INSTITUTE OF COMPUTING - UNICAMP

Graduate Program

MO417A Design and Analysis of Algorithms

2015 - Semester 1 - Jorge Stolfi

Midterm exam 1 - 2015-04-15

Name

RA Number	Signature

Item													TOT
Points													

You must do the exam by yourself, without any help.
You may not consult notes, books, tables, etc..
Computers and calculators may not be used during the exam.
Turn off and put away cellphones, music players, etc..
Do not separate the sheets of this exam booklet.
You cannot use any scratch paper besides this booklet.
Only answers in the marked-off spaces will be considered.
Purely numerical calculations need not be carried out.
Write your name legibly, and sign with a pen (not pencil or marker).
Once the exam booklet has been distributed:
*** if you leave the room, you will not be allowed to return.**
*** after someone leaves the room, no one else will be admitted.**

1. For any positive integer k, let φ_k be the number of distinct divisors of k. So, for example, $k = 24$ has 8 divisors (1, 2, 3, 4, 6, 8, 12, and 24), so $\varphi_{24} = 8$.

   The following algorithm takes a positive integer n, and retuns the list $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ of the first n values of that function. So, for example, given $n = 10$ the algorithm will return $\varphi = (1, 2, 2, 3, 2, 4, 2, 4, 3, 4)$. (The notation $[v]$ before a statement means "v is the number of times that this statement gets executed when the algorithm is executed once".)

```
        Algorithm NDivs(n)
            for k from 1 to n do
[f]             φ_k ← 1;
            endfor;
            d ← 2;
            while d² ≤ n
[g]             k ← d;
                while k ≤ n do;
[h]                 φ_k ← φ_k + 1
                    k ← k + d;
                endwhile;
                d ← d + 1;
            endwhile;
            return φ;
```

   (a) Determine formulas for the statement execution counts $f = f(n)$, $g = g(n)$, $h = h(n)$, as a function of n.

(b) What is the asymptotic class (in the $O/\Omega/\Theta$ notation) of $h(n)$, as a function of n?

2. Consider the algorithm below, that rearranges the elements of a list $x = (x_1, x_2, \ldots, x_n)$ in increasing order, assuming that each element x_i is between 0 and m, inclusive:

```
          Algorithm CtSort(n, m, x)
              for k from 0 to m do
[f]               c[k] ← 0;
              endfor;
              for i from 1 to n do
[g]               k ← x[i];
                  c[k] ← c[k] + 1;
              endfor;
              i ← 1;
              for k from 0 to m do
[r]               t ← 0;
                  while t < c[k] do
[s]                   x[i] ← k;
                      i ← i + 1;
                      t ← t + 1;
                  endwhile;
              endfor;
```

(a) Determine the maximum and minimum execution counts $f = f(n, m)$, $g = g(n, m)$, $r = r(n, m)$, $s = s(n, m)$, as a function of n and m;

(b) Would this algorithm be better than mergesort, if $n = 1\,000\,000$ and $m = 1000$? And if $n = 1000$, $m = 1\,000\,000$? (Justify the answers)

(c) What is the asymptotic class of the running time of this algorithm, as a function of n and m?

3. Find upper and lower bounds for the following summations, using the integral method. Recall that $x - 1 < \lfloor x \rfloor \le x$, $\sqrt{x} = x^{1/2}$, and $\log_a x = \ln x / \ln a$ for any x and a such that the formulas are defined. Recall also that $\int \ln x dx = x(\ln x - 1)$.

   (a) $s(n) = \sum_{k=m}^{n} \frac{(k-2)(k+2)}{k}$

   (b) $s(n) = \sum_{k=m}^{n} \ln(k+1)$

   (c) $s(n) = \sum_{k=m}^{n} \lfloor i^2/\pi \rfloor$

4. Consider the infinite sequences $x_0, x_1, \ldots, x_n, \ldots$ that satisfy the homogeneous linear recurrence

$$x_n = -x_{n-1} + 6x_{n-2} \tag{1}$$

(a) Write the characteristic polynomial of this recurrence.

(c) Determine the roots r_i of the polynomial.

(d) Determine a general non-recursive formula for the term x_n of any sequence that satisfies that recurrence.

(d) Determine a non-recursive formula for x_n, for the specific sequence such that $x_0 = 1$ and $x_1 = 2$.

(e) What is the asymptotic class for x_n, for that specific sequence?