A Review of Affine Arithmetic Methods and Applications

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4 Random thoughts as of 2022-11-20

4.1 On the origin of AA

In retrospect it is not surprising that AA was invented in the computer graphics domain. In many applications that might need self-validated computation or range estimation, the relevant functions are relatively simple, well-behaved, and statically known depending on relatively few parameters. Thus static analysis and standard IA are often sufficient. In computer graphics, on the other hand, the functions that define geometric models (such as implicit surfaces) cam be higley complicated and unpredictable.

4.2 Sources of uncertainty

- Expected variation of input data.
- Incomplete knowledge of input data.
- Approximation errors.
- Randomized algorithms.
- Roundoff errors.

When subdividing domain, the magnitude of approx errors of AA decrease like r^2 where r is the diameter of the cell. The input variation uncertaintly decreases like r, while roundoff errors remain constant.

AA is generally better than IA when the result significantly depends on difference between correlated inputs and when the approximation errors are significantly smaller than the input uncertainties. It is also better when the computation is such that there is significant correlation between intermediate values, and when the approximation errors are large but mostlu cancel out.

An advantage of AA over several other first-order encosure methods is that it treats all the above source of uncertainty in the same way, thus (for example) accounting for cancellation between approximation errors or randomization variables generated internally.

4.3 Validated numerics for CSG rendering

The idea of simplifying the CSG operation tree in each box of a space subdivision is attributed to Duff. But maybe our ZZ-buffer paper is earlier?

4.4 Using AA (or IA) for a priori error bounds.

AA can be used to estimate the max roundoff error a priori rather than a posteriori. Must redefine x it as an enclosure for all possible **FP** numbers (not real numbers) which may result from computing x with *with FP arithmentic* (not exact arithmentic) given input **FP** values (nt input real values) contained in the respective input AA forms.

Then, for example, when adding two affine forms $x + y$ the new roundoff error term must not be the error made when adding the coeffs, but the max error that could be committed when adding FP values represented by x and y.

Must think more about that. Check Eva Darulová's papers.

4.5 Matrix computations

An $m \times n$ matrix of affine forms depending on q noises can be more elegantly described as an affine form M depending on q noises whose coefficients are $m \times n$ FP matrices. In turn that can be expressed as $M = M_0 + M_* \circ$ where M_0 is an $m \times n$ FP matrix, M_* is an $m \times n \times q$ FP tensor, is the vector of q noises, and $(A \circ B)_{i\,j} = \sum_k A_{i\,j\,k k}$.

Key problems

- Compute C that encloses $C = AB$ for all $A \in A$ and $B \in B$, where is addition and multiplication.
- Compute N that encloses M^{-1} for all $M \in M$.
- Compute L and U that enclose the Gaussian factors L and U of every $M \in M$.
- Compute affine vector x that encloses the solution x of $Ax = b$, for all $A \in A$ and $b \in b$.
- Compute affine vector and affine rotation matrix R that encloses the eigenvalue vector and orthonormal eigenframe R of every matrix $M \in M$.

4.6 Meet and join

The join of two zonotopes arises when evaluating a conditional expression that cannot be resolved given the range of the variables that enter in the condition. There are algoritthms to compute the smallest enclosure trying to preserve correlation information.

The meet of two zonotopes arises when solving systems of equations.

4.7 Interval fields

An application of affine arithmetic is the concept of *interval field* as defined by Faes and Moens [**fae-moe-19-aa-intfld**]. Namely a field defined by n basis functions ϕ_k and n real intervals c_k, where the value $f(z)$ at some point z of space is $\sum_k c_k \phi_k(z)$ for some coefficients $c_k \in c_k$. If each interval c_k is represented by an affine form $c_{k\,0} + c_{k\,*k}$, then the values $f(z')$ and $f(z'')$ at any two point of space, as evaluated by AA, will be n-term affine forms that capture the dependency between the two field values.

Faes and Moens claim that AA is not good. But seems that they are thinking of using AA to compute enclosured of $f(z)$ over a sudvision of space. Their proposed “better method”, based on “convex hull pair constructions” and “inverse distance weighting interpolation”, may be what is described above. To investigate.

4.8 Simploid arithmetic?

Lucas Batista Freitas's thesis introduced *simploids*, spaces that are the Cartesian product of simplices. The canonical d-dimensional simplex K_d is the set of all $(d{+}1)$-tuples of non-negative numbers that add to 1. The canonical simploid of multidimension (d_0, d_1, d_m) is the Cartesian product of $K_{d_1} \times K_{d_2} \times \cdot \times K_{d_m}$. When all d_i are 1, this is Euclidean-equivalent to the m-dimensional hypercube. The simploid of dimension $(1, 2)$ is Euclidean-equivalent to the 3-dim prism with triangular base. And so on. Lucas defined Bézier elements on simploids and developed conditions for smooth joining between them. I wonder if we cannot define a *simploid arithmetic* whose enclosing volumes are affine projections of a canonical simploid, intead of a hypercube?

4.9 Higher-order generalizations

One could generalize IA even further than AA by using polynomials of degree 2 or more plus deviations, the idea of Taylor arithmetic. Non-affine operations, like $\sqrt{}$, can then be replaced by composition of two polynomials.

Another version uses truncated Chebyshev series instead of Taylor series [**dze-15-aa-valode**].

In these higher-order models, composition and multiplication then require approximating a high-degree polynomial by a polynomial of a prescribed maximum degree.

The advantage of these models is that they can be much more accurate. The error of an approximation of degree d is in theory proportional to h^{d+1} where h is the diameter of the enclosure of the arguments. In zero-finding or optimization problems, for example, quadratic Taylor arithmetic could eliminate larger regions earlier, compared to AA or IA.

One problem with these extensions is cost: the number of terms in a quadratic polynomial with n independent uncertainty sources would be about $n^2/2$, instead of about n for the affine forms. The the cost of multiplication and other non-affine operations would be much higher too. This cost may make it impractical to track approximation errors as individual noises, thus preventing their cancellation. Some non-affine operations, like max may be very hard to approximate even for low-degree polynomials.

Duracz *et al.* proposed to use intervals where each bound can be independently chosen from some wider class of functions of the input noises [**dur-far-kon-tah-14-aa-fu**
Besides the above problems, in this approack even the product of two interval with polynomial bounds may be hard to implement when one of the operands straddles zero.

5 Unsorted references

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