A Survey on Non-Cooperative Facility Location Games [*]

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March 4, 2015

Abstract

The Facility Location problem is a well know NP-Hard combinatorial problem. It models a diverse set of situations where one aims to provide a set of goods or services via a set of facilities F to a set of clients C. There are opening costs to each facility in F and connection costs for each pair of facility and client, if such facility attends this client. A central authority wants to determine the solution with minimum cost, considering both opening and connection costs, in such a way that all clients are attended by one facility. In this paper we are interested in the non-cooperative game version of this problem, where instead of having a central authority, each client is a player and decides where to connect himself. In doing so, he aims to minimize his own costs, given by the connection costs and opening costs of the facility, which may be shared among clients using the same facility. This problem has several applications as well, specially in distributed scenarios where a central authority is too expensive or even infeasible to exist. In this paper we present a survey describing different variants of this problem and reviewing several results about it. For some of the variants, where results were not found in the literature, we show new results concerning the existence of equilibria, PoS and PoA. We also point out open problems that remain to be addressed.

1 Introduction

The facility location class of problems models a large number of important decision problems that may occur in practice, ranging from traditional areas such as economics and urban planning, to more recent ones such as computer networking. This class of problems is concerned with the placement of facilities that will supply some demand of products or services by clients

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in order to minimize some function of cost. This function cost may be defined in different ways, depending on each specific problem. Generally the costs consider several factors such as competitors, distance from clients, and others.

A common version of the facility location problem can be stated as the problem of choosing from a set of facilities F, a subset of facilities to open and to establish a connection with each client from a set of clients T, also called terminals. The opening and connection costs must be minimized. A formal definition is given bellow.

Definition 1 (Uncapacitated Facility Location Problem)**.** Let F be a set of facilities, T a set of terminals, c_f opening costs for each facility $f \in F$ and d_{tf} connection costs for connecting terminal $t \in T$ to facility $f \in F$. The problem is to find a subset of facilities to open and establish connections from terminals to this subset such that the sum of all costs are minimized.

An integer program formulation for this problem is presented below:

$$\begin{aligned} \text{minimize} \quad & \sum_{f \in F} c_f y_f + \sum_{f \in F} \sum_{t \in T} d_{tf} x_{tf} & \\ \text{subject to} \quad & \sum_{f \in F} x_{tf} = 1 & , \forall t \in T \\ & y_f \leq x_{tf} & , \forall f \in F, \forall t \in T \\ & y_f, x_{tf} \in \{0, 1\} & , \forall f \in F, \forall t \in T \quad , \end{aligned}$$

where y_f is a boolean variable that indicates if a facility f is opened, while the variable x_{tf} represents whether terminal t is connected to facility f or not.

From this problem, several possible variants may arise. There might be capacities associated with each facility, as well as quotas for each facility. Furthermore, the opening costs of a facility f may not be constants, but a function on the number of terminals connected to f. In another possible variant the facilities can be any point in a metric space, i.e. F is infinite. All these variants mentioned and the original problem are well studied, with most of them being NP-Hard with known approximation algorithms [23, 17].

In all these problems, it is assumed that both terminals (clients) and facilities are controlled by a single central entity seeking to minimize the total cost of the system. However, in several applications the terminals or the clients may behave differently, for example being controlled by different agents. It is therefore important to analyse these problems from a game theoretic perspective.

In game theory, a *non–cooperative game* is a scenario where players or agents choose strategies independently trying to either minimize their costs or maximize their utility. For each player i there is a set A_i of actions that it can choose to play. A pure strategy S_i consists of one action from A_i, while a mixed strategy corresponds to a probability distribution over A_i. In a pure game each player choses one action to play, while in a mixed game each player randomizes his action according to the probability distribution. In this paper we assume pure strategies games unless mentioned otherwise. A set of strategies S consisting of one strategy for each player, is denominated a *strategy profile*. Let $\mathcal{S} = A_1 \times A_2 \times \ldots \times A_n$ be the set of all possible strategy profiles and let $c : \mathcal{S} \to \mathbb{R}^n$ be a cost function that attributes a cost $c_i(S)$ for each player i given a strategy profile S. Define $S_{-i} = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ a strategy profile S without i's strategy, so that we can write $S = (S_i, S_{-i})$. If all players other than i decide to play S_{-i}, then player i is faced with the problem of determining a best response to S_{-i}. A strategy S_i^* from a player i is a *best response* to S_{-i}, if there is no other strategy which could yield a better outcome for the player, i.e.

$$c(S_i^*, S_{-i}) \leq c(S_i, S_{-i})$$

for all $S_i \in A_i$. A strategy profile is in a *pure Nash equilibrium* (PNE) if no player can increase his utility or reduce his cost by choosing a different strategy, i.e. every strategy in the strategy profile is a best response.

Game theory can be used to analyze many aspects of decision problems. It may be used to help design games with desired properties, or be used to measure the inefficiency arising from players selfish behaviour. The social welfare or social cost is a function mapping a strategy profile to a real number, indicating a measure of the total cost or payoff of a game. Two of the most important concepts for efficiency analysis are the *Price of Anarchy* (PoA) and the *Price of Stability* (PoS). The PoA is the ratio between a Nash equilibrium with worst possible social cost and the strategy profile with optimal social cost, while the PoS is the ratio between the best possible Nash equilibrium to the social optimum.

One variation of a facility location game occurs when clients behave selfishly connecting to facilities opened by a central authority. If the central authority is aware of the exact location or connection costs of each client, then the problem is equal to the one presented in Definition 1. However, when clients may lie to the central authority about their location, there is a need for such authority to design mechanisms encouraging clients to be truthful. There have been several advancements in this area of mechanism design, in particular on *strategy-proof* mechanisms for these games, with seminal papers by Pal and Tardos [21], Devanur et al. [6, 7] and Leonardi and Schäfer [15] as well as complementary works [24, 8, 27].

Another variant of a facility location game considers a cooperative game, where a solution is going to be constructed attending all the clients that represent players. The problem is how to split the solution cost among all clients in such a way that no coalition of players has incentive to leave the grand coalition and form a new solution. This problem was studied by Goemans and Skutella [9] and later in the book Algorithmic Game Theory [20] (Chapter 15), with several related results presented.

These previous versions of facility location games are based on the fact that a central authority is partially present in the problem. Nonetheless, when no authority is dictating where each facility is located, several traditional games may be formed. One possibility is when facilities and clients are controlled by players. The facility players set operating prices for clients, which behave selfishly always choosing to connect to the cheapest option available. Games with these premises have been studied by Vetta [25] as valid utility games, with proof of the existence of pure Nash equilibria and bounds on the price of anarchy. Later this subject was also covered in the book Algorithmic Game Theory [20] (Chapter 19), with several results for variants of these games also explored in other works [18, 26, 13].

Perhaps the most natural game that arises from facility location problems occurs when players control terminals with the need to connect to a facility. In this case terminals connected to a facility share its opening cost. How players share the costs may vary depending on the specific version of the game being analyzed. When there is no rules on how to share opening costs, some important results have been presented by [2]. However, few direct results have been presented for other variants of this game, with most results being adaptations from other problems such as the network design problem [1]. Therefore, our focus in this work is to study Facility Location games when terminals are controlled by players. We are interested in how much this behaviour may hamper the system cost when compared to the system optimum, both optimistically by considering the Price of Stability of games and pessimistically, with the Price of Anarchy. Furthermore, we adapt and present results from the literature for games where players are not completely selfish in their behaviour.

This survey is organized as follows. In Section 2, we summarize results for facility location games with no cost sharing rules, mainly from Cardinal and Hoefer [2, 19]. In Section 3, results from network design [1] are adapted to fair cost facility location games, and a short compilation of results for the weighted version of the game is presented. In Section 4, capacity restrictions are added for the previous analyzed games, pure Nash equilibria existence and bounds for PoA and PoS are proven. Altruism in facility location games is explored in Section 5, and final conclusions are given in Section 6.

2 Facility Location without Cost Sharing Rules

The facility location problem can model several practical scenarios. Imagine a situation where some groups are interested in constructing public goods, such as libraries or museums. There is no defined rule on how these groups share the construction costs, and opened facilities do not have ties to the groups which helped build them, being available to anyone willing to use them. These scenarios may be modelled using game theory: a game where players controls terminals and need to connect to an opened facility.

Definition 2 (Facility Location Game without Cost Sharing Rules (FLG))**.** Let $(G = (T \cup F, T \times F), k, c, d)$ be an instance of FLG, where G is a bipartite graph with vertex sets F of n facilities and T of m terminals, k is the number of players and c and d are opening and connection costs, respectively. Each facility $f \in F$ has an opening cost c_f, and connection costs d_{tf} for each terminal $t \in T$. Each player $i \in [1, k]$ controls a subset of terminals $T_i \subseteq T$. These subsets form a partition of T, i.e, each terminal from T is controlled by some player and $T_i \cap T_j = \emptyset$ for $i, j \in [1, k]$ where $i \neq j$. Each terminal must be connected to exactly one opened facility. Each player i has a strategy S_i composed of a payment function $p_i^c : F \rightarrow \mathbb{R}_0^+$ indicating how much he offers for opening a facility, as well as a function $p_i^d : T \times F \rightarrow \mathbb{R}_0^+$ which indicates how much he pays for the connection costs. Let $p^c(f) = \sum_{i=1}^{k} p_i^c(f)$ be the total paid by players for a facility f. If $p^c(f)$ is greater than or equal to the cost c_f, then the facility f is considered opened. Likewise, if the total offered for connection cost of a terminal-facility pair t, f is greater than or equal to d_{tf} then the connection is bought. Each player tries to minimize their payments while ensuring that the terminals they control are connected to an opened facility.

Note that there is no rule on how players share the costs to open a facility. Therefore, how players share these costs may depend on which player need the facility the most and in how the equilibrium is reached. Consider the game in Figure 1, with two players, each controlling one terminal. One possible strategy is if t_1 offers 1 to f_1, while t_2 offers 1 to f_3. There is no incentive to any player to change their payment scheme, and thus they are in an equilibrium. Suppose now that t_1 chooses instead, as his payment function, to offer 0.75 to facility f_2 and zero to the others, while t_2 chooses to only pay 1 to f_3. The player controlling t_2 then has an incentive to change his strategy to pay 0.75 to open f_2. This strategy profile is an equilibrium in which both players share equally the opening costs of f_2. However, if t_1 had offered only $0.5 + \varepsilon$ to f_2, t_2 would still pay less by offering to pay the remaining opening cost of f_2. In fact, there is an infinite number of possible equilibria in this example, since a player may offer to pay for the opening costs of f_2 any amount in the interval $(0.5, 1.0]$ and the other player will, in an equilibrium, complete the offer to open f_2.

In [2], Cardinal and Hoefer analyzes a class of covering games which includes FLG and answer a few fundamental questions about it. They prove that there may be instances of FLG with no pure Nash equilibrium, also showing that it is NP-Complete to determine whether an instance of FLG

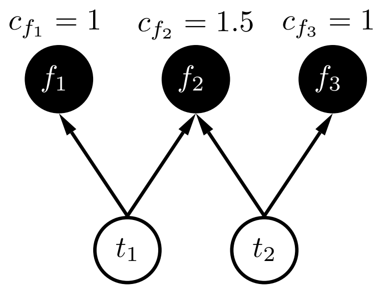

Figure 1: A example of a Facility Location Game. Connection costs are constant.

has an equilibrium or not. Furthermore, they provide bounds on the Price of Anarchy and Stability for the FLG for the instances that admit equilibria. They also presented approximation algorithms to find an approximated equilibrium based on a well known primal-dual algorithm for the facility location problem. We shortly summarize some of these results below.

Theorem 1 (Pure Nash Equilibrium existence for FLG [2])**.** *There are instances of FLG where there is no PNE.*

Proof. Consider the instance of FLG showed in Figure 2. Player A controls terminal t_1, while player B controls terminals t_2 and t_3. For all edges shown, the connection costs are constant, and infinite otherwise. Suppose facility f_1 is opened. Either A or B paid completely for it, or they shared the costs in some manner. If B paid fully, player A do not need to pay anything to fulfil his constraints, and B would need to pay for either f_2 or f_3 to attend terminal t_2. In this case B would pay less by not opening f_1 and instead only paying for f_3 for a total payment of $1 + \varepsilon$. However, player A would then need to pay fully for f_1 or f_2, which would make B chooses to use both f_1 and f_2, foregoing f_3 since it would only need to open one facility with total cost equal to 1. Player A would be free in this scenario to not pay for any facility, choosing to connect to the one B opened, and completing a best response cycle. The same occurs when they initially share the costs of f_1, since B would either choose to open fully f_3 or f_2, entering the same best response cycle. □

To determine whether the FLG game does have or not a PNE is NP-Hard. As detailed in [2], the 3-SAT problem [12] can be reduced to this problem. Given an instance to the 3-SAT it is constructed a game where an equilibrium exists if and only if there is a solution to the 3-SAT instance.

When restricted to instances of the game that admits PNE, Cardinal and Hoefer [2] show that the price of anarchy of FLG is k. The social cost $C(S)$ for a strategy profile S of an instance of FLG is defined as the sum of

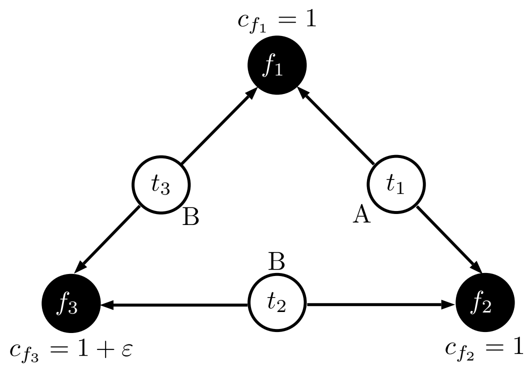

Figure 2: A game with no PNE, from [2]. Letters next to terminals indicate which player controls the terminal.

all payments made by the k players, i.e.

$$C(S) = \sum_{i=1}^{k} \sum_{f \in F} p_i^c(f) + \sum_{i=1}^{k} \sum_{(t,f) \in T \times F} p_i^d(t, f)$$

Theorem 2 (Price of Anarchy of FLG [2]). *The price of anarchy for any FLG instance is at most k, and there exists a FLG instance with price of anarchy at least k.*

Proof. Suppose that there is an equilibrium S which is more than k times the cost of a strategy profile S^* with optimal social cost. Then, at least one player in S is paying more than $c(S^*)$ to cover his terminals. This player could then simply offer the optimal solution S^* as his own payment scheme, and therefore reach a better solution, which means that S is not an equilibrium. Thus, PoA $\leq k$.

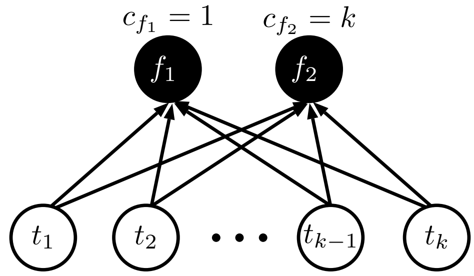

Figure 3: A game with PoA equal to k.

Now consider the game of Figure 3, with $m = k$ terminals, each one controlled by a different player, and two facilities, f_1 with opening cost 1, and f_2 with opening cost k, with no connection costs. The optimal solution is clearly to connect each terminal to f_1 with total cost 1. However, the strategy profile where each player is connected to f_2 paying 1 to open it, is an equilibrium with total cost of k. Therefore, the Price of Anarchy of FLG is k. $\square$

By exploiting the fact that there are instances with no equilibrium, Cardinal showed a game where the Price of Stability is close to the Price of Anarchy.

Theorem 3 (Price of Stability of FLG [2]). *There is an instance of the FLG with price of stability of at least $k-2$.*

Proof. Consider the game in Figure 4, where player 1 controls terminal t_1, player 2 controls terminals t_2 and t_3, and player $i \in [3, k]$ control terminal t_{i+1}. Each player i from 3 to k can connected to the center facility f_{k+1} with connection cost ε and opening cost 1, as well as their "leaf" facility f_i with connection cost 1 and opening cost ε. Note that the instance induced by players 1 and 2 is very similar to the one in Figure 2, and also does not have by itself an equilibrium. Clearly, the optimal solution is the one where players 1 and 2 connect to f_1 and f_2, while the remaining players all connect to f_{k+1}, with total cost $1 + (k-2)\varepsilon + 5\varepsilon = 1 + (k+3)\varepsilon$.

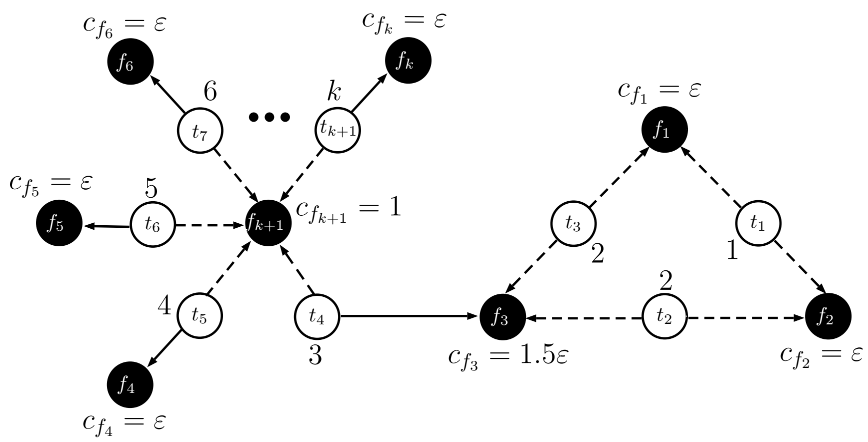

Figure 4: A game with PoS of $k-2$, from [2]. Dashed lines have connection cost ε, while full lines have cost 1. Numbers next to terminals indicate which player controls the terminal.

If any player chooses to open the center facility f_{k+1}, all other players will eventually connect to the center as well, with the exception of players 1 and 2 which would never reach an equilibrium (see Theorem 1). So in order to exist an equilibrium, player 3 must connect to f_3 paying some value in $[\varepsilon/2, \varepsilon]$ of the facility cost (say $\varepsilon/2$). Then player 2 pay the remaining cost, say ε, connecting both of its terminals to f_3. It is easy to check that this is a best equilibrium. Therefore each player $i \in [4, k]$ fully pays for facilities f_i, each paying a connection cost of 1 and opening cost of ε. Player 3 pays a connection cost of 1 to f_3 and $\varepsilon/2$ of its opening cost. Finally, player 1 will pay 2ε for opening and connecting its terminal to either f_1 or f_2, and player 2 will pay 3ε in connection costs to f_3 and the remaining opening cost. The total cost for this equilibrium is $(k-3)(1+\varepsilon)+1+5.5\varepsilon = (k-2)(1+\varepsilon)+4.5\varepsilon$. Therefore, when ε tends to 0, the price of stability of this instance tends to $k-2$.

□

Note that both the Price of Anarchy and the Price of Stability have similar values for this class of games, indicating a large gap between equilibria and social optima. However, in all theorems seen so far, the fact that players do not have a clear way to share facility opening costs plays a major role in making such big differences between the optimal welfare and pure equilibria. If global sharing rules for costs are considered, this gap, and the undesirable fact of the nonexistence of pure equilibria in some games may change.

3 Facility Location with Fair Cost Allocation

In this section, we consider facility location games where, instead of players freely coordinating on how to share facilities' opening costs, they are forced to equally share the costs for each facility they want to open. The game is defined in a similar way as in Section 2, since the only change is in how the players share the facilities opening costs.

Definition 3 (Facility Location Game with Fair Cost Allocation (FLG-FC))**.** Let $G = (T \cup F, T \times F)$, with vertex sets F of n facilities and T of m terminals. Each facility $f \in F$ has an opening cost c_f, and connection costs d_{tf} for each terminal $t \in T$. Let $K = [k]$ be the set of players. Each player i controls a subset of terminals $T_i \subseteq T$, and each terminal must be connected to exactly one opened facility. A player i chooses a strategy $S_i \subset T_i \times F$. Let $S = (S_1, ..., S_k)$ be a strategy profile and $U(S) = \bigcup_{i \in K} S_i$ be the set of all strategies chosen by players in S. Each player tries to minimize his own payment

$$p_i(S) = \sum_{f \in S_i} \frac{c_f}{x_f(S)} + \sum_{(t,f) \in S_i} d_{tf} \ ,$$

where $x_f(S) = |\{1 \leq i \leq k : f \in U(S)\}|$ is the number of players using facility f in strategy profile S.

The *social welfare cost* for a strategy S is defined as the sum of all player payments, i.e.

$$C(S) = \sum_{i \in K} p_i(S) = \sum_{f \in U(S)} c_f + \sum_{(t,f) \in U(S)} d_{tf} \ .$$

We use the expression $f \in U(S)$ to represent all facilities connected to a terminal in a strategy profile S, while $f \in S_i$ represents all facilities player i uses to connect its terminals in strategy S_i.

This game can be seen as a specialization of Network Design, first defined and explored in [1]. In the Network Design Game, it is given a graph $G = (V, E)$, where each player i has a set of terminal nodes T_i which he needs to connect, and his strategy is a set of edges $S_i \subset E$ which must form a tree connecting all nodes in T_i. Each edge e has an opening cost c_e associated with it, and players who use this edge share its cost equally.

For the network design game, it was proved in [1] that the game always have a pure NE, since it is a potential game with potential function

$$\Phi(S) = \sum_{e \in E} \sum_{x=1}^{x_e(S)} \frac{c_e}{x} \tag{1}$$

where x_e is the number of players which have the edge e in the strategy profile S.

A similar proof of the existence of PNE is possible for the Facility Location with Fair Cost Allocation. In this case we use a modified potential function

$$\Phi(S) = \sum_{f \in F} \sum_{x=1}^{x_f(S)} \frac{c_f}{x} + \sum_{(t,f) \in U(S)} d_{tf} \tag{2}$$

where $(t, f) \in U(S)$ is a pair terminal–facility used in the strategy profile S, and $x_f(S)$ is the number of players sharing facility f in S.

Theorem 4. *Every instance of the FLG-FC game admits a pure Nash equilibrium.*

We can use the same arguments and example used in Theorem 2 to show that the Price of Anarchy of the FLG-FC is equal to k, the number of players.

Theorem 5. *The price of anarchy for any FLG-FC instance is at most k, and there exists a FLG instance with price of anarchy at least k.*

As for the Price of Stability, Anshelevich et. al [1] proved that for the network design game there is an upper bound of $H(k)$. We can do a similar proof for the Facility Location game obtaining the same bound and we can also show that this bound is tight.

Theorem 6 (DRAFT Price of Stability for FLG-FC)**.** *Consider a facility location game with fair cost allocation* (FLG-FC) *with nondecreasing concave opening cost c_f for each facility f and connection costs d_{tf} for pairs of terminal-facility (t, f). Then the Price of Stability is $H(k)$, where k is the number of players of the game .*

Proof. Let $\Phi(S)$ be the potential function defined in Equation 2. Let S^* be a strategy with optimal social cost

$$C(S^*) = \sum_{f \in U(S^*)} c_f + \sum_{(t,f) \in U(S^*)} d_{tf} \ .$$

Then, $\Phi(S^*) \leq H(k)C(S^*)$, since

$$\begin{aligned}
\Phi(S^*) &= \sum_{f \in F} \sum_{x=1}^{x_f^*(S)} \frac{c_f}{x} + \sum_{(t,f) \in U(S^*)} d_{tf} \\
&\leq \sum_{f \in F} c_f H(k) + \sum_{(t,f) \in U(S^*)} d_{tf} \\
&\leq H(k) \left(\sum_{f \in F} c_f + \sum_{(t,f) \in U(S^*)} d_{tf} \right) \\
&\leq H(k)C(S^*).
\end{aligned}$$

Since FLG-FC is a potential game, we can start the game using the strategy profile S^* and let each player chooses a best response strategy in a series of rounds. After a finite number of rounds the game will reach a pure Nash equilibrium S with $\Phi(S) \leq \Phi(S^*)$. For any strategy profile S', $\Phi(S') \geq C(S')$ and therefore

$$C(S) \leq \Phi(S) \leq \Phi(S^*) \leq H(k)C(S^*) \ .$$

Reordering the inequality, we obtain that

$$\text{PoS} \leq \frac{C(S)}{C(S^*)} \leq H(k) \ .$$

□

To see that this bound is tight, consider the example in Figure 5. In this game with $n = k + 1$ facilities and $m = k$ terminals each player $i \leq k$ controls terminal t_i. Clearly the solution with optimal social welfare is the one where all players open facility f_{k+1} with a total cost of $1 + \varepsilon$. However, this strategy profile is not an equilibrium, since player k would be able to pay less by opening facility f_k. This change in strategy from player k thence would cause player $k - 1$ to also change his strategy to open facility f_{k-1}, which ultimately would cause all players to choose to not open f_{k+1}, resulting in a equilibrium of total cost $1 + \frac{1}{2} + ... + \frac{1}{k} = H(k)$. This strategy profile is the only possible equilibrium, since every terminal must connect to a facility and there are no equilibrium in which facility f_{k+1} is open.

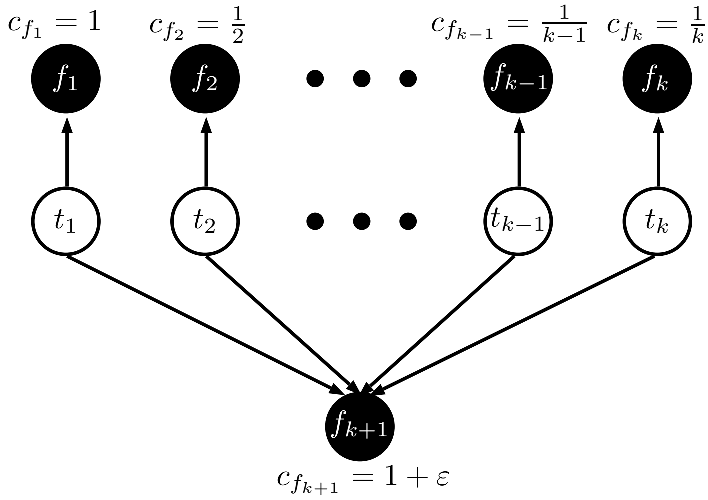

Figure 5: Game with PoS of $H(k)$. All edges have cost equal to zero.

3.1 Weighted Players

Suppose a Facility Location game where players have different demands. Suppose player i demands w_i units of some good while player j demands w_j. When sharing the cost of a common facility, this cost should be divided considering these demands.

The facility location game with fair cost allocation can be extended for cases where players may pay a larger or smaller fraction of opening costs for facilities. This is accomplished by changing the cost calculation function by adding weights for each player. Now each player i pays in a strategy profile S,

$$p_i(S) = \sum_{(t,f) \in S_i} d_{tf} + \sum_{f \in S_i} w_i \frac{c_f}{W_{f,S}} \quad ,$$

where $W_{f,S}$ is the sum of the weights of all players using f in the strategy profile S.

This extension has been studied by Hansen and Telelis [10, 11]. They prove that e-approximate equilibria exists, i.e. there is a strategy profile S where each player cannot improve by more than a factor e from what he is paying in S. Furthermore, a bound for both the PoA and PoS of $\Theta(\log W)$ is shown, where W is the sum of all player weights. In the metric case constant bounds were shown.

The Network Design Game with weights was also explored in the literature, particularly in [3]. In this paper, Chen and Roughgarden proved that this variant do not always have a Pure Nash Equilibrium. However, the example used in this proof do not translate directly to a Facility Location game with weighted players. It is currently an open problem if this extension always have a Pure Nash Equilibrium. The proof using potential functions used in the last section do not apply to this variant, since it does not possess a potential function.

4 Facility Location Games with Capacities

It is not always possible for a facility to provide goods for an unlimited number of terminals. Therefore, extended versions of facility location games where the facilities have limited capacities are also of interest. In the most natural extension, every facility f has a capacity u_f associated with it, which indicates how many terminals can be connected to this facility.

All examples seen so far can be modified to this extension, by setting the capacity of each facility equal to the number of terminals. Thus, the price of anarchy is at least the number of players k for this new game. However, with capacities restriction we can show that worse equilibria exists, even in the case of fair cost allocation.

Theorem 7 (Price of Anarchy for Capacitated Facility Location)**.** *The Price of Anarchy for the FLG and FLG-FC with capacities is unbounded.*

Proof. Consider the game in Figure 6, where player 1 controls terminal t_1 and player 2 controls terminal t_2. One possible equilibrium is the state where t_1 chooses f_2 and t_2 consequently has to open f_3, since all facilities have unitary capacities. This equilibrium is also the social optimum solution. Nonetheless, there is also another possible equilibrium for this game: when t_2 opens f_2 and t_1 is forced to open facility f_1, with an infinity opening cost. Therefore, the price of anarchy for facility location games with capacities is unbounded. Note that this example works for both games FLG and FLG-FC with capacities, since each facility can attend at most one terminal.

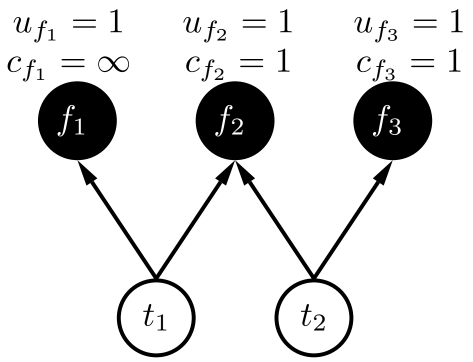

Figure 6: Game with unbounded Price of Anarchy.

□

As for the price of stability, the way facility opening costs are divided among players has a big influence on the PoS. For games with fair cost allocation with capacities, we can show that it is still a potential game, and Theorem 6 applies, giving it a $H(k)$ bound for the PoS. For games with no cost sharing rules, however, we can prove that there are games with unbounded price of stability.

Theorem 8 (Existence and Price of Stability for Capacitated FLG)**.** *There are instances of the capacitated facility location game without cost sharing rules (Capacitated FLG) without Nash equilibrium. In the case where instances admit a Nash equilibrium, the Price of Stability is unbounded.*

Proof. First we prove that there are instances with no equilibria, even when a player is only allowed to control a single terminal. Consider the game in Figure 7 where one player controls terminal t_1 and another one controls t_2. Suppose the player controlling terminal t_2 opens f_2. Then terminal t_1 must open f_1 paying the full opening cost $2+\varepsilon$. However, since now f_1 is opened, t_2 can just connect to f_1 without paying any opening costs, which causes t_1 to change its strategy by connecting to f_2. The game proceeds in this manner with t_1 and t_2 changing their connections between the facilities.

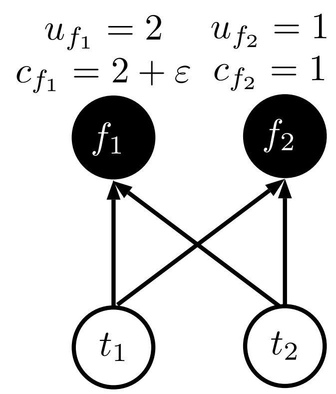

Figure 7: Game without a Nash equilibrium. Connection costs have a constant value.

Now suppose we restrict ourselves to instances of this game where a PNE exists. Based in the game of Figure 7, we can construct new instances with unbounded Price of Stability. Consider the game instance in Figure 8, where each player $i \in [1,5]$ controls the terminal t_i. Note that the subgraph induced by terminals t_4, t_5 is a game instance with no equilibrium, unless some amount of the opening cost of f_4 is paid by an external terminal. In fact, in order for this instance to have an equilibrium, terminal t_3 must connect to f_4, paying at most 1 of its opening cost. In an equilibrium, t_2 must not connect to f_3, since doing so t_3 would also choose f_3 and thus t_4

and t_5 would not reach an equilibrium. Terminal t_2 therefore must connect to f_2, and since this facility has unitary capacity, t_1 must choose facility f_1 with unbounded opening cost. This is the only possible equilibrium in this instance, therefore the PoS is unbounded.

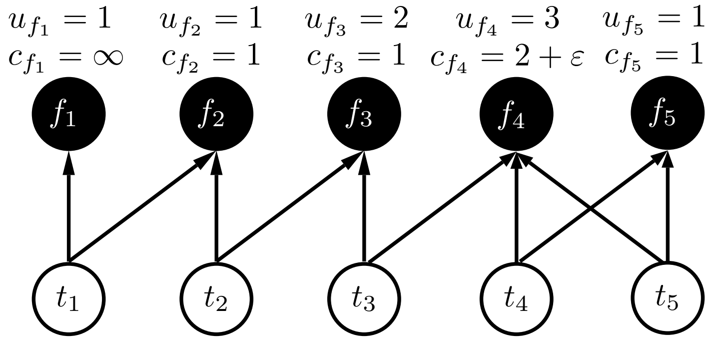

Figure 8: Game with unbounded Price of Stability.

□

When dealing with capacities, there is also a possibility of a softer approach. For example we can consider a game where the player cost function has and additional cost that increases with the number terminals using a same facility. In this variant, the cost of a terminal t connecting to a facility f is defined as $\frac{c_f}{x_f} + g(x_f) + d_{tf}$, where x_f is the number of terminals connected to f.

If the added cost function $g(x)$ is monotone increasing, and the opening function cost $c_f(x)$ is monotone increasing and concave then Anshelevich et. al [1] proved, for the network design game, that the Price of Stability is bounded by $A \times H(k)$. The parameter A depends on the type of the function g. For functions with polynomial degree at most l, this term is equal to $l + 1$. This bound extends for the facility location game with fair cost allocation and with the additional $g(x)$ costs. This occurs because the proof for such bound is based on the potential function of the network design game, and the distance costs d_{tf} that appear in the facility location game do not interfere in the proof presented in [1].

5 Altruism in Facility Location Games

All analysis seen so far for facility location games assume that players are completely selfish. However, this assumption does not always reflect what happens in practice. Players behavior in practice may be at least partially altruistic [14, 16], indicating a need to incorporate this alternate behavior for games modeling real world scenarios. In light of this, in recent years there has been increasing interest in the study of alternate models on how players behave. A model for altruistic behavior is presented by Chen et al. [5]. It changes how players perceive utility by adding a α_i parameter for each player i indicating how selfless a player behaves.

Definition 4 (Altruism)**.** Let $G = (K, (S_i)_{i\in K}, (p_i)_{i\in K})$ be a cost minimization game, where $K = [k]$ is the set of k players, $\mathcal{S} = S_1 \times ... \times S_k$ is the set of all possible strategy profiles, where S_i is the set of possible strategies for player i. The payment function $p_i : \mathcal{S} \to \mathbb{R}$ defines the cost of a strategy profile for player i.

The *α-altruistic extension* of G, for $\alpha \in [0, 1]^k$, is defined as the strategic game $G^\alpha = (K, (S_i)_{i\in K}, (p_i^\alpha)_{i\in K})$, where for every player i and $S \in \mathcal{S}$,

$$p_i^\alpha(S) = (1 - \alpha_i)p_i(S) + \alpha_i C(S) \ , \tag{3}$$

where $C : \mathcal{S} \to \mathbb{R}$ is a function mapping strategy profiles to a real number that represents a social costs. This function must satisfy the property that for any $S \in \mathcal{S}$, $C(S) \leq \sum_{i\in K} p_i(S)$.

The function $p_i^\alpha(S)$ represents the perceived cost of a strategy profile S for a player i, while $C(S)$ determines the total social cost for the game G. Note that using this model, when α_i is zero, player i is completely selfish, while a player i is completely altruistic when $\alpha_i = 1$. Therefore, if $\alpha = [0]^k$, the α-altruistic extension G^α is equal to the original game G. We say that a game G^α is *uniformly α-altruistic* when for any player i, $\alpha_i = \alpha$.

Chen et al. in [4] analyze a few classes of games using this altruistic model. Their analysis extends the definition of smooth games to incorporate altruism. Before using their model, we present some important definitions for smooth games [22].

5.1 Smooth Games and Altruism

The notion of smooth games, first defined by Roughgarden in [22], is an important tool in the analysis of inefficiency in games. It provides bounds not only for pure and mixed equilibria, but also for both correlated and coarse correlated equilibria.

Definition 5 (Smooth Games)**.**
A cost minimization game $G = (K, (S_i)_{i\in K}, (p_i)_{i\in K})$ is (λ, μ)*–smooth* if for any strategy profiles $S, S^* \in \mathcal{S}$,

$$\sum_{i\in K} p_i(S_i^*, S_{-i}) \leq \lambda \cdot C(S^*) + \mu \cdot C(S). \tag{4}$$

where $C : \mathcal{S} \to \mathbb{R}$ is again mapping strategy profiles to social costs such that for any $S \in \mathcal{S}$, $C(S) \leq \sum_{i\in K} p_i(S)$.

If a minimization game is (λ, μ)–smooth then it is possible to assert several facts about such game. Among them, a bound for the price of anarchy. If a game is (λ, μ)–smooth, with ($\lambda \geq 0$ and $\mu < 1$), then every equilibria S has cost at most $\frac{\lambda}{1-\mu}$ times that of an optimal solution S^*.

The *robust price of anarchy* is defined as the best upper bound that is possible to prove using smoothness analysis.

Definition 6 (Robust Price of Anarchy)**.** The *robust price of anarchy* of a cost-minimization game is defined as

$$\inf\left\{\frac{\lambda}{1-\mu} : (\lambda, \mu) \text{ s.t. the game is } (\lambda, \mu)\text{-smooth}\right\} \quad , \tag{5}$$

where $\lambda \geq 0$ and $\mu < 1$.

Note that Definition 5 can be relaxed by allowing the inequality to hold only for an optimal solution S^* and all other strategy profiles S, while still retaining the properties based on the smoothness property [22].

In [5], the definition of smooth games is extended to incorporate altruism, while maintaining most of the properties proved for the original concept.

Definition 7 ((λ, μ, α)-smoothness)**.** Let G^α be a α-altruistic game with social cost function C. G^α is (λ, μ, α)-smooth iff for any two strategy profiles $S, S^* \in \mathcal{S}$, the following is satisfied:

$$\sum_{i \in K}\Big[p_i(S_i^*, S_{-i}) + \alpha_i(C_{-i}(S_i^*, S_{-i}) - C_{-i}(S))\Big] \leq \lambda C(S^*) + \mu C(S) \quad , \tag{6}$$

where $C_{-i}(S) = C(S) - p_i(S)$ and for any S, $C(S) \leq \sum_{i \in K} p_i(S)$.

If a game is (λ, μ, α)-smooth with $\mu < 1$, then the price of anarchy of the game is at most $\frac{\lambda}{1-\mu}$, even for coarse correlated equilibria.

For facility location games, altruistic PoA and PoS bounds for valid utility games [25] and for fair–cost allocation games have been shown in [5].

5.2 Fair Cost Allocation Games with Altruism

A similar game to the Facility Location Game with Fair Cost Allocation (FLG-FC) seen in Section 3 has been considered in [5]. **QUAL O NOME DO JOGO ANALISADO LA???** In this game, there are no connection costs between terminals and facilities. Furthermore, while in the FLG-FC each terminal may choose any facility to open, in the game analyzed in [5] each player has to connect his clients to some subset of facilities given as an input to the game. Below we adapt the results in [5] to FLG-FC game.

Recall the FLG-FC specified in Definition 3. Here we use $d(S_i) = \sum_{(t,f) \in S_i} d_{tf}$ as the sum of all connection costs for a player i in strategy S_i, and $U(S) = \bigcup_{i \in K} S_i$ as the set with all strategies in the strategy profile S.

Theorem 9 (DRAFT Price of Anarchy for α–altruistic FLG-FC)**.** *For any FLG-FC game G with k players, the α–altruistic extension G^{α} is $(k, \hat{\alpha}, \alpha)$–smooth, where $\hat{\alpha} = \max_{i \in K} \alpha_i$.*

Proof. Let S and S^* be two strategy profiles for G. Fix an arbitrary player $i \in K$. Then,

$$\begin{aligned} C(S_i^*, S_{-i}) - C(S) &= \sum_{f \in U(S_i^*, S_{-i})} c_f + \sum_{j \in K: j \neq i} d(S_j) + d(S_i^*) - \sum_{f \in U(S)} c_f - \sum_{j \in K} d(S_j) \\ &\leq \sum_{f \in S_i^* \setminus U(S)} c_f + \sum_{(t,f) \in S_i^* \setminus U(S)} d_{tf} \ . \end{aligned}$$

This inequality can be used to establish the following bound:

$$\begin{gathered} (1 - \alpha_i) p_i(S_i^*, S_{-i}) + \alpha_i (C(S_i^*, S_{-i}) - C(S)) \\ \leq \\ (1 - \alpha_i) \left(\sum_{f \in S_i^*} \frac{c_f}{x_f(S_i^*, S_{-i})} + d(S_i^*) \right) + \alpha_i \left(\sum_{f \in S_i^* \setminus U(S)} \frac{c_f}{x_f(S_i^*, S_{-i})} + \sum_{(t,f) \in S_i^* \setminus U(S)} d_{tf} \right) \\ \leq \\ \sum_{f \in S_i^*} \frac{c_f}{x_f(S_i^*, S_{-i})} + d(S_i^*) \leq k \left(\sum_{f \in S_i^*} \frac{c_f}{x_f(S^*)} + d(S_i^*) \right) . \end{gathered}$$

Note that the first inequality follows from the fact that $x_f(S_i^*, S_{-i}) = 1$ for every facility that is in strategy profile S^* but not in strategy profile S. The last inequality follows from $x_f(S_i^*, S_{-i}) \geq x_f(S^*)/k$ for every $f \in S_i^*$.

From this, we can conclude that the game is $(k, \hat{\alpha}, \alpha)$–smooth as defined in Definition 7:

$$\begin{gathered} \sum_{i \in K} \Big[p_i(S_i^*, S_{-i}) + \alpha_i (C_{-i}(S_i^*, S_{-i}) - C_{-i}(S)) \Big] = \\ \sum_{i \in K} \Big[p_i(S_i^*, S_{-i}) + \alpha_i (C(S_i^*, S_{-i}) - p_i(S_i^*, S_{-i}) - C(S) + p_i(S)) \Big] = \\ \sum_{i \in K} \Big[((1 - \alpha_i) p_i(S_i^*, S_{-i}) + \alpha_i (C(S_i^*, S_{-i}) - C(S)) + \alpha_i p_i(S)) \Big] \leq \\ k \sum_{i \in K} \left(\sum_{f \in S_i^*} \frac{c_f}{x_f(S^*)} + d(S_i^*) \right) + \sum_{i \in K} \alpha_i p_i(S) \leq \\ k C(S^*) + \hat{\alpha} C(S) \ . \end{gathered}$$

$\square$

UMA COISA QUE PERCEBI AGORA É QUE NA ULTIMA DESIGUALDADE ACIMA VOCE USA O FATO DE QUE $\sum_{i\in K} p_i(S) \leq C(S)$ MAS EM TODAS AS SUPOSICOES MOSTRADAS ANTES ERA ASSUMIDO QUE $C(S) \leq \sum_{i\in K} p_i(S)$. CHECAR SE A SUPOSICAO EH ESTA MESMO.

Corollary 10 (Robust Price of Anarchy for α-altruistic FLG-FC)**.** *The robust price of anarchy (RPoA) of α-altruistic FLG-FC games is at most $\frac{k}{1-\hat{\alpha}}$, where $\hat{\alpha} = \max_{i\in P} \alpha_i$, and there is an α-altruistic instance of FLG-FC with RPoA $\frac{k}{1-\hat{\alpha}}$.*

Proof. As seen in Theorem 9, for any instance G of the FLG-FC game, the α-altruistic extension G^α is $(k, \hat{\alpha}, \alpha)$-smooth and therefore has a robust price of anarchy of $\frac{k}{1-\hat{\alpha}}$.

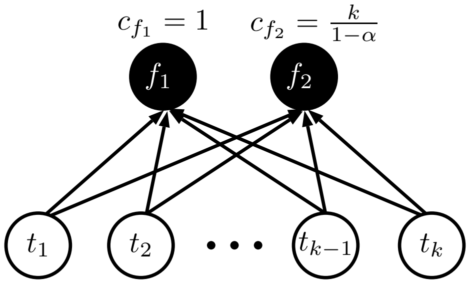

Figure 9: Instance of FLG-FC with PoA equal to $\frac{k}{1-\alpha}$ where every player is α-altruistic. Connections cost are equal to zero.

To show that this bound is tight, even for pure Nash equilibria, we can slightly alter the example of Figure 3. Instead of a facility with cost equal to the number of players, we now have a facility with cost $\frac{k}{1-\alpha}$, with every player being uniformly α-altruistic, as shown in Figure 9. In this instance each player $i \in [1, k]$ controls a terminal t_i, and can choose between facilities f_1 with cost 1 and f_2 with cost $\frac{k}{1-\alpha}$. Consider the strategy profile $S^* = ((t_1, f_1), ..., (t_k, f_1))$ where every player chooses f_1 and $S = ((t_1, f_2), ..., (t_k, f_2))$ where f_2 is chosen by all players. Clearly S^* is the strategy profile with optimal social cost, with $C(S^*) = 1$, while S has cost $C(S) = \frac{k}{1-\alpha}$. The strategy profile S is a pure Nash equilibrium of the uniformly α-altruistic extension G^α, since for any player i,

$$p_i^\alpha(S) = (1-\alpha)p_i(S) + \alpha C(S) = 1 + \alpha\frac{k}{1-\alpha} = p_i^\alpha((t_i, f_1), S_{-i}) \ .$$

Therefore, the price of anarchy of G^α is at least $\frac{k}{1-\alpha}$, and the bound for the robust price of anarchy for the α-altruistic extension of FLG-FC is tight. $\square$

The Pure Price of Stability for uniformly α-altruistic games can be determined in a similar way as was done in Theorem 6, for a bound of $(1-\alpha)H_k + \alpha$.

Theorem 11 (Pure Price of Stability for α-altruistic FLG-FC)**.** *The Pure Price of Stability for uniformly α-altruistic fair cost facility location games is at most* $(1-\alpha)H_k + \alpha$.

Proof. Let G^α be an uniformly α-altruistic facility location game. Then it is a potential game with potential function

$$\Phi^\alpha(S) = (1-\alpha)\Phi(S) + \alpha C(S) \ ,$$

where

$$\Phi(S) = \sum_{f \in F} \sum_{x=1}^{x_f(S)} \frac{c_f}{x} + \sum_{(t,f) \in S} d_{tf} \ .$$

We have that

$$\Phi^\alpha(S) = (1-\alpha)\left(\sum_{f \in F} \sum_{x=1}^{x_f(S)} \frac{c_f}{x} + \sum_{(t,f) \in S} d_{tf}\right) + \alpha C(S) \leq$$

$$((1-\alpha)H_k + \alpha)\left(\sum_{f \in S} c_f + \sum_{(t,f) \in S} d_{tf}\right) = ((1-\alpha)H_k + \alpha)C(S) \ .$$

Let S^* be the social optimum strategy profile. From this strategy, we can derive an equilibrium S by best response dynamics such that $\Phi^\alpha(S^*) \geq \Phi^\alpha(S)$. Since $C(S) \leq \Phi^\alpha(S)$, we have

$$C(S) \leq \Phi^\alpha(S) \leq \Phi^\alpha(S^*) \leq ((1-\alpha)H_k + \alpha)C(S^*) \ ,$$

which means that the price of stability is at most $((1-\alpha)H_k + \alpha)$.

$\square$

NA FUNCAO POTENCIAL ACIMA VOCE FAZ O SOMATORIO SOBRE $f \in F$. DEVERIA SER SOBRE $f \in U(S)$????

We note that in this section a tight bound of a linear function ok k was given for the robust price of anarchy, while for the price of stability a considerably more restricted bound was proven (considering only games with uniform altruism). It may be the case that this bound can indeed be much closer to the optimal social welfare if completely altruistic players are mixed with selfish players. For example, in the instance in Figure 5, used to prove the tightness of the bound $H(k)$ for the PoS of FLG-FC, if the player controlling terminal t_k is completely altruistic, the best pure Nash equilibrium is the optimal solution.

We analysed in this section altruistic versions of the FLG-FC game. It remains an open problem to explore altruism for the facility location game without cost sharing rules. An interesting question for these games, where there are instances with no pure Nash equilibria, is whether a certain amount of altruism in the game can guarantee the existence of such equilibrium. And if altruism can guarantee the existence of equilibrium, how this altruistic behaviour must be distributed between the players. It may be the case that a certain amount of completely altruistic players are needed to guarantee PNE existence, or that every player must be at least a α amount altruistic.

6 Conclusions

In this survey, we combined results from several works relating to facility location games. We focused on proving existence for pure Nash equilibria and bounds for the price of anarchy and stability.

For facility location games without cost sharing rules, we presented results from Cardinal and Hoefer [2] for the uncapacitated version, and provided examples for the capacitated game proving unbounded PoA, PoS and instances with no PNE even when players control only a single terminal. For fair cost sharing facility location games, we adapted well known results from Anshelevich et al. [1, 3] for the network design game. In these new proofs we need to take in consideration the additional connection costs and show that they do not interfere in the bounds for equilibria. For the capacitated version, we prove that the PoA can be unbounded, while the PoS has the same bound as the uncapacitated version. Furthermore, we analyze facility location using an altruistic model for player behavior [5], and adapted recent results for the fair cost sharing game to facility location games. A summary of the known bounds are presented in Table 1.

Table 1: Known results for Facility Location Games

Game	PNE	PoA	PoS	First seen
FLG (Facility Location Game)	$\times$	k	$k-2$	[2]
Capacitated FLG	$\times$	∞	∞	–
FLG-FC (Fair Cost Allocation)	✓	k	$H(k)$	[1]
FLG-FC with Weights	?	$\Theta(\log W)$	$\Theta(\log W)$	[10, 11]
Capacitated FLG-FC	✓	∞	$H(k)$	–
Altruistic FLG-FC	✓	$\frac{k}{1-\hat{\alpha}}$	$((1-\alpha)H(k)+\alpha)$	[5]

While several bounds for these games can be adapted directly from network design and fair cost games, some questions remain undiscovered. One example is the question of the existence of PNE for weighted FLG-FC. The examples of instances with no equilibria for network design do not translate to instances of FLC-FG. and there is still no proof for exact equilibria. **NAO ENTENDI O QUE VC QUIS DIZER POR EXACT EQUILIBRIA**

Since the addition of hard capacities to facility location games can lead to unbounded PoA and PoS, other models that impose capacities in facility location games may be of interest. Nonetheless, few have been explored in the context of facility location.

Another interesting question is to consider altruism for facility location games outside of the fair cost sharing model, for example with no cost sharing rules. Several questions can be explored with altruism, specially in games with no guaranteed equilibria.

Finally, while there are some research in altruism in the context of location games, no results are known when spiteful behaviour from players are considered. We note however, that there are still no completely accepted model for spiteful player behaviour. If we model players as completely spiteful, selfish or altruistic, perhaps some results can be extended from distributed networks, as disrupting agents may be considered completely spiteful players. If someone models spite in a similar manner to altruism, then solution feasibility is another possible concern.

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