Non-Cooperative Capacitated Facility Location Games☆

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Abstract

We study capacitated facility location games, where players control terminals and need to connect each one to a facility from a set of possible locations. There are opening costs and capacity restrictions for each facility. Also, there are connection costs for each pair of facility and terminal if such facility attends this terminal. This problem has several applications, especially in distributed scenarios where a central authority is too expensive or even infeasible to exist. In this paper, we analyze and present new results concerning the existence of equilibria, Price of Anarchy (PoA), and Stability (PoS) for metric and non-metric versions of this game. We prove unbounded PoA and PoS for some versions of the game, even when sequential games are allowed. For metric variants, we prove that sequentiality leads to bounded PoA and PoS.

Keywords: Algorithmic Game Theory, Facility Location, Graph Algorithms, Price of Anarchy, Theory of Computation

1. Introduction and Notation

In game theory, a *non–cooperative game* is a scenario where players or agents choose strategies independently and try to either minimize their costs or maximize their utility. For each player i there is a set A_i of actions that it can choose to play. A pure strategy S_i consists of one action from A_i, while a mixed strategy corresponds to a probability distribution over A_i. In this paper we assume players pick pure strategies unless mentioned otherwise. A strategy profile, denoted by $S = (S_1, \ldots, S_k)$, corresponds to a solution of the game where each player $i = 1, \ldots, k$ chooses a strategy S_i.

We consider capacitated facility location games with and without a cost sharing scheme, which means that the cost to open a facility can be divided equally among all terminals connected to it (fair cost sharing) or it can be divided without any rules (no cost sharing rules).

Now we give formal definitions of the games considered in this paper. Let $G = (T \cup F, T \times F)$ be a bipartite graph, with vertex sets F of n facilities and T of m terminals. Each facility $f \in F$ has an opening cost c_f and a capacity u_f indicating how many terminals can be connected to f at any given time. Furthermore, there are connection costs d_{tf} for each pair terminal $t \in T$ and facility $f \in F$. In games with general distance costs, some connections (t, f) should be avoided in any solution, because they don't exist for example. In this case we assume they have a prohibitively large constant cost $\mathcal{U}_d$. When a connection is not shown, it is assumed that it has a cost equal to $\mathcal{U}_d$, unless mentioned otherwise. Let $K = [1, \ldots, k]$ be the set of players. Each player i controls a subset of terminals $T_i \subseteq T$ forming a partition of T, and each terminal must be connected to exactly one opened facility. When a player controls only a single terminal he is denominated a *singleton* player.

In the *Capacitated Facility Location Game with no cost sharing rules (CFLG)*, the set of actions A_i of player i is composed by tuples $(\mathcal{F}_i, p_i^c)$ where $\mathcal{F}_i : T_i \to F$ maps each terminal i controls to a facility, and $p_i^c : F \to \mathbb{R}_0^+$ maps the amount i pays to open facility f if some of its terminals is connected to it. Given some strategy S_i chosen by i, we simplify the notation by writing $(t, f) \in S_i$ to represent each connection i choose to its terminals. Likewise we write $f \in S_i$ to represent each facility where some terminal of i is connected to. The total amount paid by player i in strategy profile S is

$$p_i(S) = \sum_{f \in S_i} p_i^c(f) + \sum_{(t,f) \in S_i} d_{tf} \ .$$

Let $p^c(f) = \sum_{i=1}^{k} p_i^c(f)$ be the total paid by players for a facility f. If $p^c(f)$ is greater than or equal to the cost c_f, then the facility f is considered opened. Each player tries to minimize his payment. We denote the number of players connected to a facility f in a solution S by $x_f(S) = |\{1 \le i \le k : f \in S_i\}|$.

Solutions where there are more terminals connected to some facility f than its capacity u_f should be avoided. Should also be avoided solutions where terminals do not pay enough to open the facility they are connected to. To

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avoid such solutions we add a prohibitively large constant cost $\mathcal{U}_c$ to the payment of terminals in such situations. For a player i, if there is a connection $(t, f) \in S_i$ where $p^c(f) < c_f$ or the number of players connected to f is greater than its capacity ($x_f(S) > u_f$), a prohibitively large constant cost $\mathcal{U}_c > \mathcal{U}_d$ is added to the total amount paid by i, i.e, he pays $p_i(S) + \mathcal{U}_c$.

For *Capacitated Facility Location Games with Fair-Cost sharing (CFLG-FC)*, a player i chooses a strategy $S_i \subset T_i \times F$ such that in S_i each terminal controlled by i is connected to exactly one facility. Let $S = (S_1, ..., S_k)$ be a strategy profile. Each player tries to minimize his own payment

$$p_i(S) = \sum_{f \in S_i} \frac{c_f}{x_f(S)} + \sum_{(t,f) \in S_i} d_{tf} \ ,$$

where $x_f(S) = |\{1 \le i \le k : f \in S_i\}|$ is the number of players using facility f in strategy profile S. Again, to ensure that capacity restrictions are respected, if a player i in the solution S has one of his terminals connected to f where $x_f(S) > u_f$, then a prohibitively large constant cost $\mathcal{U}_c$ is added to the payment of player i, i.e, he pays $p_i(S) + \mathcal{U}_c$.

Let $S_{-i} = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ be a strategy profile S without i's strategy, so that we can write $S = (S_i, S_{-i})$. *Pure Nash Equilibria* (PNE) are strategy profiles where no player can decrease his own costs by unilaterally changing his strategy, i.e., S is a PNE if for each player i, $p_i(S_i, S_{-i}) \le p_i(S'_i, S_{-i})$ for all $S'_i \in A_i$.

The *social cost* is a function mapping a strategy profile to a real number, indicating a measure of the total cost of a game. We use the expression $f \in S$ to represent all facilities connected to a terminal in a strategy profile S, and $(t, f) \in S$ to represent all connections established in S. The social cost of a strategy profile S is defined for this game as the sum of all player payments, i.e.

$$C(S) = \sum_{i \in K} p_i(S) = \sum_{f \in S} c_f + \sum_{(t,f) \in S} d_{tf} \ . \qquad (1)$$

Two of the most important concepts for efficiency analysis are the *Price of Anarchy* (PoA) and the *Price of Stability* (PoS). The PoA is the ratio between a Nash equilibrium with worst possible social cost and the strategy profile with optimal social cost, while the PoS is the ratio between the best possible Nash equilibrium and the social optimum. In the facility location games analyzed in this paper, the optimal social cost is the cost of an optimum solution for the corresponding optimization version of the problem.

Solution concepts such as pure Nash equilibria usually assume that players choose strategies simultaneously. This requirement can lead to unintuitive equilibria for facility location games where players choose to open expensive facilities when cheaper ones are also available. A possibility to take sequential movements in consideration is to analyze these games as *sequential games* [1, 2]. In these games, players choose their strategies in a predefined arbitrary order. In the sequential facility location games considered in this paper, we assume each player $i \in [1, k]$ chooses a strategy only once given all strategies chosen by players before, so player 1 chooses first then player 2, and so on until player k.

An alternative solution concept that aims to better represent such scenarios is *Subgame Perfect Equilibrium* (SPE). Sequential games are usually represented as extensive form games, in the form of a game tree where each node represents a player and edges represent possible actions from the player on that node. SPE is defined as a strategy profile which is a PNE in every subgame of this game tree, so a SPE is also a PNE for the entire game. The *Sequential Price of Anarchy* (SPoA) is defined as the ratio between the cost of the worst subgame perfect equilibrium and the optimal social cost, while the *Sequential Price of Stability* (SPoS) is the ratio between the best SPE and the optimal social cost. One important aspect of such games is that they always posses a SPE which can be computed using a method called backward induction. For further details on SPE and extensive form games see Chapter 4 of [2].

2. Related Work and Contributions

Facility location has been analyzed in a game-theoretic perspective from several directions. From mechanism design and strategy-proof mechanisms [3, 4, 5], to cooperative facility location [6] and valid utility games [7]. When there is competition between facilities to dominate markets, facilities may be modeled as players in a game-theoretic setting. These facility location problems are described as competitive location [8], with several relevant results in the literature [9, 10, 11].

In this paper we consider only the case where players control terminals, where each one requires a connection to an open facility. These facility location games can be viewed as connection games where every player starts from a single source vertex on a two-layered directed graph, and therefore multiple results for the uncapacitated versions of facility location games can be adapted from connection and network creation games.

For uncapacitated facility location games with fair cost sharing rules, most results can be adapted from cost sharing games and network design [12]. The PoA and PoS can be proven to be k and $H_k = \Theta(\log k)$, respectively, the same bounds obtained for network design [12].

For the metric version of this game Hansen and Telelis [13, 14] proved constant bounds both for the PoS and the strong PoA. For the non-metric case they proved a bound of $\Theta(\log k)$ for both the PoS and strong PoA. When players have no rules on how to share opening costs, the PoA and PoS have been proven to be $\Theta(k)$ [15] for uncapacitated facility location. If players are allowed to control more than a single terminal, there are games with no PNE and it is NP-hard to know if an instance has a PNE [15, 16]. When all players are singletons, a PNE with optimal social cost is guaranteed to exist for network design [17], which implies that the same is true for uncapacitated facility location games with no cost sharing rules.

On the other hand, few results are known for capacitated facility location games. One of the few results for this case is the one of Feldman and Ron [18] who proved that the PoA is unbounded for capacitated network design games unless the network topology is series-parallel. Another work is the one of Erlebach and Radoja [19] for the capacitated network design game. They prove an upper bound of $O(n)$ for the PoS assuming symmetric games, and a lower bound of $\Omega(n \log n)$ for the asymmetric version of the game.

The Sequential Price of Anarchy was used by Paes Leme et al. [1] to analyze machine cost sharing games, a game similar to facility location games. They argue that some equilibria found on these games, facility location games included, require some "unnatural" coordination from players, such as choosing a machine with high cost when there are much cheaper machines available. In the machine game, each job must choose one machine r to be scheduled, and jobs in the same machine share its cost. So if x jobs are scheduled on machine r, each one pays c_r/x, where c_r is a fixed cost of r. For this machine cost sharing game, in a special case of generic costs, a bound of $\Theta(\log k)$ is given for the SPoA. For general costs, Bilò et al. [20] prove that the SPoA is at least $(k+1)/2$. The same reasoning used by Leme et al. can be used for the capacitated facility location games, leading to question whether the SPoA can be better than the PoA for CFLG and CFLG-FC.

In this paper, for the CFLG (no cost sharing rules) we present original examples proving unbounded PoA, PoS and instances with no PNE even when players control only a single terminal. We prove that it is NP-hard to determine if an instance of CFLG has a PNE, even when all players are singletons. In addition, we prove that even when sequentiality is considered, the SPoA and SPoS are still unbounded for the general game. In the case of CFLG-FC (fair cost sharing), while the PoS is bounded by H_k, we show instances of games where the PoA, the SPoA and even the SPoS are unbounded.

We also consider the metric versions of the CFLG and CFLG-FC games. While for the Metric CFLG there are instances with no PNE, for the Metric CFLG-FC pure Nash equilibrium always exist. For both versions, with and without cost sharing rules, we show that there are scenarios where an equilibrium with unbounded cost exists, making the PoA also unbounded. For the PoS, SPoA and SPoS we prove upper and lower bounds showing that all of them are $\Theta(2^k)$.

A summary of the proved bounds is presented in Table 1. We note that the H_k bound for the PoS in the Metric CFLG-FC is not proven tight, while the others are.

Table 1: Results for Capacitated Facility Location Games. Unbounded results are represented by $\mathcal{U}$.

Game	PNE	PoA	SPoA	PoS	SPoS
Metric CFLG	$\times$	$\mathcal{U}$	$\Theta(2^k)$	$\Omega(2^k)$	$\Theta(2^k)$
CFLG	$\times$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$
Metric CFLG-FC	✓	$\mathcal{U}$	$\Theta(2^k)$	H_k	$\Theta(2^k)$
CFLG-FC	✓	$\mathcal{U}$	$\mathcal{U}$	H_k	$\mathcal{U}$

3. Metric Version of Capacitated Facility Location Games

A common restriction for location games is to require all connections to obey the triangle inequality. In this section we present results regarding this version of the game.

3.1. Existence of PNE

First of all we consider the existence of PNE for games with no cost sharing rules.

Proposition 1 (PNE existence for the Metric CFLG)**.** *There are instances of the Metric CFLG that do not admit a PNE, even when restricted to singleton players.*

Proof. Consider the game in Figure 1 where one player controls terminal t_1 and another one controls t_2. In a situation where both players are connected to f_1 at least one of them is paying a value greater than 1, so this player has an incentive to move to f_2. So suppose one player is connected to f_2, and suppose it is t_2. Then terminal t_1 must open f_1 paying the full opening cost $2+\varepsilon$. However, in this situation t_2 can just connect to f_1 without paying any opening costs. But in this case t_1 has an incentive to move to f_2 since its opening cost is 1. So the game is in a similar situation as in the beginning but with t_1 and t_2 connected to the opposite facilities. So it is easy to see that this game does not admit a PNE.

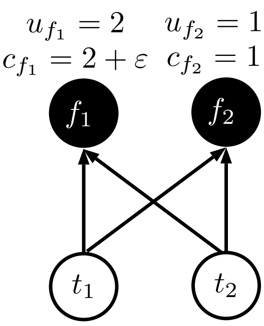

Figure 1: Game instance of the Capacitated FLG without a PNE. Connection costs have a constant value.

□

Note that this is not necessarily the case for the uncapacitated version, where results from Hoefer [16] and Anshelevich et. al [17] indicates that for singleton players a PNE with optimal social cost always exists. For the

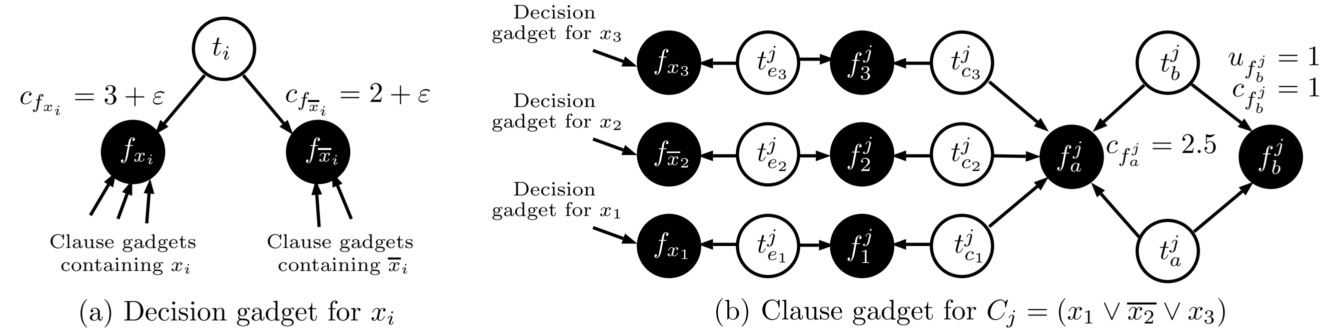

Figure 2: Decision variable gadget (a) and clause gadget (b). In (a) we assumed that the literal x_i occurred 3 times in the formula and the literal $\overline{x}_i$ occurred 2 times. All opening costs are equal to 1 unless otherwise mentioned. All connection costs for the drawn edges are equal to 1, and any edge (t, f) not drawn has cost equal to the shortest path cost from t to f.

capacitated game we can use the construction of Figure 1 to show that is NP-complete to determine if an instance of the Metric CFLG has a PNE or not when all players are singletons, and NP-hard otherwise. The argument for this proof derives from the hardness proof by Cardinal and Hoefer for vertex cover games and uncapacitated facility location games [15], where some players may control more than a single terminal. However in our case the reduction is done using exclusively singleton players.

Theorem 1. *It is NP-hard to determine if an instance of the Metric CFLG has a PNE or not, even when all players are singletons. When restricted to instances with only singleton players, it is NP-complete to determine if an instance of the Metric CFLG has a PNE.*

Proof. First notice that it is easy to verify if a given solution S to an instance of the CFLG is a PNE or not when all players are singletons, and so CFLG belongs to NP when restricted to these instances. Now we present a reduction from the 3-SAT problem to the problem of determining if an instance of the Metric CFLG has a PNE or not. We transform an instance from the 3-SAT as follows: for each decision variable x_i we introduce a terminal t_i and facilities f_{x_i} and $f_{\overline{x}_i}$ which t_i can connect as shown in Figure 2a. For each clause C_j we introduce a gadget with several terminals and facilities as shown in Figure 2b. For each literal of C_j there is an end terminal $t^j_{e_i}$ connecting to its respective facility f_{x_i} or $f_{\overline{x}_i}$ in the decision variable gadgets. For each literal of C_j there is also a center facility f^j_i where $t^j_{e_i}$ can connect to, and a center terminal $t^j_{c_i}$ that can connect to either the center facility f^j_i or to facility f^j_a. Note that terminals t^j_a and t^j_b and facilities f^j_a and f^j_b correspond to the instance in Figure 1. Each terminal is controlled by a single player. The cost of facility f_{x_i} (respectively $f_{\overline{x}_i}$) is equal to the number of occurrences of the literal x_i (respectively $\overline{x}_i$) in all clauses plus a small constant ε. Facilities f^j_a have an opening cost of 2.5 while all other facilities have unitary opening costs. All connections shown have cost equal to 1, while any edge (t, f) not shown has cost equal to the shortest path cost from t to f. For each clause C_j, facility f^j_b has unitary capacity restriction, while all others are unrestricted. Note that the facilities f_{x_i} and $f_{\overline{x}_i}$ are shared between all clauses that have one of the two literals x_i or $\overline{x}_i$, since the corresponding end terminals are connected to one of them, and they are also shared with the corresponding terminal t_i of the variable gadget.

Suppose there is a truth assignment for a given instance of the 3-SAT. Then we can construct a PNE to the corresponding CFLG instance as follows: if $x_i = 1$ connect terminal t_i to facility f_{x_i} paying ε to open it, otherwise connect t_i to $f_{\overline{x}_i}$ also paying ε to open it. For each clause C_j where literal x_i appears, connect its end terminal $t^j_{e_i}$ to f_{x_i} if $x_i = 1$, and connect $t^j_{e_i}$ to the center facility f^j_i otherwise. In both cases the end terminal pays 1 to open the facility it is connected to. Do the same thing for clauses with the literal $\overline{x}_i$. Finally each center terminal $t^j_{c_i}$ is connect to f^j_a if its respective literal is true, and pays 0.5 of its opening cost. If its corresponding literal is false then $t^j_{c_i}$ is connected to the center facility f^j_i paying nothing, since the end terminal $t^j_{e_i}$ payed to open it. Since the assignment is truth, then for each clause there is at least one center terminal connected to f^j_a paying 0.5 of its opening cost, and therefore the clause gadget can be stabilized with terminals t^j_a and t^j_b connecting to f^j_a as well.

Now suppose there is a PNE in the constructed instance. The gadget of a clause C_j is in a PNE only if some center terminal $t^j_{c_i}$ is connected to facility f^j_a paying at least 0.5 of its opening cost. In this case terminals t^j_a and t^j_b can also be connected to f^j_a remaining in equilibrium. Since terminal $t^j_{c_i}$ is connected to f^j_a, this means that its corresponding center facility f^j_i is closed. Then we must have the end terminal $t^j_{e_i}$ connected to its decision variable facility paying at most 1 of its opening cost. Since we have a PNE, then at least one center terminal $t^j_{e_i}$ of each clause is connected to its decision variable.

The truth assignments to the 3-SAT variables are done as follows: if the end terminal $t^j_{e_i}$ is connected to its corresponding decision variable then its corresponding literal is set to true, otherwise it is set to false. Notice that the PNE implies that at least one literal of each clause is set to true.

Now we have to prove that this assignment is consistent, i.e, there is no variable x_i with $x_i = 1$ and $\overline{x}_i = 1$, or $x_i = 0$ and $\overline{x}_i = 0$. Consider the decision variable gadget corresponding to x_i. In order for a facility f_{x_i} or $f_{\overline{x}_i}$ to be opened, terminal t_i must pay at least ε of its opening cost, since otherwise its always better for an end terminal $t^j_{e_i}$ to connect to its center facility f^j_i paying 1 of its opening cost. Then exactly one of the two facilities is opened in an equilibrium, and therefore it cannot be the case that $x_i = 1$ and $\overline{x}_i = 1$ at the same time. Now consider $x_i = 0$ and $\overline{x}_i = 0$ and the instance of the game is in equilibrium. Then this variable is irrelevant to obtain a truth assignment of the 3-SAT formula, and we can set either x_i or $\overline{x}_i$ to 1.

□

Now consider the game with fair cost sharing. Note that Rosenthal's potential function [21] can be adapted to model facility location games, and therefore it is not hard to see that CFLG-FC always admit a PNE, since it is a potential game.

Corollary 1 (PNE for the CFLG-FC)**.** *All instances of the CFLG-FC admit a PNE.*

3.2. Bounds for the PoA and SPoA

The known PoA lower bounds for uncapacitated facility location games [15] and network design [12] trivially carry over for the capacitated variants, by setting the capacity of each facility equal to the number of terminals. However, with capacity restrictions we can show that worse equilibria exist, even in the case of fair cost sharing.

Theorem 2 (PoA for Metric CFLG and Metric CFLG-FC)**.** *The PoA for the Metric CFLG-FC is unbounded. For the Metric CFLG, there are instances that admit PNE but have unbounded PoA.*

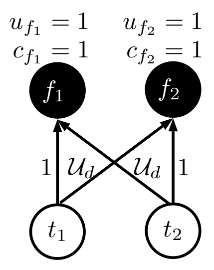

Figure 3: Metric game with unbounded Price of Anarchy.

Proof. Consider the instance depicted in Figure 3. Suppose player 1 controls terminal t_1, while player 2 controls t_2. The solution with optimal social cost is the one where t_1 connects to f_1 and t_2 to f_2. However, suppose that t_1 connects to f_2 and t_2 to f_1, each paying a connection cost $d_{t_1 f_2} = d_{t_2 f_1} = \mathcal{U}_d$. In this scenario, if a terminal switches to the alternative facility, he would pay an extra $\mathcal{U}_c > \mathcal{U}_d$ due to the capacity restrictions, and therefore the terminals are in equilibria. □

While the unbounded state described in Theorem 2 exists, it never arises from players "natural" choice. If we consider sequentiality for this game, there is no scenario where an unbounded equilibrium is reached in the metric variant.

Theorem 3 (SPoA for Metric CFLG and Metric FLG-FC)**.** *Consider an instance of the Metric CFLG-FC (or CFLG) game with k players where*

- *each player i controls one terminal t_i,*
- *players play in order $1, \dots, k$, where player i knows every action taken by players $1, \dots, i-1$,*
- *S is the SPE reached and S^* is a solution with optimum social cost.*

Then the SPoA $\leq 2^k$ and this bound is tight.

Proof. In S there are players connected to the same facilities they are connected to in S^*, and there are players connected to different facilities from the ones they are connected to in S^*. Let A be the set of the latter players.

Among these players in A let a be the last player to connect to some facility f in the solution S. We want to bound the cost of a, given by $p_a(S) = c_f/x_f + d_{af}$ (in case of the CFLG game its some other fraction of the facility cost plus the connection cost), by some value of the optimum cost $C(S^*)$. For that, consider facility f^*_a which is where a is connected to in the solution S^*.

Consider the moment a decided to connect to f in S paying $p_a(S)$ and let $p_a(S^*_a, S_{-a})$ be the amount a would pay if he had connected to f^*_a in strategy profile S. Since a is connected to f in S, then one of the two options must hold: (1) $p_a(S) \leq p_a(S^*_a, S_{-a}) \leq c_{f^*_a} + d_{af^*_a}$ or (2) in the moment a chose to connect to f, f^*_a was full and he would be incurred with cost $\mathcal{U}_c$ to connect to f^*_a. In the first case $p_a(S) \leq c_{f^*_a} + d_{af^*_a} \leq C(S^*)$. In the second case, there must exist another terminal $(a-1)$ which is connected to f^*_a, but that in the optimal solution S^* is connected to a different facility $f^*_{(a-1)}$. This must be true since in S^* there is room in f^*_a for a. We use the notation $a \rightarrow (a-1)$ to indicate that a is not connected to its optimal facility because terminal $(a-1)$ is connected to that facility. Now the same two options (1) and (2) hold for $(a-1)$. Since $(a-1)$ is connected to facility f^*_a, either $p_{(a-1)}(S) \leq p_{(a-1)}(S^*_{(a-1)}, S_{-(a-1)})$ or the facility $f^*_{(a-1)}$ was full with some terminal $(a-2)$ connected to it, but that in S^* is connected to $f^*_{(a-2)}$. Then we have $a \rightarrow (a-1) \rightarrow (a-2)$. We say that the relations $(a \rightarrow (a-1) \rightarrow (a-2))$ form a path in these terminals. This process eventually ends in some terminal, and to simplify notation, assume that it ends in terminal 1 (just rename terminals so that this is true). See Figure 4 for an example.

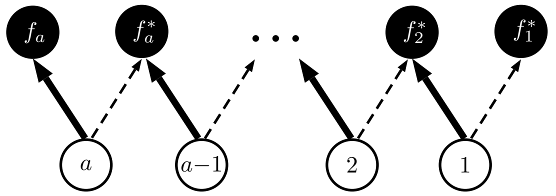

Figure 4: Path $(a \to (a-1) \to \cdots \to 2 \to 1)$, where the solid edges represent players chosen connections in the SPE S and dashed edges represent the connections chosen in the social optimum S^*.

We claim that $d_{if^*_{(i+1)}} \le p_i(S) \le 2^{i-1}C(S^*)$ for each $i = 1, \ldots, (a-1)$ in the path $(a \to (a-1) \to \ldots \to 1)$, where $p_i(S)$ corresponds to the amount terminal i is paying in S to connect to f^*_{i+1}, and $d_{if^*_{(i+1)}}$ is the cost of the edge used in this connection.

We prove the claim by induction on the index i. For the base case, since player 1 is the last of the path and is not connected to f^*_1, we must have that $p_1(S) \le c_{f^*_1} + d_{1f^*_1} \le C(S^*)$. Now consider player i which is paying

$$p_i(S) \le c_{f^*_{i+1}} + d_{if^*_{(i+1)}}$$

but could have connected to f^*_1 with a cost of at most

$$c_{f^*_1} + \sum_{j=1}^{i} d_{jf^*_j} + \sum_{j=1}^{i-1} d_{jf^*_{(j+1)}}.$$

Note that

$$c_{f^*_1} + \sum_{j=1}^{i} d_{jf^*_j} \le C(S^*)$$

since the summation only considers edges of the optimal solution. Now, by hypotheses $d_{jf^*_{(j+1)}} \le 2^{j-1}C(S^*)$. So we must have

$$\begin{aligned} d_{if^*_{(i+1)}} \le p_i(S) &\le c_{f^*_1} + \sum_{j=1}^{i} d_{jf^*_j} + \sum_{j=1}^{i-1} d_{jf^*_{(j+1)}} \\ &\le C(S^*) + \sum_{j=1}^{i-1} 2^{j-1}C(S^*) \\ &= 2^{i-1}C(S^*) \end{aligned}$$

Notice that we can bound the cost of the last edge of the path, edge d_{af_a}, in the same manner, obtaining $d_{af_a} \le p_a(S) \le 2^{a-1}C(S^*)$. So for each terminal i in the path $P = (a \to (a-1) \to \ldots \to 1)$ we have the bound $p_i(S) \le 2^{i-1}C(S^*)$, with a bound for the entire path of $C(P) \le \sum_{i=1}^{a} 2^{i-1}C(S^*) \le 2^a C(S^*)$.

We can now discard these players from the set A and construct paths with remaining players in the set. Note that these paths are vertex-disjoint, since for each player a that could not connect to an optimal facility f^*_a, there must exist exactly one player in A that is connected to f^*_a but that in the optimal solution is not.

For each terminal i that does not belong to the set A, its clear that $p_i(S) \le C(S^*)$ since it is connected in S to the same facility it is connected to in S^*. Consider these terminals as singleton paths.

So let $P_1, P_2, \ldots P_l$ be the paths formed for all terminals. These paths are vertex-disjoint and they satisfy

$$C(P_j) \le 2^{v(P_j)}C(S^*)$$

where $v(P_i)$ is the number of vertices in the path P_i. Since $C(S) = \sum_{i=1}^{k} p_i(S)$, we have $C(S) \le \sum_{j=1}^{l} C(P_j)$ and then

$$\begin{aligned} C(S) &\le \sum_{j=1}^{l} C(P_j) \\ &\le \sum_{j=1}^{l} 2^{v(P_j)}C(S^*) \\ &\le 2^{(v(P_1)+v(P_2)+\ldots v(P_l))}C(S^*) \\ &\le 2^k C(S^*) \end{aligned}$$

where the last inequality holds since the paths are vertex-disjoint.

Therefore, we can conclude that the SPoA for these games is at most 2^k, where k is the number of players in the game.

To demonstrate that this bound is tight, consider the instance in Figure 5. In this instance, let each player i control terminal t_i. Connections not shown have cost equal to the shortest path, and each facility has unitary capacity. Only f_1 has an opening cost, and it is equal to 1. In S^* each terminal t_i connects to facility f_i, with a total cost of 1. However, backwards induction may produce the SPE where each terminal t_i connects to facility f_{i+1}. To see this, when analyzing the options of player t_1 there are two minimum choices, connecting to f_1 or f_2, and so t_1 may connect to f_2. Now for each t_i, $i = 2, \ldots k$, its minimum choice is either connect to f_1 or f_{i+1} and so the solution where each t_i connects to f_{i+1} is a SPE. The cost of this strategy profile S is $2^0 + 2^1 + \ldots + 2^{k-1} = 2^k$, and thus $SPoA = C(S)/C(S^*) = 2^k$.

□

3.3. Bounds for the PoS and SPoS

In the case of the Metric CFLG-FC, the standard potential function method [22], first used by Anshelevich et al. [12] for network design, can be used to show an upper bound of H_k for the PoS.

Corollary 2 (Upper bound for PoS in Metric CFLG-FC)**.** *For the Metric CFLG-FC the PoS is at most H_k, where k is the number of players of the game.*

This bound is not tight when considering the uncapacitated case, as shown by Hansen and Telelis [13], but is tight in the case of CFLG-FC, as shown below.

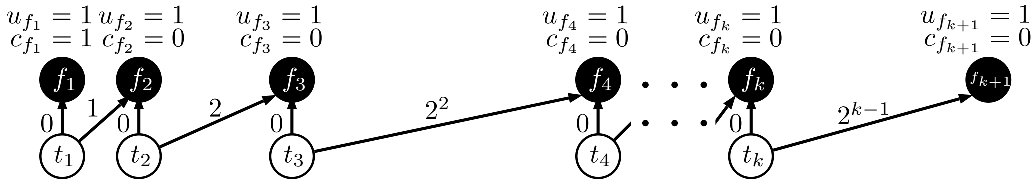

Figure 5: Metric CFLG and Metric CFLG-FC instance with Sequential PoA equal to 2^k.

Proposition 2 (PoS for Metric CFLG-FC)**.** *For the Metric CFLG-FC the Price of Stability is H_k, where k is the number of players of the game.*

Proof. As shown by Anshelevich et al. [12], the PoS for network design is at most H_k, which is true for CFLG-FC as well. Furthermore, the same example used to prove that such lower bound is tight can be adapted to facility location as well, as shown in Figure 6. Note that in any possible equilibrium, f_{k+1} is closed, and since all other facilities have unitary capacity restrictions, the cost of any equilibria is the sum of the opening costs of facilities $f_1, \ldots, f_k$, for a total social cost of $1 + \frac{1}{2} + ... + \frac{1}{k} = H_k$, while in the optimal solution only f_{k+1} is opened for a total cost of $1 + \varepsilon$.

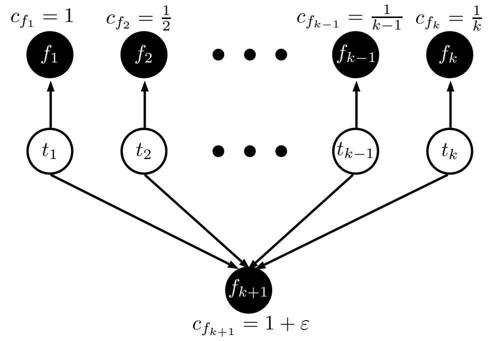

Figure 6: Game instance of Metric CFLG-FC with PoS of H_k. Any possible connection has zero cost. Facilities $f_1, \ldots, f_k$ have unitary capacities, while f_{k+1} has capacity k.

□

For the Metric CFLG, we can show that the PoS is $\Omega(2^k)$.

Theorem 4 (PoS for the Metric CFLG)**.** *There are instances of the Metric CFLG that admits a PNE and have a PoS in $\Omega(2^k)$.*

Proof. Consider the instance in Figure 7. Let player i control terminal t_i, player a control t_a and player b control t_b. Notice that if terminals $t_1, \ldots, t_k$ connect to facilities $f_1, \ldots, f_k$ somehow, it is not possible for an equilibrium to exist since the remaining game involving a and b never reaches an equilibrium. So, in order for an equilibrium to exist terminal t_k must connect to facility f_a, paying at least 0.5ε of its opening cost. The only scenario where this happens is when player 1 chooses to connect t_1 to f_2, player 2 chooses to connect t_2 to f_3 and so forth until t_k connects to f_a. The cost of this unique equilibrium is the same as the one for the instance in Figure 5, added the price for connecting t_a and t_b, for a total of $2^k + 1.5\varepsilon$. In the strategy profile with optimal cost, t_i connects to f_i, and t_a and t_b both share f_a, for a total cost of $1 + 2.5\varepsilon$. Therefore, the PoS for this instance is $\frac{2^k+1.5\varepsilon}{1+2.5\varepsilon} = \Omega(2^k)$. □

In the case of the SPoS, the results match the SPoA.

Theorem 5 (SPoS for Metric CFLG and CFLG-FC)**.** *There are instances for Sequential Metric CFLG and CFLG-FC with SPoS in $\Theta(2^k)$.*

Proof. Take the instance shown in Figure 5 and alter the cost of f_1 to $c_{f_1} = 1 + \varepsilon$. As before, each player i controls terminal t_i, connections not shown have cost equal to the shortest path, and each facility has unitary capacity. Now let players choose their strategies in order $1, \ldots, k$. It is easy to see that the cost to connect and open f_1 will always be ε higher for a player i than the alternative of opening f_{i+1}, and therefore the unique SPE has cost $2^0 + 2^1 + \ldots + 2^{k-1} = 2^k$ while the optimal social cost is $1 + \varepsilon$. Therefore, since the SPE is unique for this instance, we have SPoS $= \Omega(2^k)$. Since the SPoA is upper bounded by 2^k we have SPoS $= \Theta(2^k)$. □

4. Non-Metric Capacitated Facility Location Games

In this section we consider the versions of the games CFLG and CFLG-FC with general distance costs. In Section 3, we showed that for Metric CFLG there are instances with no PNE, even with singleton players, while the Metric CFLG-FC is a potential game, and therefore always posses a PNE. Clearly the same results apply when considering general distance costs.

4.1. Bounds for the PoA and PoS

The bounds for PoA for both CFLG and CFLG-FC follow directly from Theorem 2.

Corollary 3 (PoA for CFLG and CFLG-FC)**.** *The PoA for the CFLG-FC is unbounded. For the CFLG, there are instances that admits a PNE but whose PoA is unbounded.*

Note that while the example used for the metric case is a fringe instance with unbounded distance costs, there are

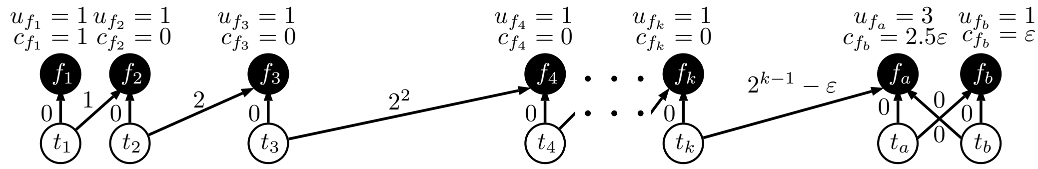

Figure 7: Metric Capacitated FLG instance with PoS equal to $\Theta(2^k)$.

non-metric examples with bounded distances costs and yet with unbounded PoA, such as the example in Figure 9.

As for the Price of Stability, the way opening costs are divided among players has a big influence on the PoS. For games with fair cost sharing, the results follow directly from Proposition 2.

On the other hand, without cost sharing rules we can prove that there are games with unbounded price of stability.

Theorem 6 (PoS for CFLG)**.** *There are instances of the CFLG that admit a PNE but have unbounded PoS.*

Proof. Suppose we restrict ourselves to instances of this game where a PNE exists. Based in the game of Figure 1, we can construct new instances with unbounded PoS. Consider the game instance in Figure 8, where each player $i \in [1,5]$ controls terminal t_i. Note that the subgraph induced by terminals t_4, t_5 is a game instance with no equilibrium, unless some amount of the opening cost of f_4 is paid by an external terminal. In fact, in order for this instance to have an equilibrium, terminal t_3 must connect to f_4, paying at most 1 of its opening cost. In an equilibrium, t_2 must not connect to f_3, since doing so t_3 would also choose f_3 and thus t_4 and t_5 would not reach an equilibrium. Terminal t_2 thus must connect to f_2, and since this facility has unitary capacity, t_1 must choose facility f_1 with opening cost $\mathcal{U}$, since otherwise he would either pay the greater penalty $\mathcal{U}_c$ to connect to f_2 or need to connect to f_3 or f_4 using infeasible connections with cost at least $\mathcal{U}_d$. Note that $\mathcal{U}$ can be as high as the arbitrarily big cost $\mathcal{U}_d$, making it unbounded. This is the only possible equilibrium in this instance, therefore the PoS is unbounded. □

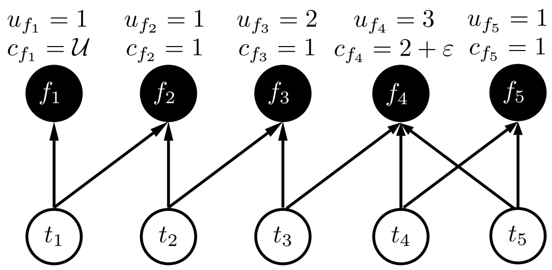

Figure 8: Game instance of the CFLG-FC with unbounded PoS. All shown connections have cost zero. Any connection not shown has a prohibitively large cost $\mathcal{U}_d$.

4.2. Bounds for the Sequential PoA and Sequential PoS

Perhaps surprisingly, for the sequential versions of capacitated facility location games, the instance depicted in Figure 9 proves that both the SPoA and the SPoS are unbounded, even for games with fair cost sharing.

Proposition 3 (SPoA and SPoS for CFLG and CFLG-FC)**.** *The SPoA and SPoS for the sequential CFLG and the sequential CFLG-FC is unbounded.*

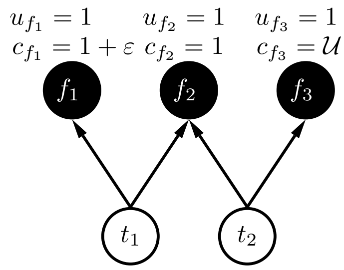

Figure 9: Game instance of the FLG and FLG-FC with unbounded Price of Anarchy. All shown connections have cost zero. Any connection not shown has a prohibitively large cost $\mathcal{U}_d$.

Proof. Consider the game in Figure 9, where player 1 controls terminal t_1 and player 2 controls terminal t_2, and all facilities have unitary capacities. In the strategy with optimal social cost, t_1 must connect to f_1 and t_2 to f_2, with a total cost of $2+\varepsilon$. For this instance, assume player 1 plays first and player 2 plays afterwards. Note that since only one terminal can be connected to a facility, player 1 will always pay less choosing f_2, and therefore player 2 has no choice but to connect to f_3 paying $\mathcal{U} \le \mathcal{U}_d$. Since player 1 will always play first, the unique SPE of the game is for player 1 to play (t_1, f_2) and player 2 to play (t_2, f_3), and therefore both the SPoA and the SPoS are unbounded for the sequential versions of CFLG and CFLG-FC. □

5. Conclusions

In this paper, we analyzed the efficiency of capacitated facility location games. Simultaneous games as well as sequential ones were considered, and results for PoA and PoS were established for these classes of capacitated facility location games.

Among the main results we show that the PoA is unbounded for metric capacitated facility location games, while the PoS for capacitated facility location games with no cost sharing rules is also unbounded. For capacitated games with no cost sharing rules, we show that it is NP-complete to decide if an instance of the game possess a PNE or not, when all players are singleton, which is not the case for the uncapacitated version. We show that even for more natural solution concepts for facility location, such as subgame perfect equilibria, SPoA and even SPoS may still be unbounded. For the metric capacitated facility location games, we show that the SPoA is bounded by 2^k, and that this result is tight.

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