \lambda(z)$, a contradiction. On the other hand, if all three labels are numeric, then we have

$$\lambda(x) + \lambda(y) \equiv \lambda(x) + \lambda(z) \pmod{\Delta},$$

which implies $\lambda(y) = \lambda(z)$, and this is also impossible.

We conclude that the colours of edges xy and xz cannot be equal and that our assignment is indeed an edge-colouring. $\square$

3 Graphs with three maximal cliques

In this section, we study graphs with three maximal cliques. The problem has already been solved if there is a universal vertex [10] or if the maximum degree is odd (see Section 2). Therefore, we concentrate our attention on graphs with even maximum degree and no universal vertex. As we mentioned in the introduction, these graphs are necessarily indifference graphs.

Let G be such a graph. Consider an indifference order of the vertices of G. This order induces a linear order on the maximal cliques of G in a natural way. We denote by C_1, C_2, and C_3 the three maximal cliques of G in the order induced by the particular indifference order considered above (see Figure 4). Notice that $C_1 \cap C_3 = \emptyset$; otherwise, we would have a universal vertex.

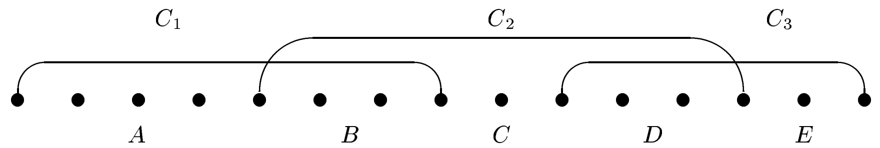

Fig. 4. The maximal cliques and clique partition of G.

There are at most five sets in the clique partition of G. We name them as follows:

$$\begin{aligned}
A &= C_1 \setminus C_2,\\
B &= C_1 \cap C_2,\\
C &= C_2 \setminus (C_1 \cup C_3),\\
D &= C_3 \cap C_2,\\
E &= C_3 \setminus C_2,
\end{aligned}$$

and denote by a, b, c, d, and e, respectively, the sizes of each set. Of these sets, only C may be empty. Hence, the graph G has at least four vertices.

Conversely, any quintuple of integers $[a, b, c, d, e]$ with $a, b, d, e \geq 1$ and $c \geq 0$ characterizes an indifference graph with three maximal cliques and no universal vertex. We therefore denote our graphs by these quintuples from here on. Without loss of generality, we may assume $a \geq e$ (otherwise, take the reverse order).

Our goal in this section is to establish the following result.

Theorem 2. *A graph with three maximal cliques and no universal vertex is Class 2 if and only if it is neighbourhood-overfull.*

Since neighbourhood-overfullness implies Class 2, the "if" part is immediate. Therefore, we concentrate on the "only if" part, which can be restated as follows.

Lemma 3. *If a graph G with three maximal cliques and no universal vertex is not neighbourhood-overfull, then there is an edge-colouring of G with $\Delta(G)$ colours.*

We begin by computing the degrees of vertices in each set of the clique partition:

$$\begin{aligned}
x \in A &\Rightarrow deg(x) = a + b - 1\\
x \in B &\Rightarrow deg(x) = a + b + c + d - 1\\
x \in C &\Rightarrow deg(x) = b + c + d - 1\\
x \in D &\Rightarrow deg(x) = b + c + d + e - 1\\
x \in E &\Rightarrow deg(x) = d + e - 1.
\end{aligned}$$

Therefore, a maximum degree vertex cannot belong to either A, C, or E, since the vertices in B and D will always have more neighbours. Furthermore, since $a \geq e$, vertices in B have maximum degree. If $a > e$, then these are the only ones. If $a = e$, then the nodes in D have maximum degree as well. In any case, the maximum degree is $\Delta = a + b + c + d - 1$.

We consider separately the cases $a = e$ and $a > e$.

3.1 The case $a = e$

Without loss of generality, we shall assume $b \geq d$.

Ortiz [8] shows that Lemma 3 is valid when $a = e = 1$. Hence, we treat the case $a \geq 2$ in the sequel.

Our strategy here will be as follows. Among all graphs of the form $[a, b, c, d, a]$ which are not neighbourhood-overfull, some are subgraphs of others with the same Δ. When this happens, it suffices to colour the supergraphs, since the removal of some edges and vertices will never create conflicts. We use this argument to show that it suffices to colour the graphs with $c = 0$.

Lemma 4. *Suppose $a \geq 2$. Every graph of the form $[a, b, c, d, a]$ which is not neighbourhood-overfull is a subgraph of a graph of the form $[a', b', 0, d', a']$ which is not neighbourhood-overfull and which has the same maximum degree.*

Proof. When $c \leq b - d$, the graph $G = [a, b, c, d, a]$ is a subgraph of $G' = [a, b, 0, c + d, a]$, both have the same Δ, and if G is not neighbourhood-overfull, then G' is not neighbourhood-overfull.

When $c > b - d$, we can first transfer vertices from C to D until D and B have the same size, and then split the remaining elements in C equally between B and D. More precisely, $G = [a, b, c, d, a]$ is a subgraph of $G' = [a, b, c - b + d, b, a]$ with the same Δ and if G is not neighbourhood-overfull, then G' is not neighbourhood-overfull. In turn, $G' = [a, b, c', b, a]$, where $c' = c - b + d$, is a subgraph of $G'' = [a, b + \lceil c'/2 \rceil, 0, b + \lfloor c'/2 \rfloor, a]$, $\Delta(G'') = \Delta(G)$ and G'' is not neighbourhood-overfull if $a \geq 2$. □

The colouring of graphs of the form $[a, b, 0, d, a]$ are done with the several results below. The cases considered are:

- $d = 1$;
- $d \geq 2$ and $b < a + d$;
- $d \geq 2$ and $b \geq a + d$.

Lemma 5. *A graph of the form $[a, b, 0, 1, a]$ with $a \geq 2$ which is not neighbourhood-overfull is Class 1.*

Proof. The subgraph induced by $A \cup B \cup D$ is a neighbourhood since it is induced by the neighbourhood of any vertex in B. By hypothesis, the neighbourhood of a maximum degree vertex is not overfull. Thus we can colour the neigbourhood $A \cup B \cup D$ with Δ colours. There are exactly a colours missing in the single vertex $u \in D$. Use them to colour the edges between u and the vertices in E.

Without loss of generality, assume these colours are $0, 1, 2, \ldots, a-1$ and label each node v in E with the colour of the edge uv: $\lambda(v) = \kappa(uv)$. Now, let q be an odd number such that $a \leq q \leq a + 1 \leq a + b = \Delta$ and assign

$$\kappa(vw) = \frac{(q+1)(\lambda(v) + \lambda(w))}{2} \bmod q,$$

for $v, w \in E$. Notice that the division by 2 is well defined since q is odd. At most Δ colours are used, because $q \leq \Delta$. Each node v uses colour $\lambda(v)$ to connect to $u \in D$ and colours in the range $0, 1, \ldots, q-1$ other than $\lambda(v)$ to connect to the other vertices in E. This is a valid edge-colouring and proves that G is Class 1. □

The following result deals with the case where $d \geq 2$ and $b < a + d$. Notice that $b < a + d$ is equivalent to $b \leq \Delta/2$. Recall that we are assuming that $b \geq d$ and that Δ is even.

Theorem 6. *A graph of the form $[a, b, 0, d, a]$ with $2 \leq d \leq b \leq \Delta/2$ is Class 1.*

Proof. We shall label the vertices of $G = [a, b, 0, d, a]$ using labels from the set $\{0, 1, \ldots, \Delta - 1, \mathcal{D}_A, \mathcal{D}_E\}$. A numeric label (or a colour) l will be called *low* if $l < \Delta/2$ and *high* otherwise. There are exactly $\Delta/2$ low labels and $\Delta/2$ high labels.

Let p be the smallest high, even label, and q be the largest low, odd label.

We will now describe the placement of labels in the graph. The following rules must be applied in the given order.

1. Label one of the vertices in A with $\mathcal{D}_A$. This will be the only vertex with this label.
2. Label two vertices in B with labels $\Delta - 1$ and p. Without repeating labels, label the rest of B with high labels. This is possible because $b \leq \Delta/2$.
3. Label two vertices in D with labels 0 and q. Without repeating labels, label the rest of D with low labels. This is possible because $d \leq b \leq \Delta/2$.
4. Label the rest of A with the numeric labels not used in either B or D, without repeating labels. Note that these labels used for vertices in A may be low or high.
5. Copy the labels in A to E, except that $\mathcal{D}_A$ will be replaced by $\mathcal{D}_E$ in E.

Colours are assigned to edges as follows:

$$\kappa(xy) = (\lambda(x) + \lambda(y)) \bmod \Delta, \text{ if } \lambda(x) \text{ and } \lambda(y) \text{ are numeric.}$$

$$\kappa(\mathcal{D}_A x) = \begin{cases} (2\,\lambda(x)) \bmod \Delta, & \text{if } \lambda(x) \text{ is high;} \\ (\lambda(x) + 0) \bmod \Delta, & \text{if } \lambda(x) \text{ is low and odd;} \\ (\lambda(x) + q) \bmod \Delta, & \text{if } \lambda(x) \text{ is low and even.} \end{cases}$$

$$\kappa(\mathcal{D}_E x) = \begin{cases} (2\,\lambda(x)) \bmod \Delta, & \text{if } \lambda(x) \text{ is low;} \\ (\lambda(x) - 1) \bmod \Delta, & \text{if } \lambda(x) \text{ is high and even;} \\ (\lambda(x) + p) \bmod \Delta, & \text{if } \lambda(x) \text{ is high and odd.} \end{cases}$$

The colours assigned to edges incident to $\mathcal{D}_A$ are all distinct. In fact, it is easy to see that there are no conflicts within each of the three categories ($\lambda(x)$ high, low and odd, low and even). To see that there are not conflicts across categories, notice that colour categories are distinct for each of the three label categories, as shown in Table 1. A similar argument holds for $\mathcal{D}_E$. The only tricky part in verifying this table is the third line. For $\mathcal{D}_A$, we used the fact that $\lambda(x)$ low and even means in fact $\lambda(x) \geq 2$, since $\mathcal{D}_A$ is not adjacent to the vertex labeled 0. Similarly, for $\mathcal{D}_E$, $\lambda(x)$ high and odd actually means $\lambda(x) \leq \Delta - 3$, since $\mathcal{D}_E$ is not adjacent to $\Delta - 1$.

$\kappa(\mathcal{D}_A x)$ is ...	when $\lambda(x)$ is ...	$\kappa(\mathcal{D}_E x)$ is ...	when $\lambda(x)$ is ...
even	high	even	low
low and odd	low and odd	high and odd	high and even
high and odd	low and even	low and odd	high and odd

Table 1. Colour categories for edges incident to $\mathcal{D}_A$ and $\mathcal{D}_E$.

Now let us see why vertices with numeric labels are satisfied.

If x is a vertex with a numeric label, the potential danger is the existence of two neighbours y and z of x which produce the same colour. This cannot happen if both y and z have numeric labels, because the only repetition of labels occurs between vertices in A and E, which have no common neighbours. For the same reason, at most one of y, z can have a special label ($\mathcal{D}_A$ or $\mathcal{D}_E$).

Notice that the special label $\mathcal{D}_A$ acts sometimes as a "doubler" and sometimes as if its label was 0 or q. Acting as a doubler is safe because adjacent vertices have distinct labels, so the only way to get $\kappa(xy) = 2\lambda(x)$ is through the doubler, that is, $y = \mathcal{D}_A$. Therefore, if x is adjacent to $\mathcal{D}_A$ and $\lambda(x)$ is high, then x is satisfied. If $\lambda(x)$ is low, then x must be in A and is therefore not adjacent to either 0 or q, so $\mathcal{D}_A$ is free to play the role of 0 or q.

A similar reasoning can be carried out for neighbours of $\mathcal{D}_E$. If they are low, the "doubler" character of $\mathcal{D}_E$ assures there are no colour conflicts. If they are high, they must be in E and $\mathcal{D}_E$ is free to act as either $\Delta - 1$ or p, which are not adjacent to any vertex in E. □

The following results are needed for the last possibility, namely, $d \geq 2$ and $b \geq a + d$. This is the most difficult part of this section and will be established in two steps. First, we transfer vertices from A to B, decreasing E simultaneously, until B is as large as possible without creating an overfull neighbourhood. This is accomplished with Lemma 4 and its corollary. Then, we show how to colour the graphs with large B using Lemma 5 and Theorem 4.

Lemma 7. *If $[a-1, b+1, 0, d, a-1]$ is Class 1 and $b \geq a + d$, then $[a, b, 0, d, a]$ is also Class 1.*

Proof. Let $G = [a, b, 0, d, a]$ and $G' = [a-1, b+1, 0, d, a-1]$. Notice that $\Delta(G) = \Delta(G') = a + b + d - 1$.

Define a mapping $f\colon V(G) \to V(G')$ as follows. Let A, B, D, and E be the nonempty sets in the clique partition of G as defined in the beginning of this section, and let A', B', D', and E' be the ones in G'. Select elements $u \in A$,

$v \in E$, and $w \in B'$, and let $A_1 = A \setminus \{u\}$, $E_1 = E \setminus \{v\}$, and $B'_1 = B' \setminus \{w\}$. The map f acts as follows (see Figure 5):

$$\begin{aligned} f(u) &= w, \\ f(v) &= w, \\ f(A_1) &= A' \text{ in a 1-1 fashion,} \\ f(B) &= B'_1 \text{ in a 1-1 fashion,} \\ f(D) &= D' \text{ in a 1-1 fashion,} \\ f(E_1) &= E' \text{ in a 1-1 fashion.} \end{aligned}$$

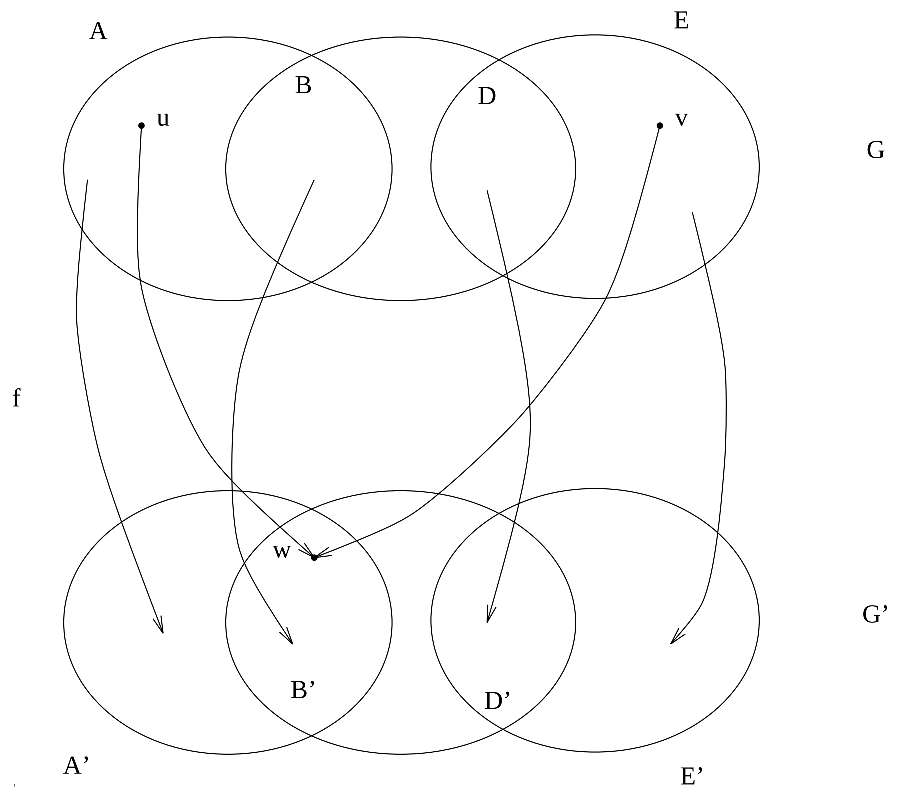

Fig. 5. Colouring “pull back” by the mapping f.

An edge-colouring of G' can be “pulled back” to G through this mapping if we set:

$$\kappa_G(xy) = \kappa_{G'}(f(x)f(y)).$$

This edge-colouring of G has no conflicts since f is injective *except* that $f(u) = f(v)$, but u and v have no common neighbour in G. It is easy to see that this procedure colours all edges of G with the exception of the edges vx for $x \in E_1$ (since there is no edge $wf(x)$ in G'). However, these edges can be easily assigned colours with no conflicts, as shown by the simple counting argument below.

Let $x_1, x_2, \ldots, x_{a-1}$ be the elements of E_1. To colour vx_1 we need to select a colour that is not being used in either v or x_1. Each x_i is using already $a+d-2$ colours, and v is using d colours. Since the total number of available colours is $\Delta = a+b+d-1$, we have at least $\Delta-(a+d-2)-d = b-d+1 \geq a+1$ colours to choose from for vx_1. Now we do the same for $x_2, \ldots, x_{a-1}$ keeping in mind that each time v will have one less free colour available. But, even for the last vertex x_{a-1} we will have left at least $b-(a+d)+3 \geq 3$ colours, which means that there is a way (in fact, more than $a!$ ways) of colouring the remaining edges. □

Corollary 8. *For any $l \geq 0$, if $[a-l, b+l, 0, d, a-l]$ is Class 1 and $b \geq a+d$, then $[a, b, 0, d, a]$ is Class 1.*

Proof. By induction on l. The base case $l = 0$ is obvious. For the induction step, we use Lemma 7. Notice that $b \geq a+d$ implies $(b+l) \geq (a-l)+d$ for all $l \geq 0$. □

The next result is used in Theorem 10 below. Notice that the neighbourhood $A \cup B \cup D$ in a graph of the form $[a, b, 0, d, a]$ is overfull if and only if $0 > ad - \Delta/2$. Moreover, taking $l = \lfloor (ad - \Delta/2)/d \rfloor$ in Corollary 8 above, we may restrict ourselves to the case where $ad - \Delta/2 < d$.

Lemma 9. *Suppose Δ is even and $0 \leq ad - \Delta/2 < d$. Then the neighbourhood $A \cup B \cup D$ in a graph of the form $[a, b, 0, d, a]$ admits an edge-colouring using Δ colours such that each colour is missing from at most two vertices in D.*

Proof. Choose a vertex $u \in A$. Add $k = ad - \Delta/2 \geq 0$ edges in the graph H induced by $A \cup B \cup D$ with one end in u and one end in D. This can be done because $k < d = |D|$. Call the resulting graph H'.

Colour the edges of H' using Δ colours. This is possible because H' is not overfull and has a universal vertex.

Notice that $|V(H')|$ is odd. Since $|E(H')| = \Delta \lfloor |V(H')|/2 \rfloor$, each colour has to be used exactly $\lfloor |V(H')|/2 \rfloor$ times, that is, it is missing from exactly one vertex.

Removing the k edges added in the beginning, we end up with an edge-colouring of the neighbourhood $A \cup B \cup D$ where each colour is missing at most twice in D. □

Theorem 10. *Let $G = [a, b, 0, d, a]$ with Δ even, $a \geq 2$, $b \geq d$, and $0 \leq ad - \Delta/2 < d$. Then G is Class 1.*

Proof. The neighbourhood $A \cup B \cup D$ is not overfull (since $\Delta/2 \leq ad$) and hence can be coloured using Δ colours. Moreover, by Lemma 9 we may assume that each colour is missing from at most two vertices in D.

Construct now a bipartite graph H that represents the vertices in D and the colours missing from each one as follows:

$$V(H) = D \cup \{0, 1, \ldots, \Delta - 1\},$$
$$E(H) = \{ux \mid \text{colour } x \text{ is missing from vertex } u \in D\}.$$

It is easy to see that $\deg_H(u) = a$ for $u \in D$ and that $\deg_H(x) \leq 2$ because each colour is missing from at most two vertices in D. Since $a \geq 2$ we have that $\Delta(H) = a$ and there is an edge-colouring of H using a colours $\alpha_1, \alpha_2, \ldots, \alpha_a$.

Use this colouring to construct a colouring of the bipartite subgraph $D \times E$ of G induced by the edges having one end in D and one end in E, as follows. Each colour α_i will correspond to a vertex $v_i \in E$. The colour of edge uv_i in G with $u \in D$ and $v_i \in E$ will be determined by the relationship:

$$\kappa_G(uv_i) = x \Leftrightarrow \kappa_H(ux) = \alpha_i.$$

Since there are a colours in the edge-colouring of H and each $u \in D$ has degree a in H, each colour α_i must be present in an edge incident to u. Hence, this colouring is well defined.

We now show that no conflicts exist. If $\kappa_G(uv_i) = \kappa_G(uv_j) = x$, then $\kappa_H(ux) = \alpha_i = \alpha_j$, so $i = j$ and $v_i = v_j$. If, on the other hand, $\kappa_G(uv_i) = \kappa_G(wv_i) = x$, then $\kappa_H(ux) = v_i = \kappa_H(wx)$ implies $u = w$.

Therefore, the bipartite $D \times E$ is coloured without conflicts with the previous colouring of $A \cup B \cup D$. It remains to colour the clique E. But notice that the elements of E are using a total of at most ad colours in their edges to D. There are still at least $\Delta - ad$ colours available. These are enough to colour E, since

$$\Delta - ad = ad - 2(ad - \Delta/2) \geq ad - 2(d-1) = a + (d-1)(a-2) \geq a.$$

So, we have enough extra colours to edge-colour E with no conflicts with the previous assignment. $\square$

3.2 The case $a > e$

Here we use a strategy similar to the one employed in the case $a = e$.

First, note that if $a > e$, then a graph of the form $[a, b, c, d, e]$ which is not neighbourhood-overfull is a subgraph of a graph of the form $[a, b, c, d, e + (a - 1 - e)]$, which in turn is not neighbourhood-overfull and which has the same maximum degree. Hence, we may assume that $a = e + 1$. Furthermore, since B always contains the vertices with degree Δ in a graph G of the form $[a, b, c, d, a-1]$, this graph is a subgraph of $G' = [a, b, 0, c+d, a-1]$ with $\Delta(G) = \Delta(G')$ and if G is not neighbourhood-overfull, then G' is not neighbourhood-overfull.

We may therefore restrict our attention to graphs of the form $[a, b, 0, d, a-1]$. To settle this case, we consider two subcases: $b \geq d$, and $b < d$.

The subcase $b \geq d$ can be reduced to the case $a = e$. Indeed, if a graph of the form $G = [a, b, 0, d, a-1]$ is not neighbourhood-overfull, then it satisfies

$ad \geq \Delta/2$. Hence, $G' = [a, b, 0, d, a]$ has vertices with degree Δ in D and the neighbourhood $B \cup D \cup A$ is not overfull either, because $ab \geq ad \geq \Delta/2$.

The next theorem deals with the subcase $b < d$. We shall use the same idea as in the proof of Theorem 6. Note that $1 \leq b < d$ implies $b < \Delta/2$ and $d \geq 2$.

Theorem 11. *A graph of the form $[a, b, 0, d, a-1]$ with $b < d$ is Class 1.*

Proof. We shall label the vertices of $G = [a, b, 0, d, a-1]$ using labels from the set $\{0, 1, \ldots, \Delta - 1, \mathcal{D}_A\}$. Using the same notation of the proof of Theorem 6, let q be the largest low, odd label.

The following rules describe the placement of labels in the graph:

1. Label one of the vertices in A with $\mathcal{D}_A$. This will be the only vertex with this label.
2. Label the vertices in B with high labels. This is possible because $b < \Delta/2$.
3. Label two vertices in D with labels 0 and q. Without repeating labels, start numbering the rest of the vertices in D with low labels and use high labels not used in B, when the low labels finish and D still has vertices without label.
4. Label the rest of A with the numeric labels not used in either B or D, without repeating labels. Note that these labels used for vertices in A may be low or high.
5. Copy the numeric labels in A to E.

The colours assigned to edges in the graph of the form $[a, b, 0, d, a-1]$ are equal to the colours for the corresponding vertices in Theorem 6.

An argument similar to the one used in that theorem shows that this is a valid edge-colouring. □

4 Conclusions

We believe our work makes a significant contribution to the problem of edge-colouring in three respects.

First, our results on the colouring of indifference graphs show that, in all cases we studied, neighbourhood-overfullness is equivalent to being Class 2. In particular, we conjecture that for any indifference graph neighbourhood-overfullness is equivalent to being Class 2. It would be interesting to extend these results to larger classes. We established recently that every odd maximum degree doubly chordal graph is Class 1 [3].

Second, we considered the relationship between subgraph-overfullness and neighbourhood-overfullness. For general graphs, the latter implies the former; for indifference graphs, we conjecture they are equivalent. Again, it would be interesting to learn for which other, larger classes this equivalence holds.

We state these two conjectures formally:

Conjecture. *The following statements are equivalent for a given indifference graph:*

- *The graph is neighbourhood-overfull;*
- *The graph is subgraph-overfull;*
- *The graph is Class 2.*

It should be noted that Hilton's lower bound $\Delta > n/3$ does not hold for all indifference graphs. In addition, this conjecture, if true, provides a way of deciding in polynomial time whether an indifference graph is Class 1 or Class 2. We have proposed a linear-time algorithm for indifference graph recognition in [2].

Third, and perhaps more important, the techniques we use to show these results are new and proved to be powerful tools. In particular, the use of modular arithmetic, colouring "pull-back" (as in Lemma 7), and duality in edge-colouring of bipartite graphs provide exciting new ways of colouring the edges of graphs.

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