BOUNDS ON THE RANK OF THE SUM
OF MATRICES

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1. INTRODUCTION

The principal result of this paper is the following:

THEOREM 1. *Let A and B be two matrices of the same size. Let their row spaces be $\mathcal{R}_1$ and $\mathcal{R}_2$; their column spaces $\mathcal{C}_1$ and $\mathcal{C}_2$. If*

$$d = dimension\,(\mathcal{R}_1 \cap \mathcal{R}_2),$$
$$c = dimension\,(\mathcal{C}_1 \cap \mathcal{C}_2),$$

and if $r(A)$ designates rank of A, then

$$r(A) + r(B) - c - d \leq r(A + B) \leq r(A) + r(B) - \max\,(c, d).$$

The proof of this theorem is in Section 4. Section 5 discusses the application of this theorem to multivariate analysis, particularly to Cochran's theorem [2], which asserts that *if $A_1, \ldots, A_m$ are symmetric $n \times n$ matrices for which*

(I) $$A_1 + \ldots + A_m = I,$$
(II) $$r(A_1) + \ldots + r(A_m) = n$$

then $A_i A_j = 0$ for $i \neq j$.

Various versions of Cochran's theorem have been considered, [1], [3], [4], and [5]. The key point of the theorem seems to be that the rank of the sum is the sum of the ranks. Theorem 1 enables us to easily characterize the condition that $r(A + B) = = r(A) + r(B)$. (Theorem 2.) This then provides an extension of Cochran's result, removing the condition that the matrices be symmetric and that their sum be the identity. (Theorem 3.)

As indicated above, we use $r(A)$ to designate the rank of the matrix A, and $d(\mathcal{A})$ for the dimension of the vector space $\mathcal{A}$. We also use I and 0 indiscriminately to

2. THE PROJECTOR OF A SPACE

Given a vector subspace $\mathscr{A}$ of the space of $1 \times n$ vectors, there is a unique symmetric, idempotent $n \times n$ matrix $\bar{A}$ satisfying

(I) $\qquad\qquad\qquad\qquad \alpha\bar{A} = \alpha$ if and only if $\alpha \in \mathscr{A}$.

(II) $\qquad\qquad\qquad\qquad$ rank of $\bar{A} =$ dimension of $\mathscr{A}$.

This symmetric, idempotent matrix is called the *projector* of $\mathscr{A}$. If $\alpha_1, \alpha_2, ..., \alpha_t$ are an orthonormal basis of $\mathscr{A}$, then $\bar{A} = \alpha_1'\alpha_1 + ... + \alpha_t'\alpha_t$, or if A is a matrix whose rows are a basis of $\mathscr{A}$, then $\bar{A} = A'(AA')^{-1}A$. Any $1 \times n$ vector β may be represented uniquely as the sum of a vector in $\mathscr{A}$ and a vector in the orthogonal complement of $\mathscr{A}$, by writing $\beta = \beta\bar{A} + \beta(I - \bar{A})$. The projector of the orthogonal complement of $\mathscr{A}$ is $I - \bar{A}$, and $r(\bar{A}) + r(I - \bar{A}) = n$.

We will use an overscore to represent the projector — given a matrix B, the notation $\bar{B}$ will mean the projector of the row space of B; thus $\bar{B}$ is the unique matrix satisfying

  (I) $\bar{B}' = \bar{B}$,

 (II) $\bar{B}\bar{B} = \bar{B}$.

(III) $\bar{B} = TB$ for some T, i.e., the rows of $\bar{B}$ are in the row space of B.

(IV) $B\bar{B} = B$,

 (V) $r(\bar{B}) = r(B)$.

The projector provides a convenient method for representing the rank of a composite matrix as a sum — for example, we write

$$r\begin{pmatrix} A \\ B \end{pmatrix} = r\begin{pmatrix} A\bar{B} + A(I - \bar{B}) \\ B \end{pmatrix}$$

and since the rows of $A\bar{B}$ are in the row space of B, they may be removed by elementary row operations; thus

(1) $\qquad\qquad r\begin{pmatrix} A \\ B \end{pmatrix} = r\begin{pmatrix} A(I - \bar{B}) \\ B \end{pmatrix} = r(B) + r[A(I - \bar{B})] ,$

since the rows of $A(I - \bar{B})$ are orthogonal to the rows of B.

For the column version of this formula, we write

$$r(R, S) = r(S) + r[(I - \overline{S}') R]$$

where $\overline{S}'$ is the projector of the column space of S, or the projector of the row space of S'.

A slightly more complicated application of this device gives this formula:

(2) $$r\begin{pmatrix} RS \\ T0 \end{pmatrix} = r(T) + r(S) + r[(I - \overline{S}') R(I - \overline{T})]$$

where $\overline{S}'$ is the projector of the column space of S and $\overline{T}$ the projector of the row space of T. To prove this, we write

$$r\begin{pmatrix} RS \\ T0 \end{pmatrix} = r\begin{pmatrix} R\overline{T} + R(I - \overline{T}) S \\ T \qquad 0 \end{pmatrix} = r\begin{pmatrix} R(I - \overline{T}) S \\ T \qquad 0 \end{pmatrix} =$$

$$= r(T) + r(R(I - \overline{T}), S) = r(T) + r(S) + r[(I - \overline{S}') R(I - \overline{T})] .$$

3. BOUNDS FOR THE RANK OF A PRODUCT

We need the results of this section to prove the main theorem. We want bounds on the rank of AB; to get them we write

$$r(B) = r\begin{pmatrix} B \\ AB \end{pmatrix}.$$

then use formula (1) to get

(3) $$r(B) = r(AB) + r[B(I - \overline{AB})],$$

where $\overline{AB}$ is the projector of the row space of AB. Now if $\overline{A}$ is the projector of the row space of A, then $\overline{A}B(I - \overline{AB}) = 0$, since for some T, $\overline{A} = TA$. Thus (3) may be written

$$r(B) = r(AB) + r[(I - \overline{A}) B(I - \overline{AB})],$$

which gives these bounds on $r(AB)$,

$$r(B) - r[(I - \overline{A}) B] \leq r(AB) \leq r(B),$$

and the weaker result,

$$r(B) - r(I - \overline{A}) \leq r(AB) \leq r(B).$$

The latter is known as Sylvester's law of nullity, usually written as

(4) $$r(A) + r(B) - n \leq r(AB) \leq r(B)$$

where $r(I - \overline{A}) = n - r(\overline{A}) = n - r(A)$, assuming that A is $p \times n$ and B is $n \times q$.

4. BOUNDS FOR THE RANK OF A SUM

This theorem gives upper and lower bounds for the rank of a sum.

THEOREM 1. *Let A and B be two matrices of the same size, let their row spaces be $\mathscr{R}_1$ and $\mathscr{R}_2$, their column spaces $\mathscr{C}_1$ and $\mathscr{C}_2$. Then*

$$(5) \qquad r(A) + r(B) - d(\mathscr{R}_1 \cap \mathscr{R}_2) - d(\mathscr{C}_1 \cap \mathscr{C}_2) \leq r(A + B)$$

and

$$(6) \qquad r(A + B) \leq r(A) + r(B) - \max\{d(\mathscr{R}_1 \cap \mathscr{R}_2), d(\mathscr{C}_1 \cap \mathscr{C}_2)\}.$$

We first prove (6), which is quite easy, then (5), which is not so easy. We have

$$r(A + B) \leq d(\mathscr{R}_1 + \mathscr{R}_2) = d(\mathscr{R}_1) + d(\mathscr{R}_2) - d(\mathscr{R}_1 \cap \mathscr{R}_2).$$

Since $d(\mathscr{R}_1) = r(A)$ and $d(\mathscr{R}_2) = r(B)$, we have

$$r(A + B) \leq r(A) + r(B) - d(\mathscr{R}_1 \cap \mathscr{R}_2),$$

and a similar argument on the column spaces yields (6).

Using the fact that

$$r\binom{A}{B} = d(\mathscr{R}_1 + \mathscr{R}_2) = r(A) + r(B) - d(\mathscr{R}_1 \cap \mathscr{R}_2)$$

and

$$r(A, B) = d(\mathscr{C}_1 + \mathscr{C}_2) = r(A) + r(B) - d(\mathscr{C}_1 \cap \mathscr{C}_2),$$

we may write (5) in the form

$$(7) \qquad r\binom{A}{B} + r(A, B) \leq r(A + B) + r(A) + r(B).$$

Now none of the five ranks in (7) is changed if we replace A and B by PAQ and PBQ with P and Q non-singular. Thus we may assume A and B have any form obtained by performing identical elementary row and column operations on each of them. We may, for example, assume that A and B have this form

$$A = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} R & S & 0 \\ T & 0 & 0 \\ 0 & 0 & I_2 \end{pmatrix}.$$

In that case, we have

$$r\binom{A}{B} = r(I_1) + r(S) + r(I_2),$$

and

$$r(A, B) = r(I_1) + r(T) + (I_2),$$

so that we will prove (7) if we can prove that

(8) $$r(I) + r(S) + r(T) \leq r\begin{pmatrix} R & S \\ T & 0 \end{pmatrix} + - r\begin{pmatrix} I+R & S \\ T & 0 \end{pmatrix},$$

dropping the subscript on I, assumed to be $n \times n$. Using formula (2), we write

$$r\begin{pmatrix} R & S \\ T & 0 \end{pmatrix} + r\begin{pmatrix} I+R & S \\ T & 0 \end{pmatrix} = 2r(S) + 2r(T) + r[(I - \overline{S'})(I + R)(I - \overline{T})] +$$
$$+ r[(I - \overline{S'}) R(I - \overline{T})]$$

and this leads to the inequality

(9) $$r\begin{pmatrix} R & S \\ T & 0 \end{pmatrix} + r\begin{pmatrix} I + R & S \\ T & 0 \end{pmatrix} \geq 2r(S) + 2r(T) + r[(I - \overline{S'})(I - \overline{T})],$$

since

$$r[(I - \overline{S'}) R(I - \overline{T})] = r[(I - \overline{S'})(-R)(I - \overline{T})].$$

Using (4) with $n = r(I)$, we have

$$r[(I - \overline{S'})(I - \overline{T})] \geq n - r(S) + n - r(T) - n$$

and putting this in (9) we have

$$r\begin{pmatrix} R & S \\ T & 0 \end{pmatrix} + r\begin{pmatrix} I + R & S \\ T & 0 \end{pmatrix} \geq r(S) + r(T) + n.$$

This establishes formula (8), and hence (7), which is equivalent to (5); the proof of Theorem 1 is complete.

5. THE RANK OF THE SUM AND THE SUM OF THE RANKS

Theorem 1 enables us to characterize the condition that rank be additive for a pair of matrices:

THEOREM 2. *Let A and B be two matrices of the same size, with row spaces $\mathscr{R}_1$, $\mathscr{R}_2$ and column spaces $\mathscr{C}_1$, $\mathscr{C}_2$. Then*

$$r(A + B) = r(A) + r(B)$$

if, and only if

$$\dim (\mathscr{R}_1 \cap \mathscr{R}_2) = \dim (\mathscr{C}_1 \cap \mathscr{C}_2) = 0.$$

The proof follows immediately from (5) and (6).

The fact that rank is additive for a set of matrices is a strong condition. We will develop some consequences of this condition in a few conclusions leading up to Theorem 3, which is a sort of generalized Cochran's Theorem. First, we point out that if rank is additive on a set of matrices, it is additive on subsets:

LEMMA 1. *If rank is additive for a set of matrices,*

$$r(A_1 + A_2 + \ldots + A_m) = r(A_1) + r(A_2) + \ldots + r(A_m),$$

then rank is additive for sets of matrices formed by adding distinct A's from that set — for example, if

$$r(A_1 + \ldots + A_{12}) = r(A_1) + \ldots + r(A_{12}),$$

then

$$r(A_1 + A_5) + r(A_3 + A_8 + A_{10}) + r(A_6 + A_9) =$$
$$= r(A_1 + A_5 + A_3 + A_8 + A_{10} + A_6 + A_9).$$

To give the gist of the proof, consider the example of the theorem. Let

$$A = A_1 + A_5, \quad B = A_3 + A_8 + A_{10}, \quad C = A_6 + A_9,$$

and let D be the sum of the matrices in $A_1, \ldots, A_{12}$ not included in A, B, or C, that is,

$$D = A_2 + A_4 + A_7 + A_{11} + A_{12}.$$

Then

$$r(A_1) + \ldots + r(A_{12}) = r(A + B + C + D) \leq r(A + B + C) +$$
$$+ r(D) \leq r(A) + r(B) + r(C) + r(D) \leq r(A_1) + \ldots + r(A_{12}).$$

Thus all inequalities are equalities, and

$$r(A + B + C) = r(A) + r(B) + r(C).$$

The converse of this lemma is not true — rank can be pairwise additive but yet not finitely additive, for example, for these three positive semi-definite matrices:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 \\ 4 & 16 \end{pmatrix},$$

rank is pairwise additive but not additive over all three.

LEMMA 2. *If* $r(A + B) = r(A) + r(B)$ *and if A and B each commute with their sum,* $S = A + B$, *then* $AB = BA = 0$.

Proof: Since A commutes with S, it commutes with $S - A = B$. Since the row space of $AB = BA$ is in both the row space of A and the row space of B, Theorem 2 shows that $AB = BA = 0$.

We are now able to give this generalized version of Cochran's Theorem mentioned in the Introduction. We find that symmetry has no essential role in the theorem, and Condition II, that $\sum A_i = I$, can be replaced by the condition that each A_j commutes with $\sum A_i$.

THEOREM 3. *Let* $A_1, A_2, \ldots, A_m$ *be square matrices for which rank is additive,*

$$r(A_1 + \ldots + A_m) = r(A_1) + \ldots + r(A_m),$$

and let S be their sum:

$$S = A_1 + \ldots + A_m.$$

Then

(10) $$A_i A_j = 0 \quad for \quad i \neq j$$

if and only if each of the A's commutes with the sum:

(11) $$A_i S = SA_i, \quad i = 1, 2, \ldots, m.$$

Proof: Condition (10) certainly implies (11). To prove that (11) implies (10) we prove that $A_1 A_2 = A_2 A_1 = 0$. Let $A = A_1$, $B = A_2$, and $C = A_3 + \ldots + A_m$. Using Lemmas 1 and 2, we know that $A(B + C) = (B + C) A = 0$, and hence $A^2 = AS = SA$. Going to a similarity transformation if necessary, we may assume that A has the form $\begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}$, with G non-singular. Then we write $A + B + C = S$ in the form

$$\begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} + \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}.$$

Since $A = AS = SA$ and G is non-singular, it follows that S must have the form $\begin{pmatrix} G & 0 \\ 0 & S_4 \end{pmatrix}$. Thus

$$\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} + \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & S_4 \end{pmatrix}.$$

Now according to Lemma 1, $r(B + C) = r(B) + r(C)$, and it follows that the rows

of $(B_1 \ B_2)$ and of $(C_1 \ C_2)$ must all be zero, or else Theorem 2 would be violated. By the same argument, the columns of $\begin{pmatrix} B_1 \\ B_3 \end{pmatrix}$ and $\begin{pmatrix} C_1 \\ C_3 \end{pmatrix}$ must be zero. Thus $B = \begin{pmatrix} 0 & 0 \\ 0 & B_4 \end{pmatrix}$, and $AB = BA = 0$.

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