1 Introduction

The goal of this note is to study, mathematically and computationally, the dynamics of a simple neuronal net model. While the model is far from realistic, especially with regards to topology and synaptic stengths, it is hoped that its study will give some useful insight about the behavior of real nets.

2 The model

The evolution of a real neuronal net through time is a continuous process. Even the firing events, which happen at arbitrary (non-synchronized) times, are fast but not instantaneous changes in the potentials of the affected neurons. The model we describe below is supposed to approximate this continuous process by a discrete-time process, with an implicit time step of 1. This time step is assumed to be large enough for each firings to be modeled as an instantaneous event that causes discontinuous step-like changes in the potentials; but also small enough for the probability of the same neuron firing twice in the same time step to be negligible.

The network model $\mathcal{N}$ has N abstract neurons, identified by indices in $\{0..N-1\}$. The state of neuron i at some time t is a real variable $U_i(t)$, its *potential* — which is assumed to model the the electric potentia difference across the membrane of a real neuron. The state of the network at time t is the vector $U(t)$ of those N potentials.

The network's state, by definition, evolves autonomously in a discrete, synchronous, non-deterministic fashion. That is, the state $U(t+1)$ at each integer time $t+1$ is a random variable with a probability distribution that depends on the state $U(t)$ only.

Specifically, the evolution of the network from $U(t)$ to $U(t+1)$ is modeled as the result of zero or more *firing* events. Between those two times, each neuron i may fire (at most once) with a probability that depends only on its potential at time t. Namely, let $X_i(t)$ be a variable that is 1 if neuron i fires, and 0 otherwise. We assume that the N random variables $X_i(t)$, for each time t, are independent, and

$$\Pr(X_i(t) = 1) = \Phi(U_i(t)) \tag{1}$$

where Φ is some monotonic function from $\mathbb{R}$ to $[0 \_ 1]$.

After a neuron fires, its potential is assumed to be reset to zero, and the potentials of all other neurons that did inot fire are assumed to increase by a fixed constant. In biological terms, this would correspond to each neuron having an excitatory synaptic connection to every other neuron, with the same synaptic strength. To simplify the scaling laws (see section 3.2), the increment is written w/N where w is the *total synaptic weight.* Note that, in this model, the increment does not depend on the potentials of the two neurons, until the post-synaptic neuron reaches the saturation potential.

On the other hand, if a neuron does not fire between times t and $t+1$, its potential is assumed to decay by some factor μ in $(0 \_ 1]$ (modeling the leakage current across the membrane of the neuron).

Combining these cases, the state of a neuron i at time $t+1$ can be expressed by the formula

$$U_i(t+1) \;=\; \begin{cases} 0 & \text{if } X_i(t) = 1, \\ \mu\left(U_i(t) \;+\; \frac{w}{N} X_{\text{tot}}(t)\right) & \text{otherwise.} \end{cases} \tag{2}$$

where

$$X_{\text{tot}}(t) = \sum_{j=0}^{N-1} X_j(t) \tag{3}$$

is the number of neurons that fired between t and $t = 1$.

Note that the potential increment $w/N X_{\text{tot}}(t)$ resulting from other neurons firing is assumed to be lost if neuron i fires between times t and $t+1$; otherwise, all those increments are reduced by decay factor μ.

Some of the theoretical analysis is simpler if $\Phi(U)$ is assumed to be linear on the neuron potential U up to some *saturation potential* U_{max}. That is

$$\Phi(U) \;=\; \begin{cases} 0 & \text{if } U \leq 0, \\ 1 & \text{if } U \geq U_{\text{max}}, \\ \frac{U}{U_{\text{max}}} & \text{if } 0 < U < U_{\text{max}}. \end{cases} \tag{4}$$

With proper choice of measurement unit for the potential, we can assume that $U_{\text{max}} = 1$.

A more realistic choice for Φ would be a steep sigmoid that is essentially zero for U below some *threshold potential* $U_{\text{min}} > 0$, and essentially 1 above the staturation potential U_{max}. Many authors assume an infinitely steep sigmoid with $U_{\text{min}} = U_{\text{max}}$, namely a step function that is 0 for $U < U_{\text{max}}$ and 1 for $U \geq U_{\text{max}}$

2.1 Limitations of the model

The behavior of the model differs in many ways from that of a real neural net. In the latter:

- Each neuron has synapses to and from a proper subset of the other neurons.
- Each synapse has a different strength, which is believed to change with time.
- When a neuron j fires, the change in the potential of a post-synaptic neuron i is not a fixed increment, but decreases as the potential U_i increases. A more realistic model could be $U_i = U_{\mathrm{lim}} - \lambda(U_{\mathrm{lim}} - U_i)$ for some $U_{\mathrm{lim}} \geq U_{\max}$ and some $\lambda \in (0 \_ 1)$.
- The probability of a neuron firing depends on its recent firing history as well as its current potential $U_i(t)$.
- The firing probability function Φ of real neurons is usually a very sharp sigmoid, almost a step function that jumps from 0 to 1 at potential $U = U_{\max}$. The use of a smoother function Φ could make sense, however as a way to compensate for the discretization of time.
- When a neuron i fires, it loses only the potential increments due to those neurons j that fired before it. Moreover, those increments that are not lost will decay by different factors depending on when exactly they occurred within the time interval.
- If a neuron j fires sufficiently early between times t and $t + 1$, the potential increment that it contributes to another neuron i may cause the latter to fire before time $t+1$. Thus, firing avalanches can be faster in the real network than in the simulated one. If realistic times are desired, the time step must be comparable or smaller than the natural firing delay in such avalanches.

2.2 Modeling neurons with refractory state

Optionally we may assume that neurons enter a *refractory* state after firing, that lasts a certain number $h > 0$ of time steps. During the refractory period, their potential is assumed to remain at zero, and further firings are inhibited.

That is, we introduce another state variable $R_i(t)$ for each neuron, the *refractory counter*, with integer values in $\{0, 1, \ldots, h\}$; and we replace equations (1) and (2) by

$$R_i(t+1) \;=\; \begin{cases} 0 & \text{if } R_i(t) = 0 \text{ and } X_i(t) = 0 \\ h & \text{if } R_i(t) = 0 \text{ and } X_i(t) = 1 \\ R_i(t) - 1 & \text{otherwise} \end{cases} \tag{5}$$

$$\Pr(X_i(t) = 1) \;=\; \begin{cases} 0 & \text{if } R_i(t) > 0 \\ \Phi(U_i(t)) & \text{otherwise} \end{cases} \tag{6}$$

and

$$U_i(t+1) \;=\; \begin{cases} 0 & \text{if } R_i(t+1) > 0 \\ \mu U_i(t) \;+\; \frac{w}{N} \sum j \neq i X_j(t) & \text{otherwise} \end{cases} \tag{7}$$

It remains to be seen whether the inclusion of refractory states in the model leads to a qualitatively different behavior. ★[I would guess that it does not make much difference to the qualitative dynamics, but makes the math a lot more complicated.]

3 Scaling laws

3.1 Changing the time step

We now consider the effect of changing that time step. That is, we consider how the parameters Φ, μ and w should be changed to yield a network $\mathcal{N}'$ whose evolution in one time step is as close as possible to that of $\mathcal{N}$ in r time steps, for some real number $r > 0$. For simplicity we consider first the case when r is a small integer.

Obviously, in the absence of firings the neuron potentials U'_i in $\mathcal{N}'$ should decay at each time step by a factor $\mu' = \mu^r$.

The total synaptic weight w' should be almost the same as w. The main difference is that firings that are assumed to occur simultaneously in $\mathcal{N}'$ may occur during r different time steps in $\mathcal{N}$, and their effect will be modified differently by the natural decay. To improve the match between $\mathcal{N}'$ and $\mathcal{N}$, we could set

$$w' = \cdots \,. \tag{8}$$

The probability that a neuron i will *not* fire during a time step of $\mathcal{N}'$ is the probability that it will not fire during any of the r corresponding steps of $\mathcal{N}$. To a first approximation, assuming that the potential U_i in $\mathcal{N}$ during those r steps is constant and equal to the potential U'_i in $\mathcal{N}'$, we get

$$\Phi'(U) = 1 - (1 - \Phi(U))^r \tag{9}$$

for all U in $[0 \,\_\, +\infty)$. Note that if Φ is linear up to $U = U_{\max}$ then Φ' is a complemented power, $\Phi(U) = 1 - (1 - U/U_{\max})^r$. If Φ is a sigmoid between $U_{\min}$ and $U_{\max}$ then Φ' too is a sigmoid, albeit with a different shape.

These formulas can be used for any real-valued step scale r, including less than 1. For realistic results, however, one must ensure that the step remains in the range where the basic assumptions are plausible: namely, the step is small enough for the probability of a neuron firing twice in the same step is negligible, and for the extra delay introduced in chained firing to be reasonable; but large enough for the firings to be considered instantaneous events.

3.2 Scaling for the number of neurons

To a first approximation, the model's behavior sould be fairly independent of the number of neurons N because of the factor $1/N$ in formula (2) — at least for large N. That is, the behavior of a net with 2000 neurons whould be very similar to that of a net with 1000 neurons, except that all statistics that refer to neuron count should be doubled. The scaling is expected to break down at very small N, however.

4 Neuron age distribution

Since every neuron (or subset of k neurons) in the model is equivalent to any other, the state $U(t)$ of the net can be represented, without loss of information, by its *potential distribution*, namely the number of neurons that have a given potential u, for each u in $[0 \,\_\, +\infty)$.

Indeed, let define the (*firing*) *age* of a neuron i as the number $\tau_i(t)$ of time steps that elapsed since its last firing, that is

$$\tau_i(t+1) = \begin{cases} 0 & \text{if } X_i(t+1) = 1 \\ \tau_i(t) + 1 & \text{otherwise} \end{cases} \tag{10}$$

The age $\tau_i(t)$ is undefined from the start of the simulation until the first time that the neuron i fires. After every neuron has fired at least once, the exact potential $U_i(t)$ of each neuron can be computed from its age and the ages of all other neurons, by the formula

$$U_i(t) = \frac{w}{N} \sum_j (\tau_j(t) < \tau_i(t)) \mu^{\tau_i(t) - \tau_j(t)} \tag{11}$$

Therefore, after every neuron has fired at least once, all the information about the state of the net at time t can be represented by the *age distribution*, the sequence of natural numbers $S(t) = (S_0(t), S_1(t), \ldots)$ where $S_\tau(t)$ is the number of neurons with age τ at the integer time t.

Note that an infinite sequence S of natural numbers is a possible age distribution for a network of N neurons if and only if $\sum_{\tau=0}^{\infty} S_\tau = N$. Therefore an age distribution has at most N non-zero elements. We will denote by $\mathcal{S}_N$ the set of all such sequences.

From now on we will use the term *network state* to mean the age distribution of the neurons, rather than the vector of potentials U. Using the age distribution to represent the state has both mathematical and computational advantages. For example, any sequence of $\mathcal{S}_N$ is a valid the initial state of a simulation, in the sense that it could possibly arise after the network has been evolving for a long time. In contrast, not every vector U of real numbers in $[0 \mathbin{\_} +\infty)$ is valid in this sense. Indeed, after a neuron has fired at least once, its potential is restricted to a countable subset of that interval.

4.1 Potentials and probabilities from age distribution

Note that, by formula (11), the potential $U_i(t)$ of a neuron i depends only on its age $\tau_i(t)$ and the ages of the other neurons. Therefore, the potential of all neurons with a certain age τ can be computed from the age distribution alone. In fact, we can define a linear operator $\mathcal{V}$ that, applied to an age distribution S, yields the infinite list $\mathcal{V}\,S$ of the corresponding potentials, in $[0 \mathbin{\_} +\infty)$:

$$(\mathcal{V}\,S)_\tau = \frac{w}{N} \sum_{\sigma=0}^{\tau-1} S_\sigma \mu^{\tau-\sigma} \tag{12}$$

Since all the neurons with the same age have the same potential, they all have the same firing probability. These probabilities can be expressed as another operator $\mathcal{P}$ that, applied to the age distribution, returns the corresponfing list of firing probabilities:

$$(\mathcal{P}\, S)_\tau \;=\; \Phi((\mathcal{V}\, S)_\tau) \tag{13}$$

This operator is non-linear in general, since $\Phi(U)$ must be a number in $[0\_1]$. However, if Φ is linear for potentials below a certain bound $U_{\max}$, then $\mathcal{P}$ is linear for all states S where all neurons have potential less than $U_{\max}$.

5 Certain death

5.1 Evolution of the age distribution

The evolution of the network then can be described by a stochastic mapping between age distributions. Namely, the distribution $S(t+1)$ is obtained from $S(t)$ by selecting a certain number $K_\tau(t) \leq S_\tau(t)$ of neurons with each age τ, and simulating their firing. Those neurons have their ages reset to zero, while all other neurons get their ages incrementd by 1. That is,

$$S_\tau(t+1) \;=\; \begin{cases} K_{\mathrm{tot}}(t) & \text{if } \tau = 0, \\ S_{\tau-1}(t) - K_{\tau-1}(t) & \text{if } \tau \geq 1. \end{cases} \tag{14}$$

where $K_{\mathrm{tot}}(t)$ is the total number of neurons that fired at time t, that is, $K_{\mathrm{tot}}(t) = X_{\mathrm{tot}}(t) = \sum_{\sigma=0}^{\infty} K_\sigma(t)$.

6 Network death

If S be an infinite sequence of natural numbers, we define the *right shift of* S as the infinte sequence $\triangleright S$ such that

$$(\triangleright S)_\tau \;=\; \begin{cases} 0 & \text{if } \tau = 0, \\ S_{\tau-1} & \text{if } \tau \geq 1. \end{cases} \tag{15}$$

If the network is in a certain state $S(t)$ at time t, and no neuron fires between t and $t+1$ (that is, $X_{\mathrm{tot}}(t) = 0$), the state $S(t+1)$ will be simply $\triangleright S(t)$. The potentials $V_\tau(t+1)$ of that state, as a function of neuron age, will be the same as before, except that shifted by 1 and scaled by μ:

$$V_\tau(t+1) \;=\; \begin{cases} 0 & \text{if } \tau = 0, \\ \mu V_{tau-1} & \text{if } \tau \geq 1. \end{cases} \tag{16}$$

I.e., $\mathcal{V} \rhd S = \mu \rhd \mathcal{V}\, S$; that is, if the network is in state S and no neuron fires in the next time step, the potentials of all states are reduced by teh factor μ. By induction, $\mathcal{V} \rhd^k S = \mu^k \rhd^k \mathcal{V}\, S$ for any state S and any natural k.

The probability that no neuron fires between t and $t+1$ is

$$\Pr(X_{\mathrm{tot}}(t) = 0) \;=\; \prod_{\tau=0}^{\infty} (1 - \Phi(V_t au(t)))^{S_\tau(t)} \tag{17}$$

Therefore, the probability that the network never fires again once it reaches state S is

$$\Pr(\text{net dies from } S) \;=\; \prod_{k=0}^{\infty} \prod_{\tau=0}^{\infty} (1 - \Phi((\mathcal{V} \overset{k}{\rhd} S)_\tau))^{(\rhd^k S)_\tau} \tag{18}$$

Note that $(\rhd^k S)_\tau$ is zero for all $\tau < k$; and, for $\tau \geq k$, we have $(\rhd^k S)_\tau = S_{\tau-k}$, $(\mathcal{V} \rhd^k S)_\tau = (\mathcal{V}\, S)_{\tau-k}$. Therefore we can replace τ by $\sigma + k$ and write

$$\begin{aligned} \Pr(\text{net dies from } S) \;&=\; \textstyle\prod_{k=0}^{\infty} \prod_{\sigma=0}^{\infty} (1 - \Phi((\mathcal{V} \rhd^k S)_{\sigma+k}))^{(\rhd^k S)_{\sigma+k}} \\ &=\; \textstyle\prod_{\sigma=0}^{\infty} \left(\prod_{k=0}^{\infty} (1 - \Phi(\mu^k (\mathcal{V}\, S)_\sigma)) \right)^{S_\sigma} \end{aligned} \tag{19}$$

Now suppose that Φ is sub-linear for small enough arguments. That is, there is some potential U_{inf} and some constant $\alpha > 0$ such that $\Phi(U) \leq \alpha U \leq 1$ for all $U \leq U_{\mathrm{inf}}$. Let S be a state for which all neurons have potential $(\mathcal{V}\, S)_\sigma$ at most U_{inf}. In that case, $\mu^k (\mathcal{V}\, S)_\sigma \leq U_{\mathrm{inf}}$, and $\Phi(\mu^k (\mathcal{V}\, S)_\sigma) \leq \alpha \mu^k (\mathcal{V}\, S)_\sigma \leq 1$. Then the innermost product in formula (19) can be bounded by $P(\alpha (\mathcal{V}\, S)_\sigma)$ where

$$P(x) \;=\; \prod_{k=0}^{\infty} (1 - \mu^k x)) \tag{20}$$

and therefore

$$\Pr(\text{net dies from } S) \;\geq\; \prod_{\sigma=0}^{\infty} (P(\alpha (\mathcal{V}\, S)_\sigma))^{S_\sigma} \tag{21}$$

It can be shown that $P(x)$ is strictly positive for any $\mu < 1$ and $x \leq 1$. $\bigstar$[(Is it true?)] It follows that formula (19) is strictly positive; that is, once all neurons have potential U_{inf} or less, there is a positive probability that no neuron will ever fire again.

To conclude the argument, observe that for any state S', and any $m \in \mathbb{N}$, there is a nonzero probability that the network in state S' will remain without firing for m steps; after which all neuron potentials will be reduced by a factor μ^m. Therefore, if $\mu < 1$, from any state S' there is a positive probability that the network will reach a state S where all neurons have potentials $U_{\inf}$ or less.

★ [Does that mean that, if $\mu < 1$, the network will die with probability 1?]

7 Metastable states

There is empirical evidence of long-lived state sets; that is, the network occasionally evolves into a cluster $\mathcal{P} \subseteq \mathcal{S}_N$ of similar states, and remains in that cluster for a long time.

To study the limit when $N \to \infty$, is is convenient to replace the age distribution S by the *normalized age distribution* $s = S/N$. Namely, weconsider the state of the network to be an infinite sequence of non-negative real numbers such that $\sum_\tau s_\tau$ converges to 1. Let $\mathcal{S}_\infty$ be set of all such sequences. In the limit of infinite N, the evolution of the network then becomes a deterministic operator from $\mathcal{S}_\infty$ to $\mathcal{S}_\infty$. Namely, from each normalized state s we can compute the potentials of neurons of any age τ, as in formula (12)

$$(\mathcal{V}\, s)_\tau \;=\; w \sum_{\sigma=0}^{\tau-1} s_\sigma \mu^{\tau-\sigma} \tag{22}$$

Then, the firing probability $(\mathcal{P}\, S)_\tau = \Phi((\mathcal{V}\, S)_\tau)$ for the neurons with a certain age τ becomes the fraction $(\mathcal{P}\, s)_\tau = \Phi((\mathcal{V}\, s)_\tau)$ of neurons in the infinite net with that age that fire between time t and time $t+1$. Therefore, a network in state $s(t)$ evolves deterministically to the unique state $s(t+1) = \mathcal{E}(s(t))$, where $\mathcal{E}$ is the operator on $\mathcal{S}_\infty$ defined by

$$(\mathcal{E}\, s)_\tau \;=\; \begin{cases} \sum_{\tau=0}^{\infty} (\mathcal{P}\, s)_\tau & \text{if } \tau = 0, \\ s_{\tau-1} - (\mathcal{P}\, s)_{\tau-1} & \text{for } \tau \geq 1. \end{cases} \tag{23}$$

We can then ask whether the $\mathcal{E}$ operator has any fixed points