A greedy method for edge-colouring odd maximum degree doubly chordal graphs

Celina M. H. de Figueiredo* João Meidanis† Célia Picinin de Mello†

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Abstract

We describe a greedy vertex colouring method which can be used to colour optimally the edge set of certain chordal graphs. This new heuristic yields an exact edge-colouring algorithm for odd maximum degree doubly chordal graphs. This class includes interval graphs and strongly chordal graphs. This method shows that any such graph G can be edge-coloured with maximum degree $\Delta(G)$ colours, i.e., all these graphs are Class 1. In addition, this method gives a simple $\Delta(G) + 1$ edge-colouring for any doubly chordal graph.

1 Introduction

An *edge-colouring* of a graph is an assignment of colours to its edges such that no adjacent edges have the same colour. The *chromatic index* of a graph is the minimum number of colours required to produce an edge-colouring for that graph.

An easy lower bound for the chromatic index is the maximum vertex degree. A celebrated theorem by Vizing states that these two quantities differ by at most one [4]. Graphs whose chromatic index equals the maximum degree are said to be *Class* 1; graphs whose chromatic index exceeds the maximum degree by one are said to be *Class* 2. Very little is known about the complexity of computing the chromatic index in general.

By definition of edge-colouring, each colour determines a matching and can cover at most $\lfloor n/2 \rfloor$ edges, where n is the number of nodes. Therefore, if the total number of edges is greater than the product of the maximum degree by $\lfloor n/2 \rfloor$, then the graph is necessarily

*Universidade Federal do Rio de Janeiro, Instituto de Matemática, Caixa Postal 68530, 21944 Rio de Janeiro, RJ, Brasil. `celina@cos.ufrj.br`. Partially supported by CNPq, grant 30 1160/91.0.

†Universidade Estadual de Campinas, Departamento de Ciência da Computação, Caixa Postal 6065, 13081-970 Campinas, SP, Brasil. {`meidanis,celia`}`@dcc.unicamp.br`. Partially supported by FAPESP.

Class 2. Graphs to which this argument can be applied are called *overfull*. More generally, if a graph has an overfull subgraph with the same maximum degree, then the same counting argument shows that the supergraph is Class 2. We call such graphs *subgraph-overfull*.

We considered in [1] classes of graphs for which being Class 2 is equivalent to being subgraph-overfull. For such graphs the chromatic index problem is in P: a decomposition algorithm due to Padberg and Rao [3] checks in polynomial time whether a graph is subgraph-overfull. We note that historical results of König on bipartite graphs and of Tait on 3-regular planar graphs show that these two classes of graphs are trivial examples: any set of Class 1 graphs satisfies vacuously the above equivalence [2].

We considered a new version of overfullness that is not as powerful as subgraph-overfullness but is trivially checkable: a graph is said to be *neighbourhood-overfull* when it has a maximum degree vertex whose neighbourhood induces an overfull subgraph. We proved that every indifference graph with odd maximum degree is Class 1. Since graphs with an even number of vertices cannot be overfull, graphs with odd maximum degree cannot be neighbourhood-overfull. Hence, being Class 2 and being neighbourhood-overfull are vacuously equivalent for indifference graph with odd maximum degree.

In this paper, we extend these results by considering classes of graphs that are defined by special vertex *perfect elimination orders.* As a result, we prove that every doubly chordal graph with odd maximum degree is Class 1.

2 Definitions and Notations

General terms

In this paper, G denotes a simple, undirected, finite, connected graph. $V(G)$ and $E(G)$ are the vertex and edge sets of G. A *clique* is a set of vertices pairwise adjacent in G. A *maximal clique* of G is a clique not properly contained in any other clique. A *subgraph* of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, we denote by $G[X]$ the *subgraph induced by* X, that is, $V(G[X]) = X$ and $E(G[X])$ consists of those edges of $E(G)$ having both ends in X.

For each vertex v of a graph G, $Adj(v)$ denotes the set of vertices which are adjacent to v. In addition, $N(v)$ denotes the *neighbourhood* of v, that is, $N(v) = Adj(v) \cup \{v\}$. A subgraph which is induced by the neighbourhood of a vertex is simply called a *neighbourhood.* We denote by $N^2(v)$ the family $\{N(u) : u \in N(v)\}$. Given a graph G and $k \geq 1$, we denote by G^k the graph having $V(G^k) = V(G)$ and satisfying $xy \in E(G^k)$ if and only if x and y are distinct and their distance in G is at most k. A vertex v is *simplicial* if $N(v)$ is complete. A vertex v is *simple* if $N^2(v)$ is totally ordered by set inclusion. A vertex $u \in N(v)$ is a *maximum neighbour of* v if and only if for all $w \in N(v)$, $N(w) \subseteq N(u)$ holds. A vertex is *doubly simplicial* if it is simplicial and has a maximum neighbour.

A *perfect elimination order* of a graph G is a total order on its vertex set $v_1, v_2, \ldots, v_n$ such that for each i, $1 \leq i \leq n$, v_i is simplicial in $G[v_1, v_2, \ldots, v_i]$. A graph is *chordal* if it admits a perfect elimination order.

A *simple perfect elimination order* of a graph G is a total order on its vertex set $v_1, v_2, \ldots, v_n$ such that for each i, $1 \leq i \leq n$, v_i is simple in $G[v_1, v_2, \ldots, v_i]$. A *doubly perfect elimination order* of a graph G is a total order on its vertex set $v_1, v_2, \ldots, v_n$ such that for each i, $1 \leq i \leq n$, v_i is doubly simplicial in $G[v_1, v_2, \ldots, v_i]$. A graph is *strongly chordal* if it admits a simple perfect elimination order. A graph is *doubly chordal* if it admits a doubly perfect elimination order. We note that, by definition, every strongly chordal graph is doubly chordal.

A *interval perfect elimination order* of a graph G is a total order on its vertex set $v_1, v_2, \ldots, v_n$ such that for every choice of vertices v_i, v_j, v_k with $i < j < k$, $v_i v_k \in E$ implies $v_i v_j \in E$. An *interval graph* is the intersection graph of a set of intervals of the real line. Alternatively, a graph is interval if it admits a interval perfect elimination order. We note that, by definition, every interval perfect elimination order is a perfect elimination order. In Section 4 we prove that every interval perfect elimination order is a doubly perfect elimination order.

The *degree* of a vertex v is $\deg(v) = |\mathrm{Adj}(v)|$. The maximum degree is then $\Delta(G) = \max_{v \in V(G)} \deg(v)$. A vertex u is *universal* if $\deg(u) = |V(G)| - 1$.

For us, K_n denotes the complete graph on $k \geq 1$ vertices.

Colouring

An *assignment of colours* to the vertices of G is a function $\lambda: V(G) \to S$. The elements of the set S are called *colours*. A *conflict* in an assignment of colours is the existence of two adjacent vertices with the same colour. A *vertex-colouring* of a graph is an assignment of colours such that there are no conflicts. The *chromatic number* of a graph G is the minimum number of colours used among all vertex-colourings of G and is denoted by $\chi(G)$.

An *assignment of colours* to the edges of G is a function $\kappa: E(G) \to S$. Again, the elements of the set S are called *colours*. A *conflict* in an assignment of colours is the existence of two edges with the same colour incident to a common vertex. A vertex u is said to be *satisfied* when $\kappa(uv) = \kappa(uw)$ implies $v = w$, for all neighbours v, w of u. An *edge-colouring* of a graph is an assignment of colours such that every vertex is satisfied or, equivalently, such that there are no conflicts. The *chromatic index* of a graph G is the minimum number of colours used among all edge-colourings of G and is denoted by $\chi'(G)$.

A graph G is said to be *Class* 1 if $\chi'(G) = \Delta(G)$ and *Class* 2 if $\chi'(G) = \Delta(G) + 1$. Vizing's theorem states that there are no other possibilities: all graphs are either Class 1 or Class 2 [4].

The *greedy method* for vertex-colouring considers the vertex set of a graph according to a total order and assign to the current vertex the smallest available colour that does not create conflicts. A *perfect order* is a total order on the vertex set of a graph such that the greedy method vertex-colours optimally all its induced subgraphs. A graph is *perfectly orderable* if it admits a perfect order. Every chordal graph is perfectly orderable as every perfect elimination order is a perfect order.

3 Pull back of complete graphs

Given two graphs G and G', a *pull back* is a function $f\colon V(G) \to V(G')$, such that

- if $xy \in E(G)$ then $f(x)f(y) \in E(G')$.
- f is injective when restricted to $N(v)$, for all $v \in V(G)$.

Lemma 1 *If $f\colon V(G) \to V(G')$ is a pull back and κ' is an edge-colouring of G' then the colour assignment κ defined by*

$$\kappa(xy) = \kappa'(f(x)f(y))$$

is an edge-colouring of G.

Proof: Each edge of G has a colour defined by κ. This is because $f\colon V(G) \to V(G')$ is a pull back function.

Moreover, the colour assignment κ has no conflits. In fact, suppose we have two distinct edges xy and xz. Assume for a moment that $\kappa(xy) = \kappa(xz)$. Hence, $\kappa'(f(x)f(y)) = \kappa'(f(x)f(z))$. Since f is a pull back function and $y \neq z \in N(x)$, we have $f(y) \neq f(z) \in N(f(x))$. Thus, κ' is not an edge-colouring of G', a contradiction. ■

An edge-colouring κ for G is said to be a *pull back* from an edge-colouring κ' for G', if κ is defined from κ' as in Lemma 1.

We begin by showing that, if G^2 admits a vertex-colouring with l colours, then there is a pull back $f\colon V(G) \to V(K_l)$. We note that, by definition, any vertex-colouring of G^2 needs $l \geq \Delta(G) + 1$ colours. Hence, this result actually says that, if we have a vertex-colouring of G^2 with $\Delta(G) + 1$ colours, then there is a natural way of getting an edge-colouring of G with $\Delta(G) + 1$ colours.

Theorem 1 *There is a pull back function $f\colon V(G) \to V(K_l)$ if and only if $\chi(G^2) \leq l$.*

Proof: For the "if" part, let $\lambda\colon V(G^2) \to S$ be a vertex-colouring of G^2 with $|S| = l$.

Consider a bijection $g\colon S \to V(K_l)$. We shall show that the composition $f = g \circ \lambda$ is a pull back function $f\colon V(G) \to V(K_l)$ (remember that $V(G) = V(G^2)$).

It is clear that the first condition of a pull back function is satisfied by f. Now, notice that λ is also a vertex-colouring of G with l colours, and that all vertices which have distance in G at most two, have distinct colours. Hence, f is injective when restricted to $N(v)$ for all $v \in V(G)$.

For the "only if" part, let $f: V(G) \rightarrow V(K_l)$ be a pull back function. This can be viewed as a colour assignment with $S = V(K_l)$. We are using $|S| = l$ colours.

If $uv \in E(G)$ then $f(u)f(v) \in E(K_l)$, that is, $f(u) \neq f(v)$. If u and v are at distance 2 in G then there is $w \in V(G)$ with $u, v \in N(w)$. Since f is injective in $N(w)$, $f(u)$ is again distinct from $f(v)$. Hence, there are no conflicts and f is a vertex-colouring of G^2. It follows that $\chi(G^2) \leq l$. ∎

The chromatic indices of complete graphs are well-known: K_l is Class 1 if $l > 0$ is even and Class 2 if l is odd. In particular, we get sufficient conditions for a graph G to be Class 1: G has odd maximum degree and G^2 admits a vertex-colouring with $\Delta(G) + 1$ colours.

Corollary 1 *Suppose $\chi(G^2) \leq l$. Then*

- $\chi'(G) \leq l - 1$ *if l is even , $l > 0$*
- $\chi'(G) \leq l$ *if l is odd.*

Proof: It follows immediatly from Lemma 1, Theorem 1, and the above remarks.

∎

4 Doubly chordal graphs

In this section we shall prove that any odd maximum degree doubly chordal graph is Class 1. We denote by $G_i = G[v_1, \ldots, v_i]$ the subgraph induced by $\{v_1, \ldots, v_n\}$ and $N_i(v)$ the neighbourhood of v in G_i.

Lemma 2 *If G is doubly chordal then $\chi(G^2) = \Delta(G) + 1$.*

Proof: Let $v_1, \ldots, v_n$ be a doubly perfect elimination order of G.

For each i, $1 \leq i \leq n$, let u_i be a maximum neighbour of v_i in G_i. By definition, for all $w \in N_i(v_i)$, $N_i(w) \subseteq N_i(u_i)$. In other words, $N_i^2(v_i) \subseteq N_i(u_i)$. Hence, there are at most $\Delta(G) + 1$ vertices in $N_i^2(v_i)$, for each i.

Thus, the greedy method uses at most $\Delta(G) + 1$ colours to colour the vertices of G^2. Since we need at least $\Delta(G) + 1$, the result follows. ∎

Corollary 2 *All doubly chordal graphs with odd maximum degree are Class* 1.

We note that in particular Corollary 2 implies that any strongly chordal or interval graph with odd maximum degree is Class 1. The fact that interval graphs are doubly chordal follows from the result below.

Lemma 3 *Every interval perfect elimination order is a doubly perfect elimination order.*

Proof: Let G be an interval graph with interval order $O = v_1, v_2, \ldots, v_n$. We denote by $f(v_i)$ the leftmost neighbour of v_i, $1 < i \leq n$. We shall prove that for each i, $1 < i \leq n$, $N_i^2(v_i) \subseteq N_i(f(v_i))$. This says that O is actually a doubly perfect elimination order.

For consider $w \in N_i^2(v_i)$. We denote by $d_i(w,x)$ the distance between $w, x \in V(G_i)$ in the graph G_i.

Case 1: $d_i(w,v_i) = 0$. This says that $w = v_i$ which implies $wf(v_i) \in E(G_i)$.

Case 2: $d_i(w,v_i) = 1$. This says that $wv_i \in E(G_i)$. Hence $f(v_i) \leq w$. If $f(v_i) = w$, then $w \in N_i(f(v_i))$. Otherwise, $f(v_i) < w$, which implies $wf(v_i) \in E(G_i)$ as O is a interval order.

Case 3: $d_i(w,v_i) = 2$. Let x be a vertex satisfying: $wx, vx \in E(G_i)$. If $x < w < v_i$, then $f(v_i) \leq x$ implies $wf(v_i) \in E(G_i)$. If $w < x < v_i$, then $f(v_i) < w$ implies $wf(v_i) \in E(G_i)$ and $f(v_i) > w$ implies $w < f(v_i) \leq x$, which in turn implies $wf(v_i) \in E(G_i)$.

■

Corollary 3 *All strongly chrodal graphs with odd maximum degree are Class* 1. *All interval graphs with odd maximum degree are Class* 1.

5 Conclusions

Consider the chordal graph G depicted in Figure 1. This graph has diameter equals to 2, i.e., every pair of vertices is at distance at most 2, or more precisely $G^2 = K_7$. In this case, we need 7 colours for any vertex-colouring of G^2. On the other hand, this graph has no universal vertex, $\Delta(G) = 5$ and its chromatic index is 5. Therefore, Lemma 2 does not generalize to chordal graphs.

On the other hand, we were unable to find any evidence that Corollary 3 does not hold for chordal graphs (and perhaps even for larger classes). We conjecture that all chordal graphs with odd maximum degree are in fact Class 1. A more general question would be to determine the largest graph class for which this is true.

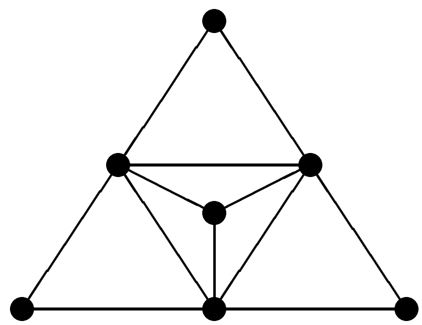

Figure 1: A chordal graph with $\chi(G^2) > \Delta(G) + 1$.

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