Topological Data Structures

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Abstract

We describe in detail a novel data structure for d-dimensional triangulations. In an arbitrary d-dimension triangulation, there are $d!$ ways in which a specific facet of an simplex can be glued to a specific facet of another simplex. Therefore, in data structures for general d-dimensional triangulations, this information must be encoded using $\lceil \log_2(d!) \rceil$ bits for each adjacent pair of simplices. We study a special class of triangulations, called the *colored triangulations*, in which there is a only one way two simplices can share a specific facet. The *gem data structure*, described here, makes use of this fact to greatly simplify the repertoire of elementary topological operators.

GEMS: A GENERAL DATA STRUCTURE FOR d-DIMENSIONAL TRIANGULATIONS

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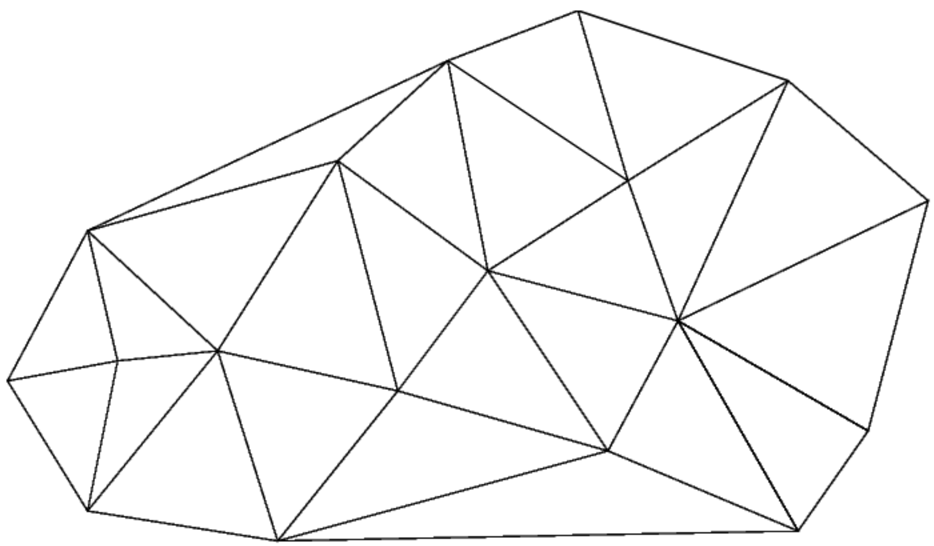
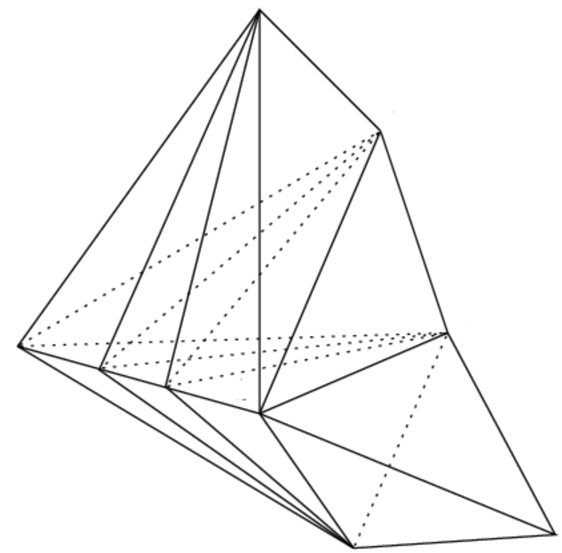
SUMMARY

- Triangulations.
- Winged-edge, half-edge, etc..
- Quad-edge.
- Facet-edge.
- N-G-maps/cell-tuple.
- Corner-stitching, 4-8, SMC, ...
- The gem data structure.
- Conclusions ans future work.

Triangulations

Triangulation: set of d-simplices, glued by facets.

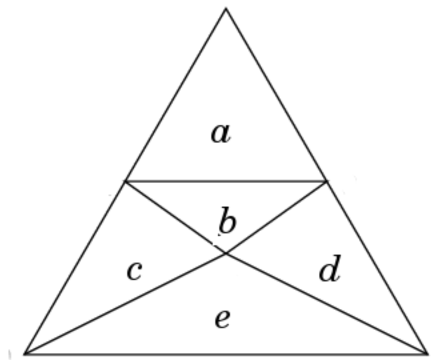


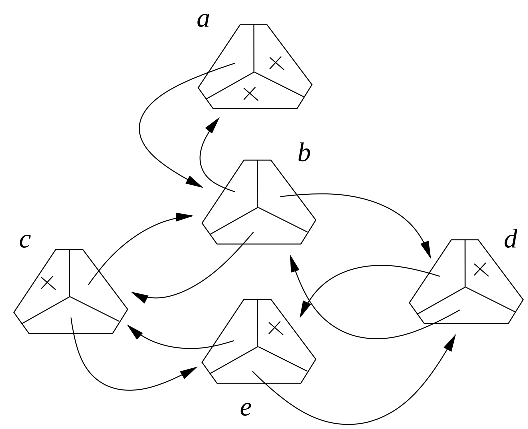


Triangulation data structures

Pointer data structures:

- One record per cell.
- One pointer per facet, to adjacent cell.

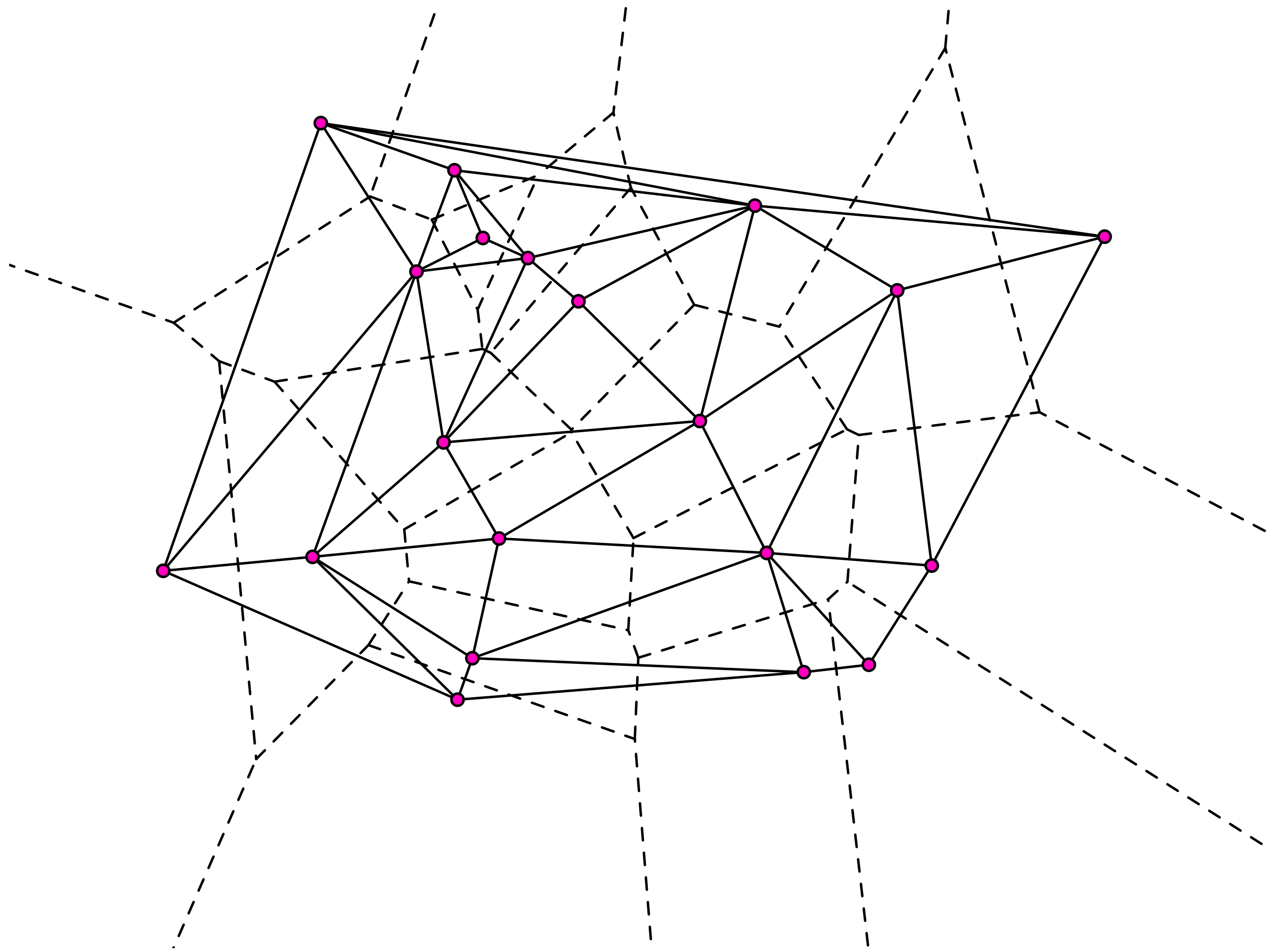




Problem: which pointer is the right one?

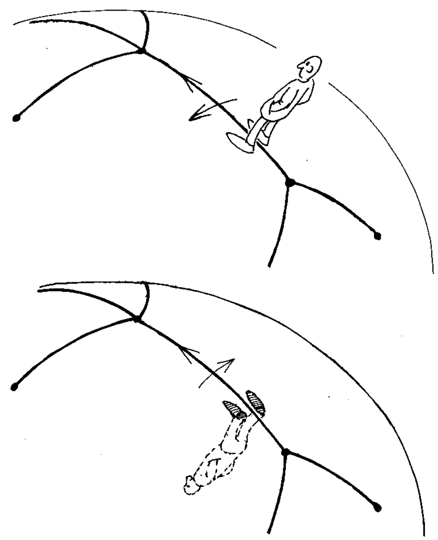
- Check all links (D. T. Lee & B. J. Schachter 1980 [7]).
- Add $\lceil \log_2((d+1)!) \rceil$ permutation bits per link
  (J. R. Shewchuck 1996 [10], J.-D. Boisonnat & al. 2002 [1], . . .)

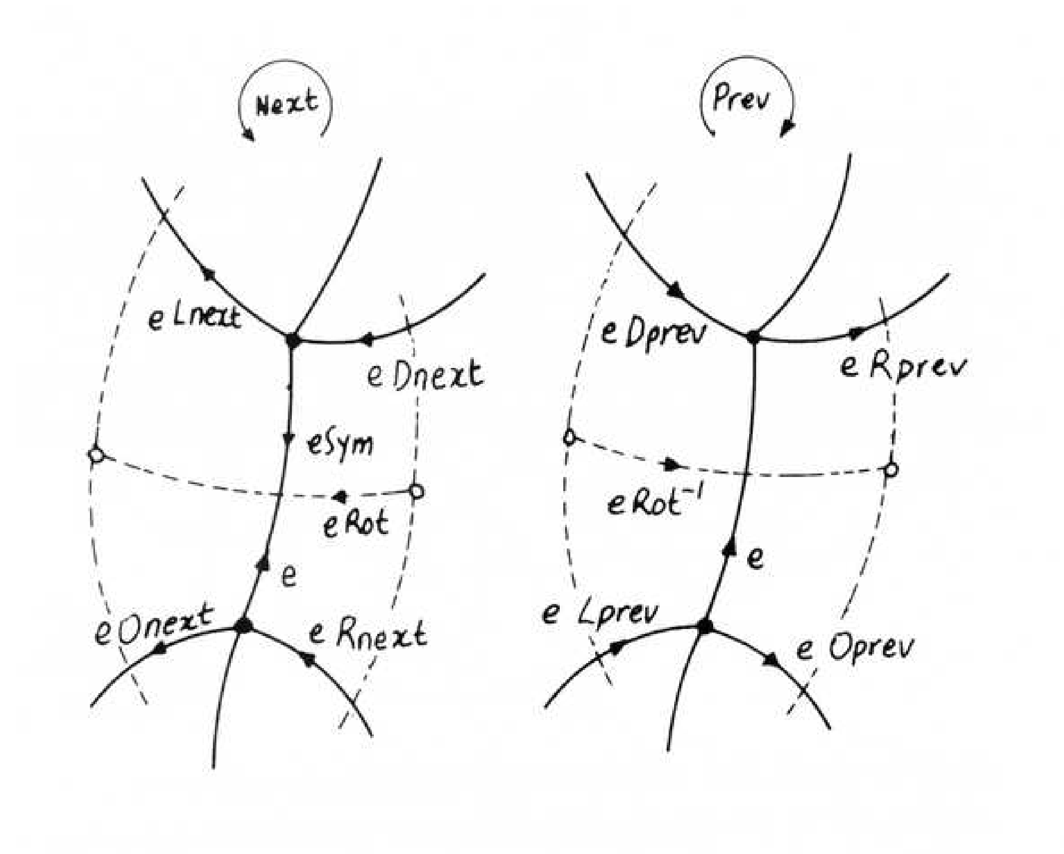
Quad-Edge and Voronoi



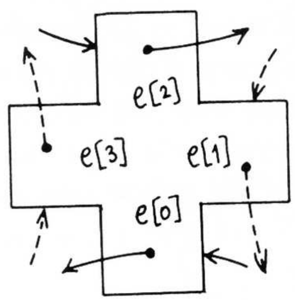
With Leo Guibas (TOG 4(2), 1985).

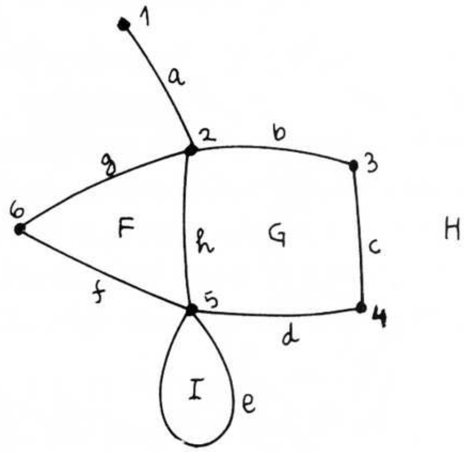
Quad-Edge and Voronoi (2)

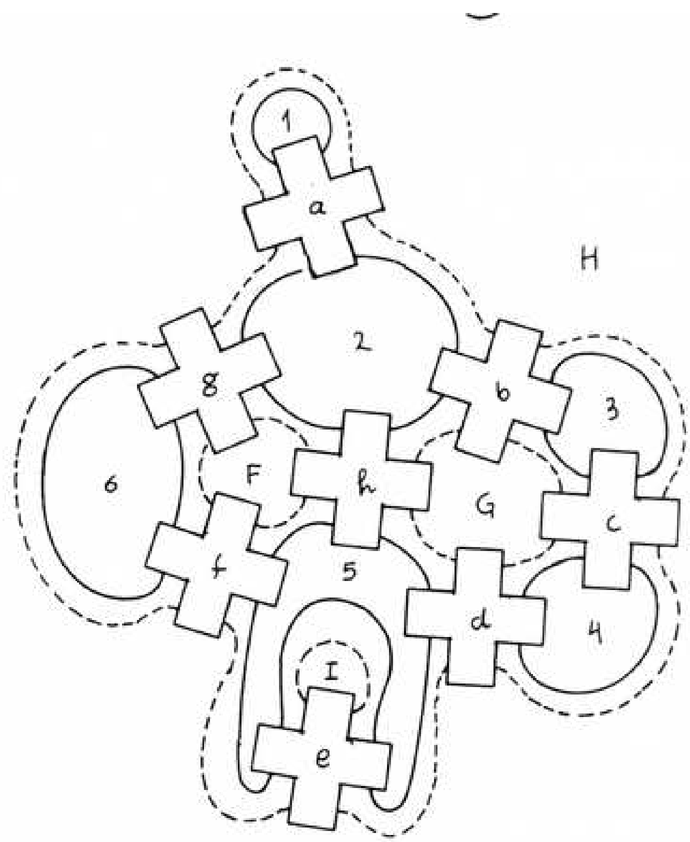




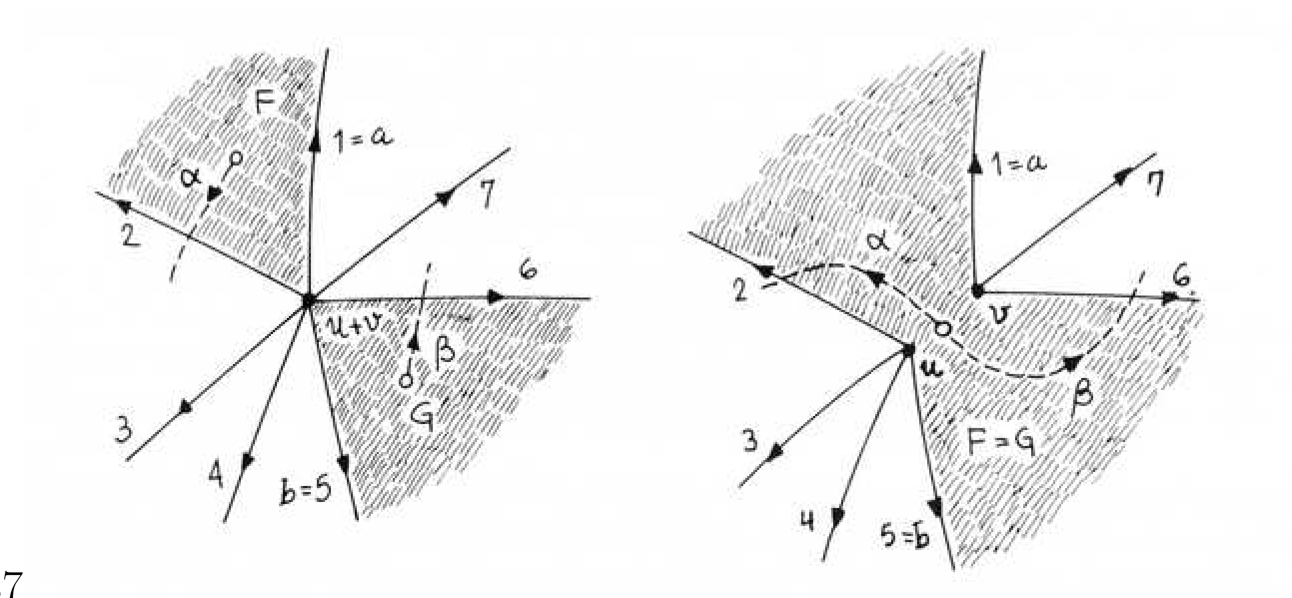
Quad-Edge and Voronoi (3)







Quad-Edge and Voronoi (4)



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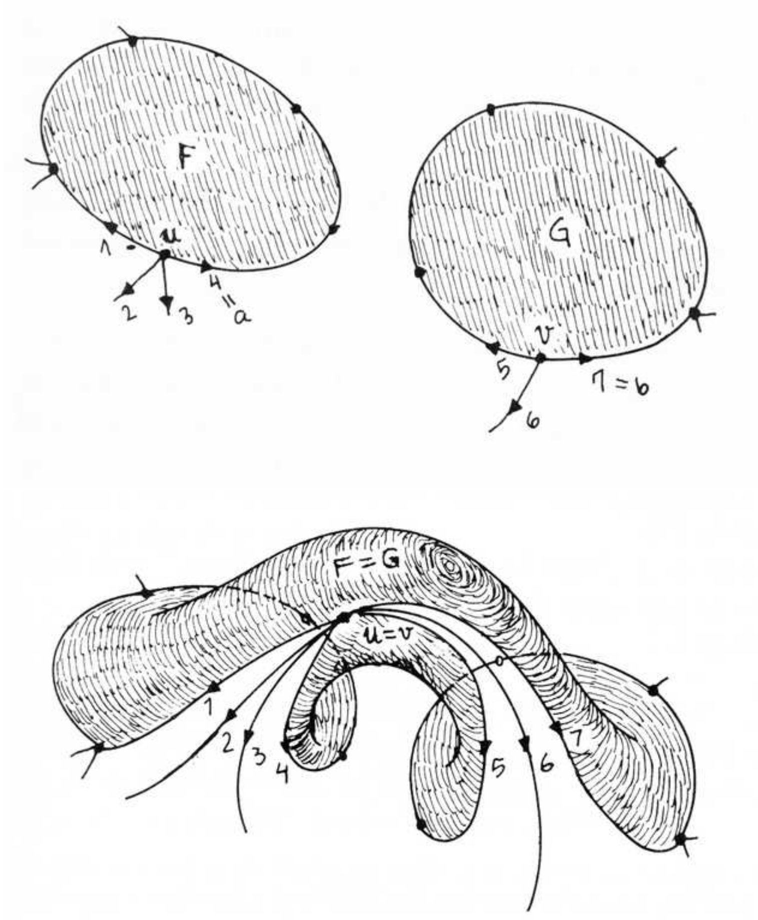
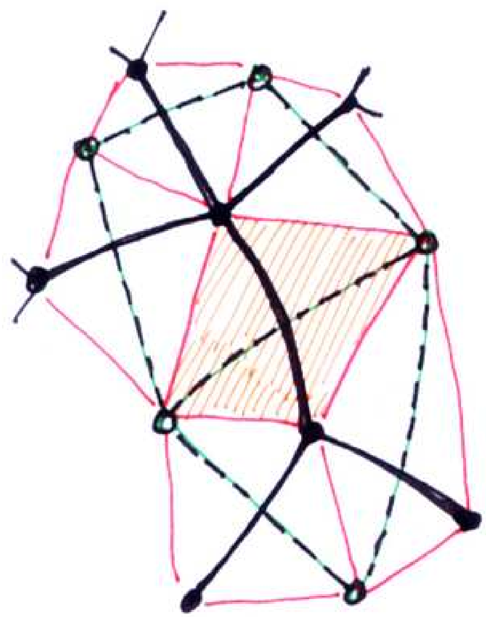
Quad-Edge and Voronoi (5)

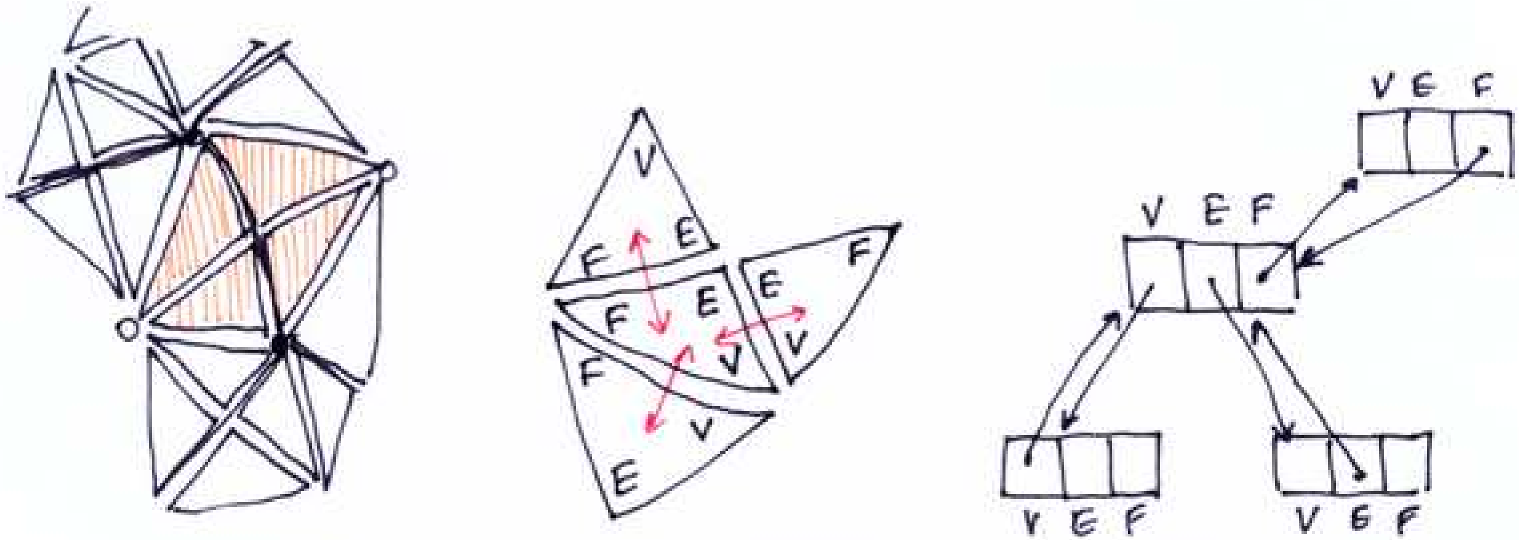


Quad-Edge and Voronoi (6)



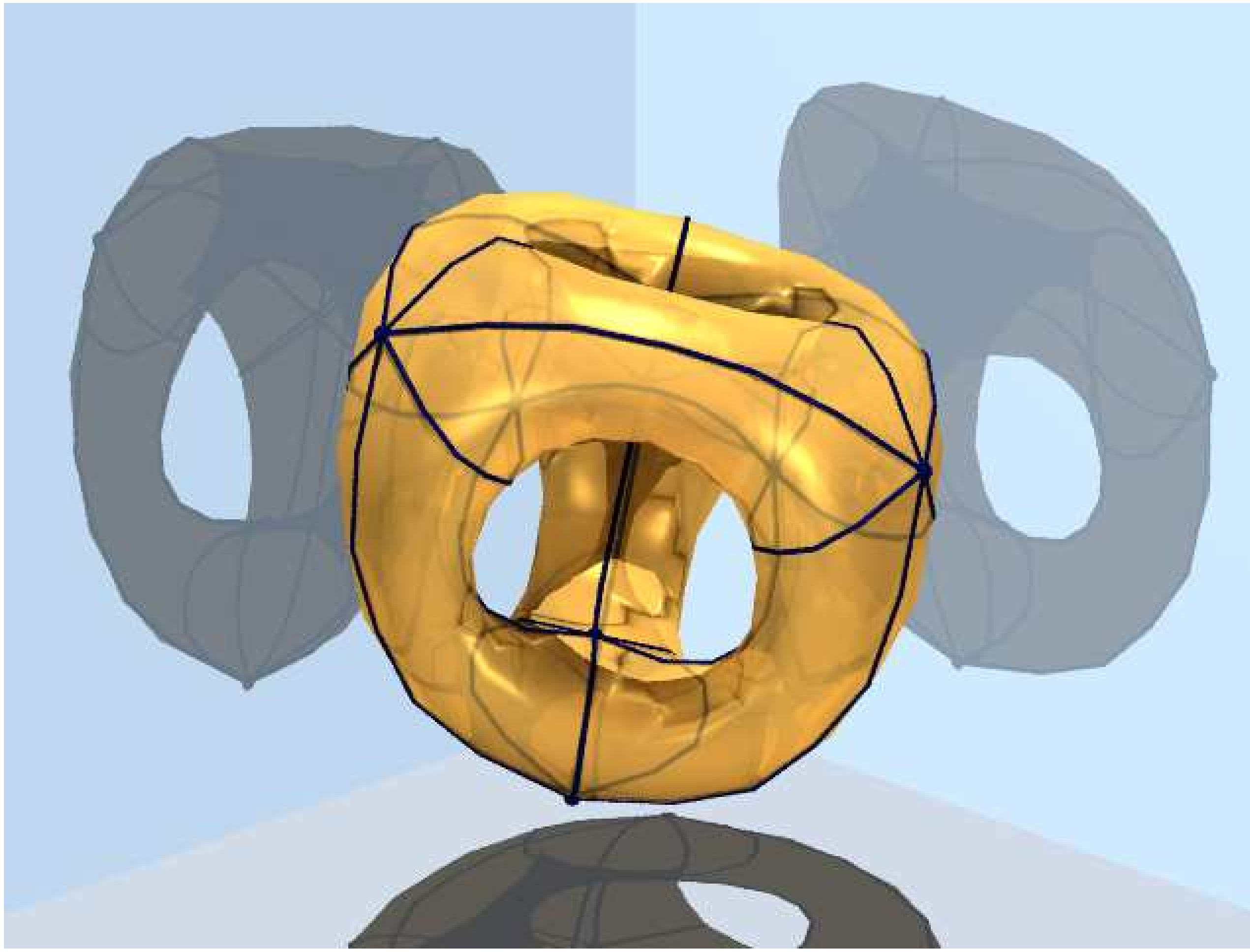
Quad-edge

Quad-Edge and Voronoi (7)



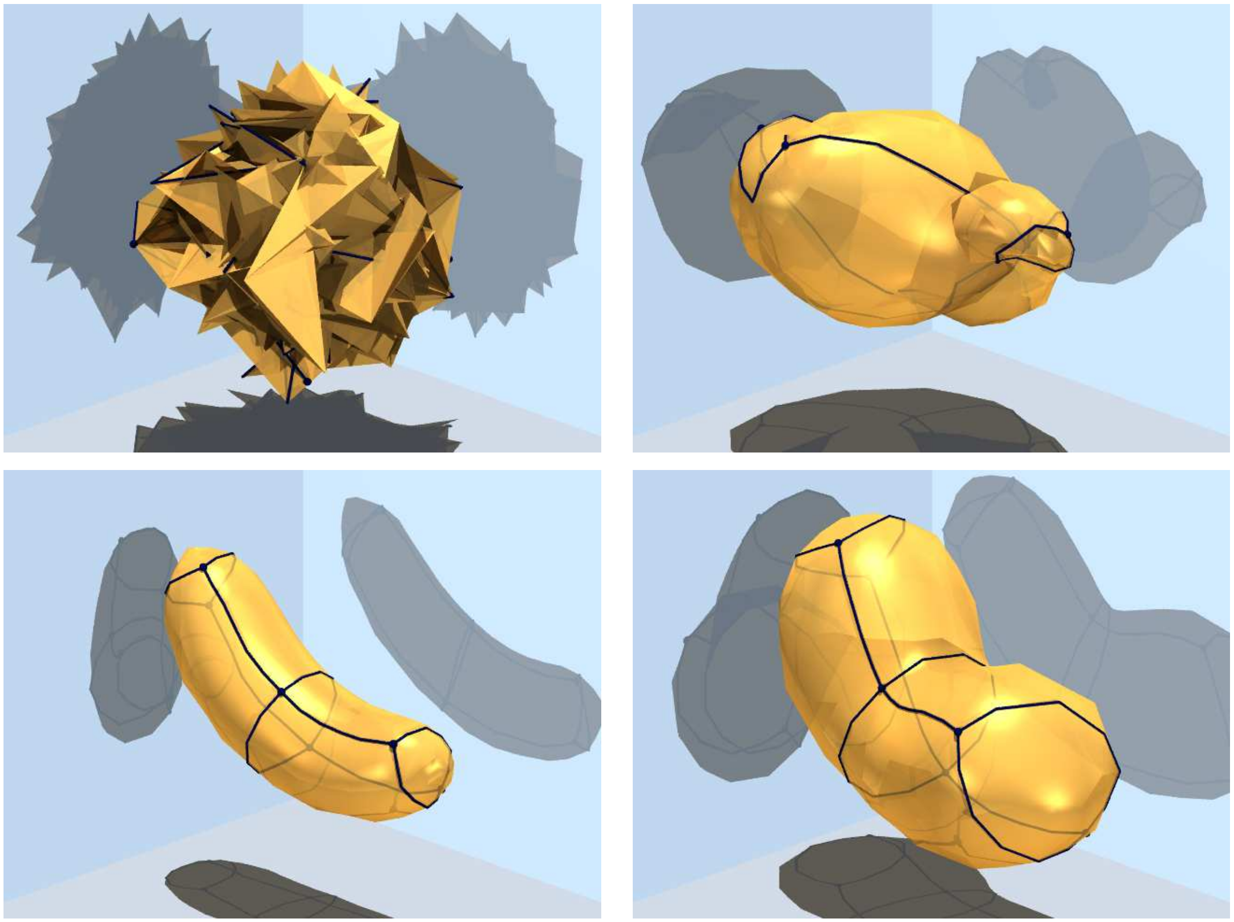
Barycentric Subdivision and Gem

Toposcope

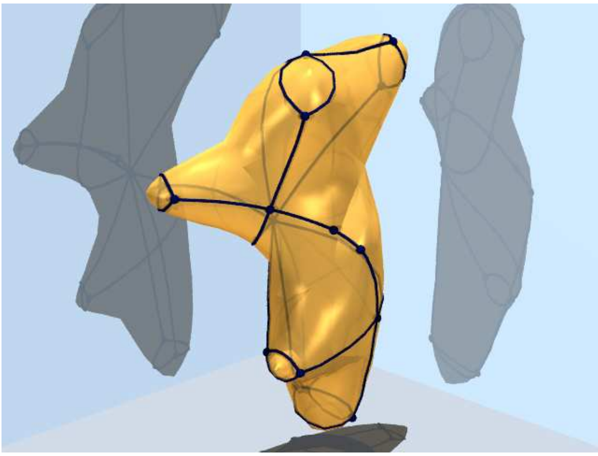


With Rober M. Rosi (Graph Drawing '96).

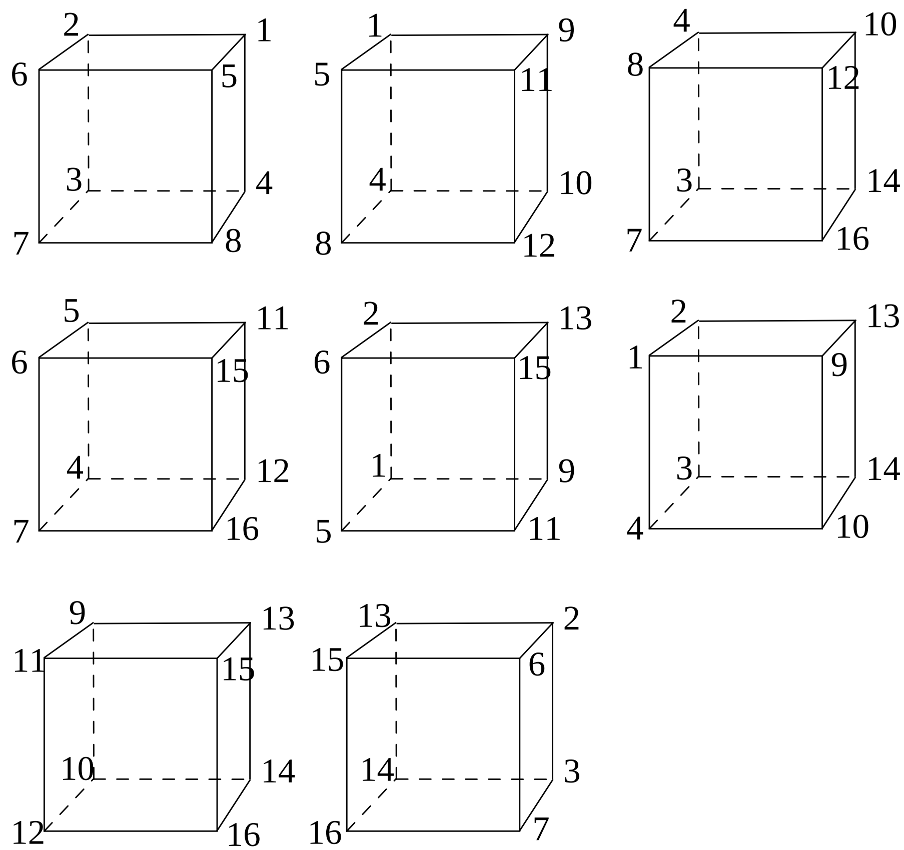
Toposcope (2)



Toposcope (3)

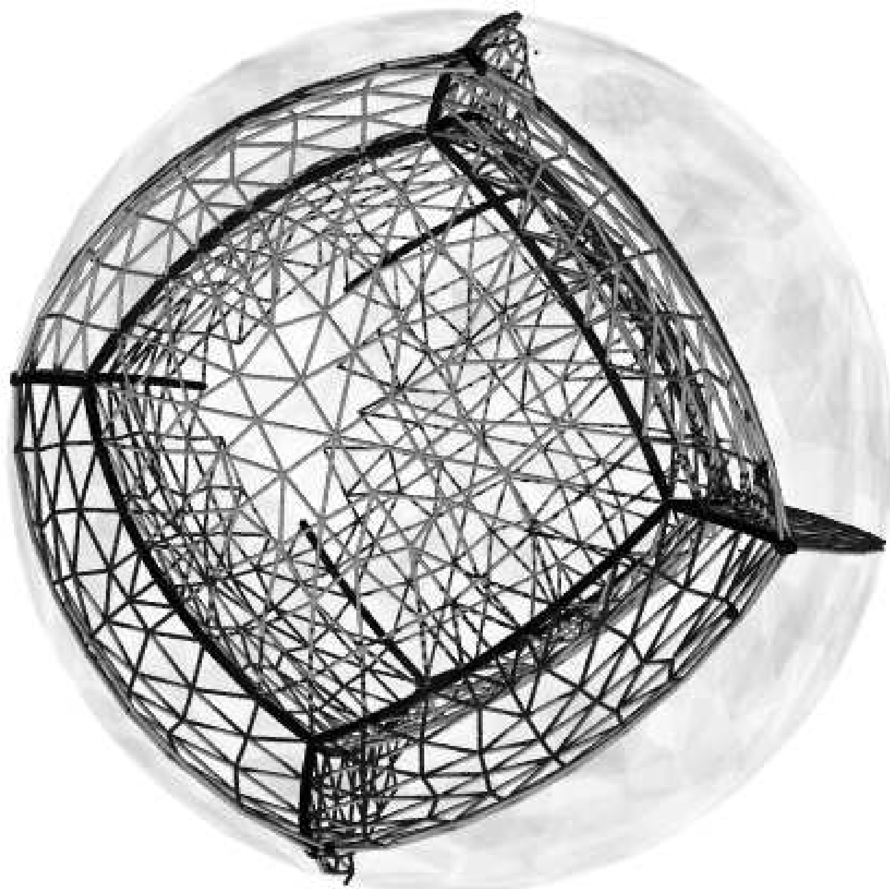
 

Toposcope (4)



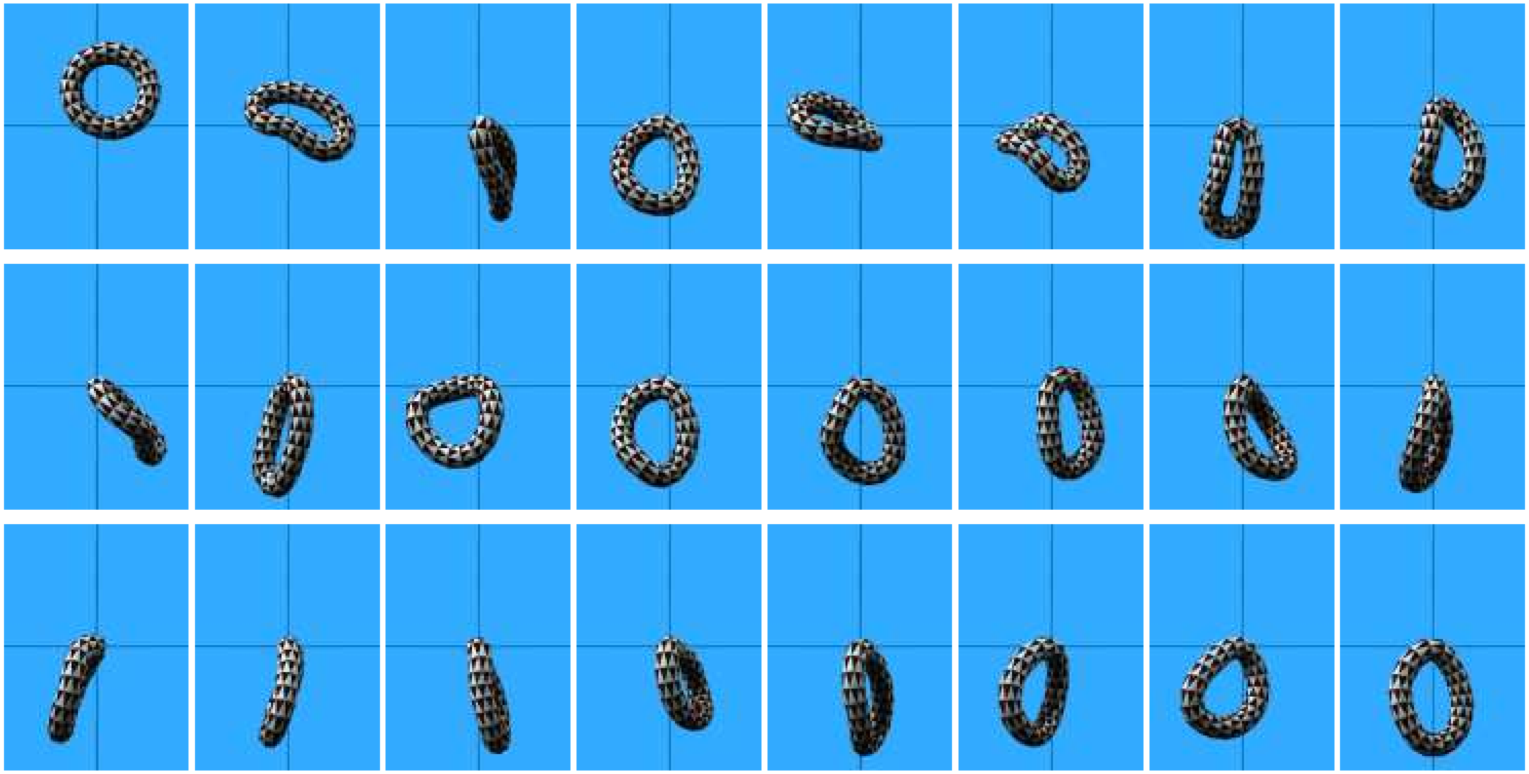
With Luís A. P. Lozada (SIBGRAPI 2000).

Toposcope (5)

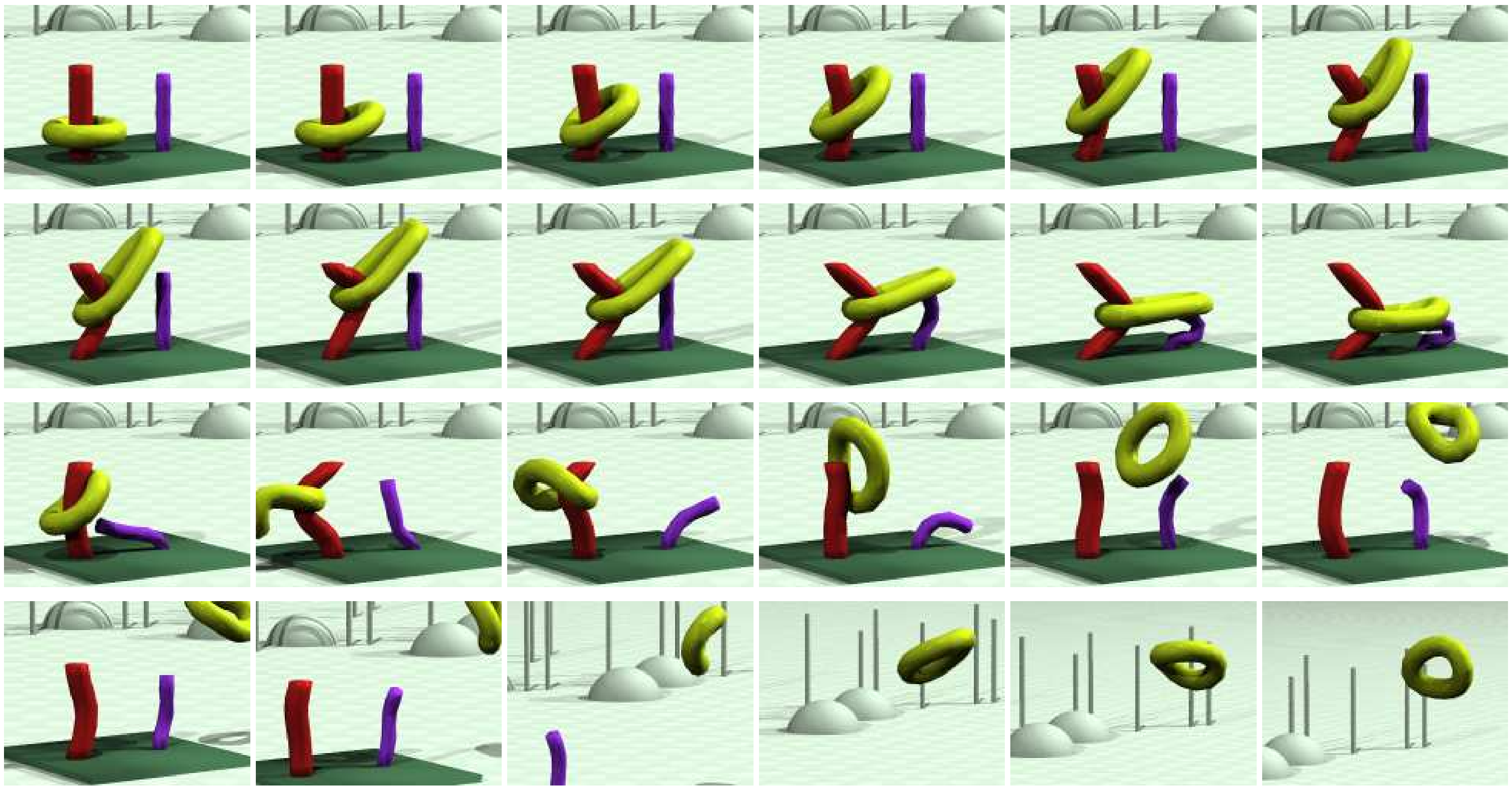


3D Triangulations (1)

3D Triangulations for animation of elastic objects.
Rogério L. W. Liesenfeld, IC-UNICAMP, 1994



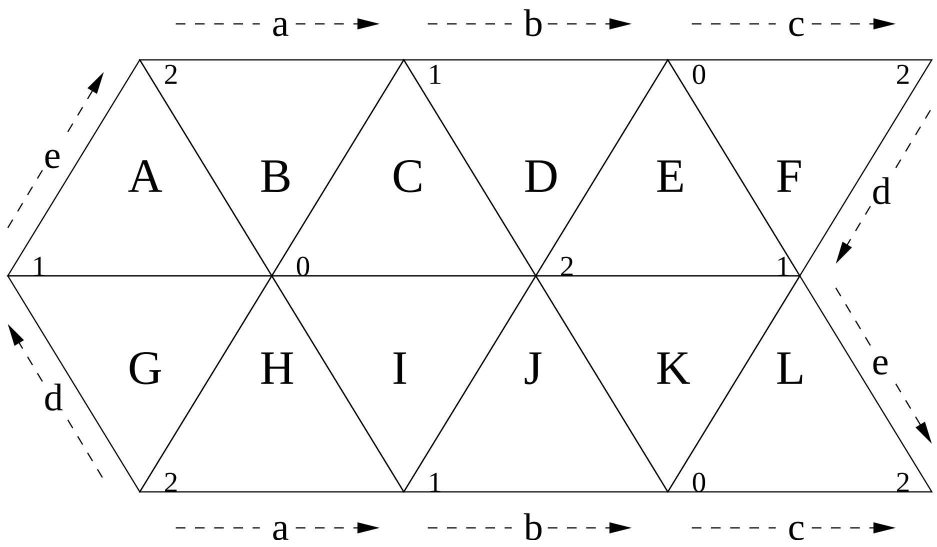
3D Triangulations (2)

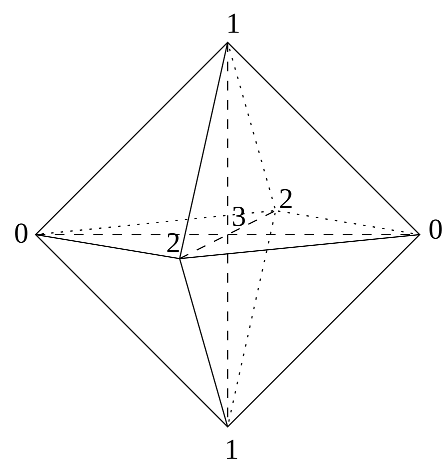


Colored triangulations

Colored triangulation:

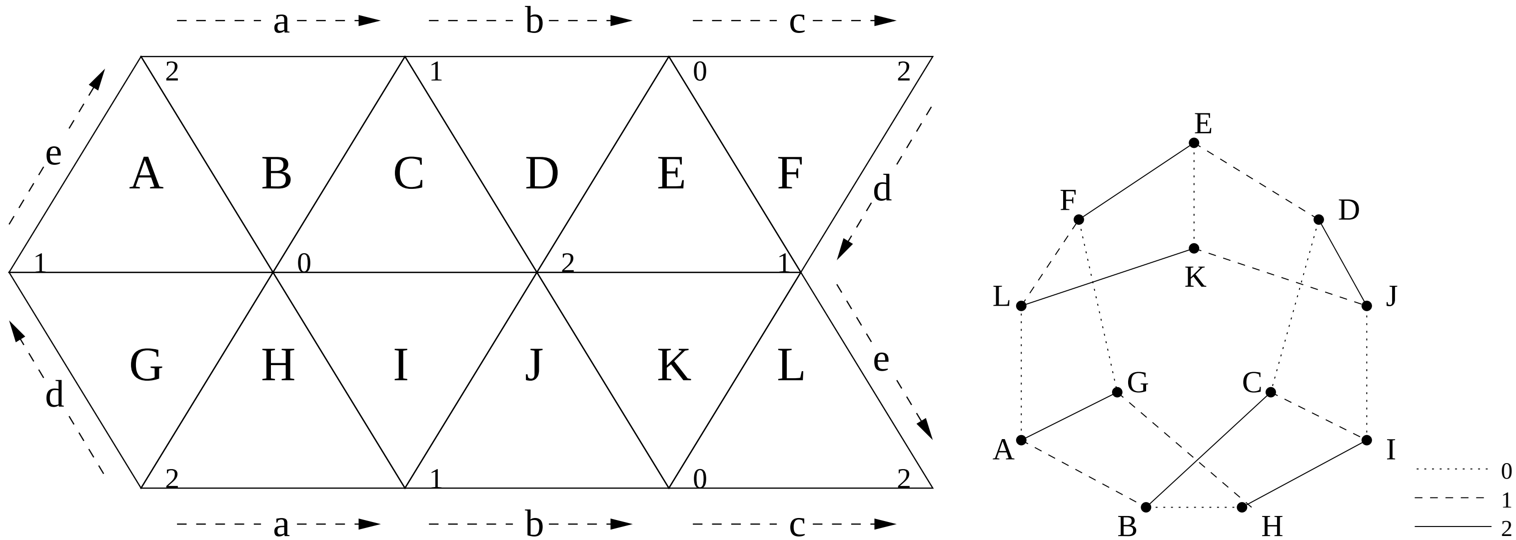
- Vertices are labeled with "colors" $0, 1, \ldots, d$.
- Each element has at most one vertex of each color.





Gems (1)

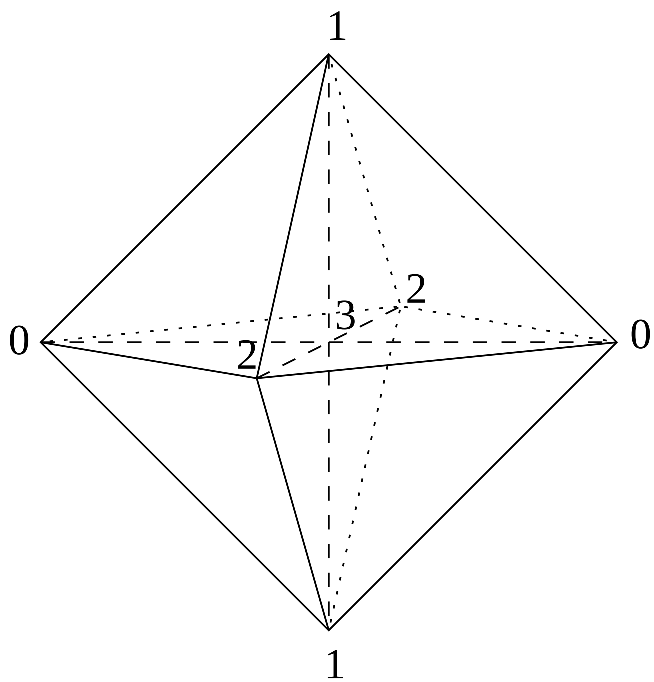
Gem $=$ the dual graph of a colored triangulation:

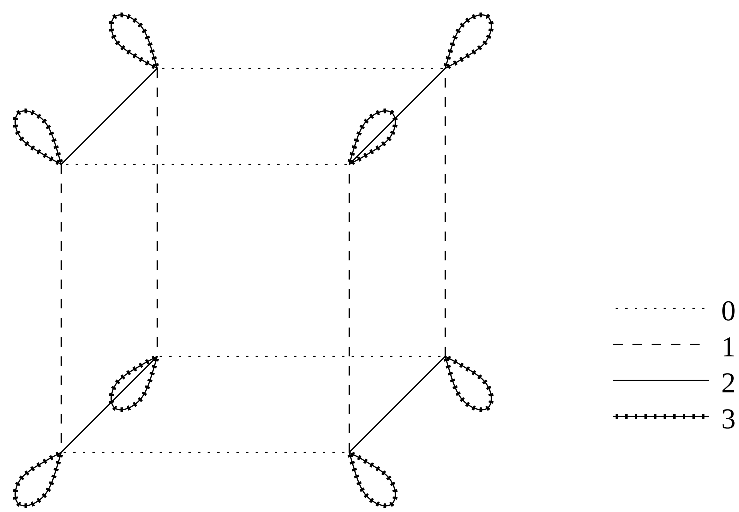


A regular graph, edge-colored with colors $0, 1, \ldots, d$.
(M. Ferri 1976 [5], S. Lins 1982 [9].

Gems (2)

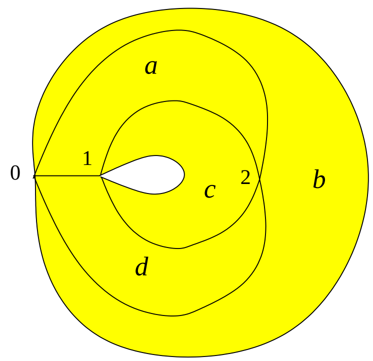
Self-loops denote unglued facets (free border).

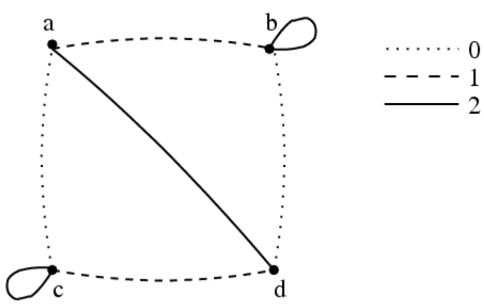


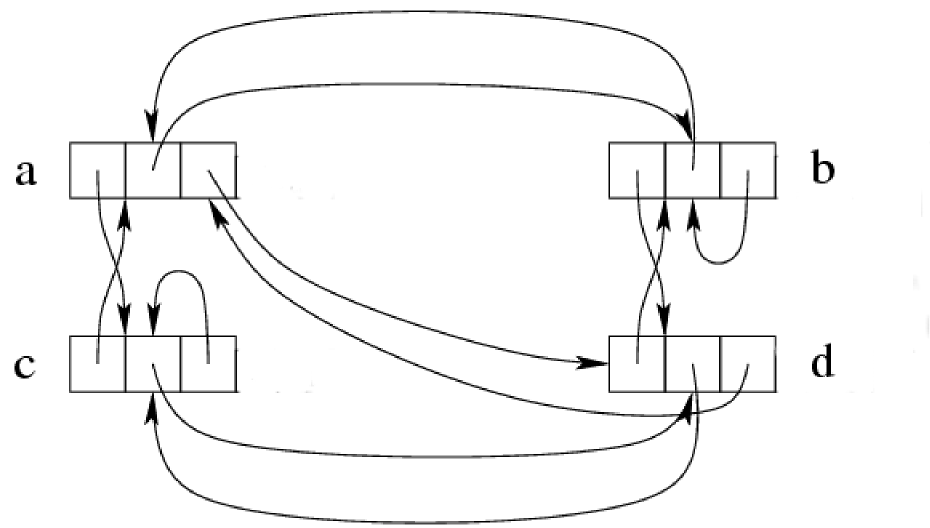


Data structure

The *Gem* data structure:





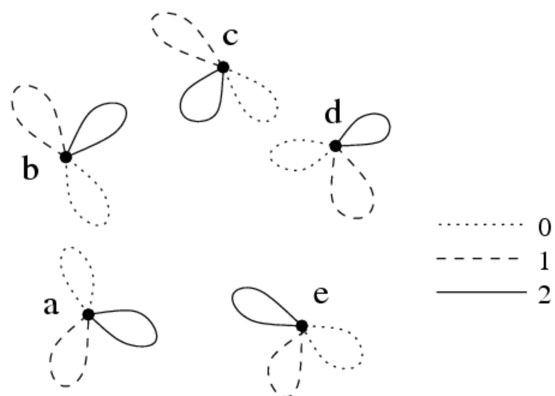


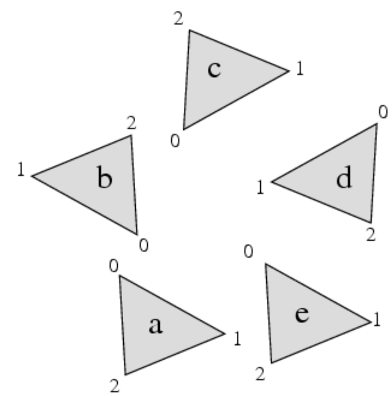
$\texttt{Step(}a\texttt{,}i\texttt{)} \;=\; \phi_i(a) \;=\;$ follow pointer i of node a.

Gem structure operations: Makenode

`Makenode()` creates an unattached simplex:

```
a = MakeNode();
b = MakeNode();
c = MakeNode();
d = MakeNode();
e = MakeNode();
```

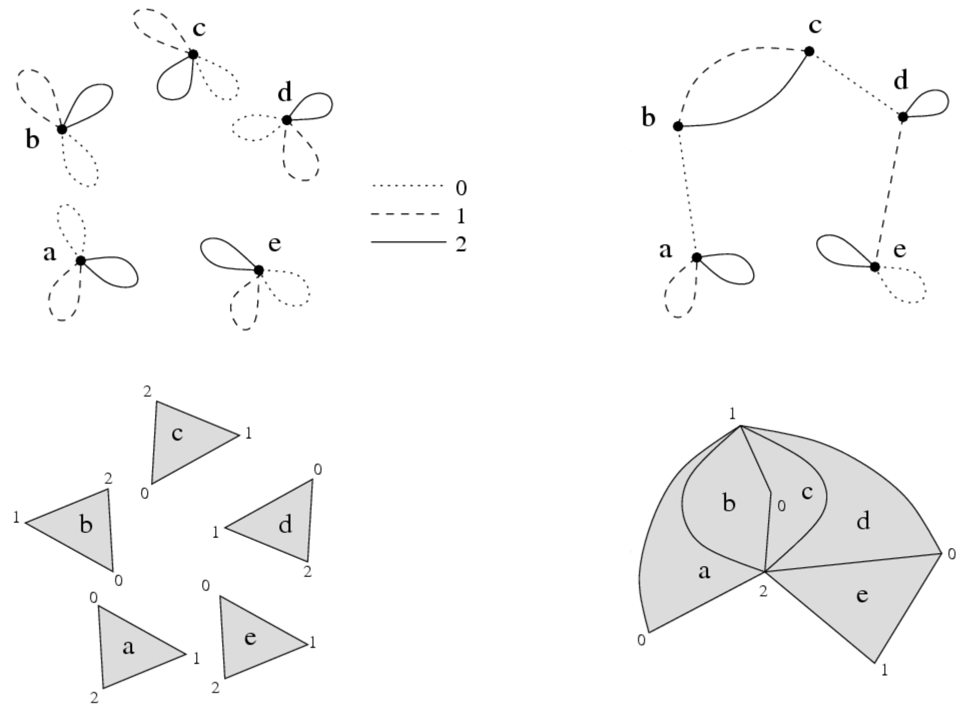




Gem structure operations: Swap

`Swap(`a`,`b`,`i`)` exchanges the i-pointers of a and b:

```
Swap(a,b,0);
Swap(d,e,1);
Swap(d,c,0);
Swap(b,c,1);
Swap(c,b,2);
```

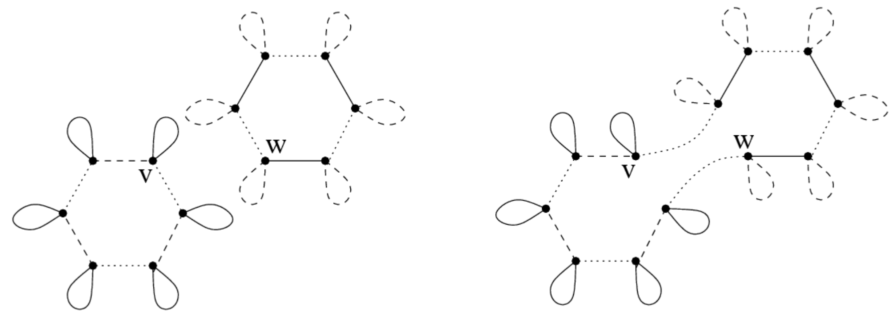


Unsafe - to be used by authorized personnel only!

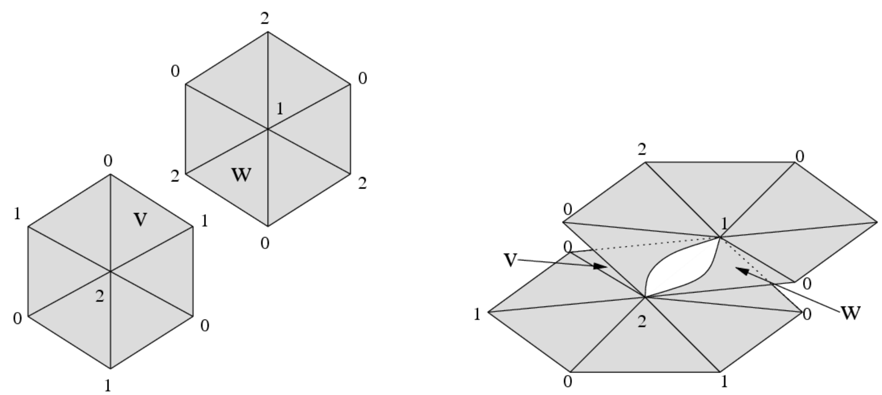
Gem structure operations: Splice

`Splice(`a`,`b`,`i`)` exchanges four pointers of color i:

`Splice(`v`,`w`,`i`):`
    $v' \leftarrow \phi_i(v)$;
    $w' \leftarrow \phi_i(w)$;
    `Swap(`v'`,`w'`,i);`
    `Swap(`v`,`w`,i).`



`Splice(`v`,`w`,0):`

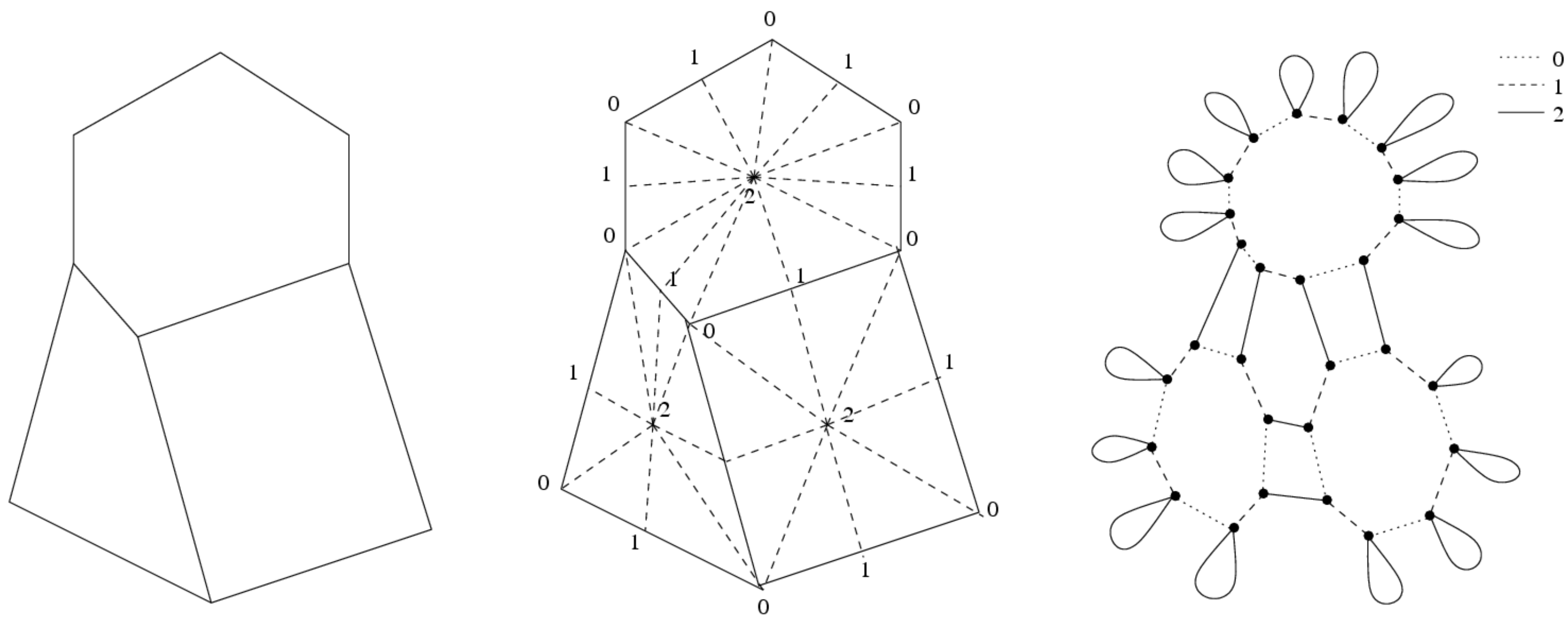


Safe for any parameters!

Barycentric subdivision

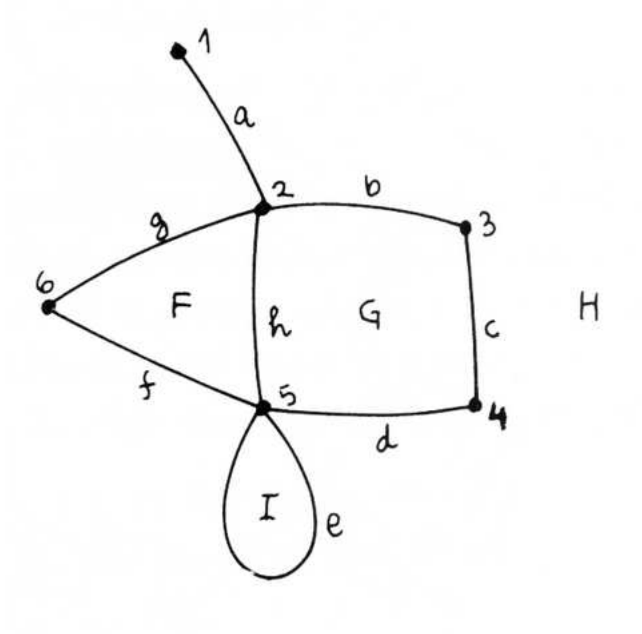
Barycentric gems and representation of general maps:

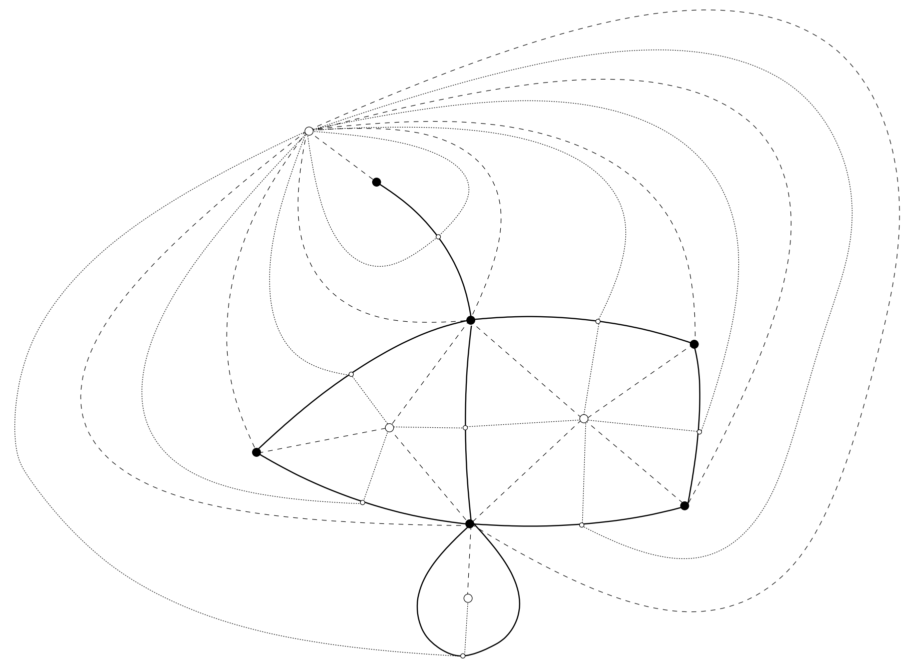
- n-G-maps (P. Lienhardt 1989 [8]).
- Cell-tuple structure (E. Brisson 1989 [2]).



Quad-edge data structure for 2D maps

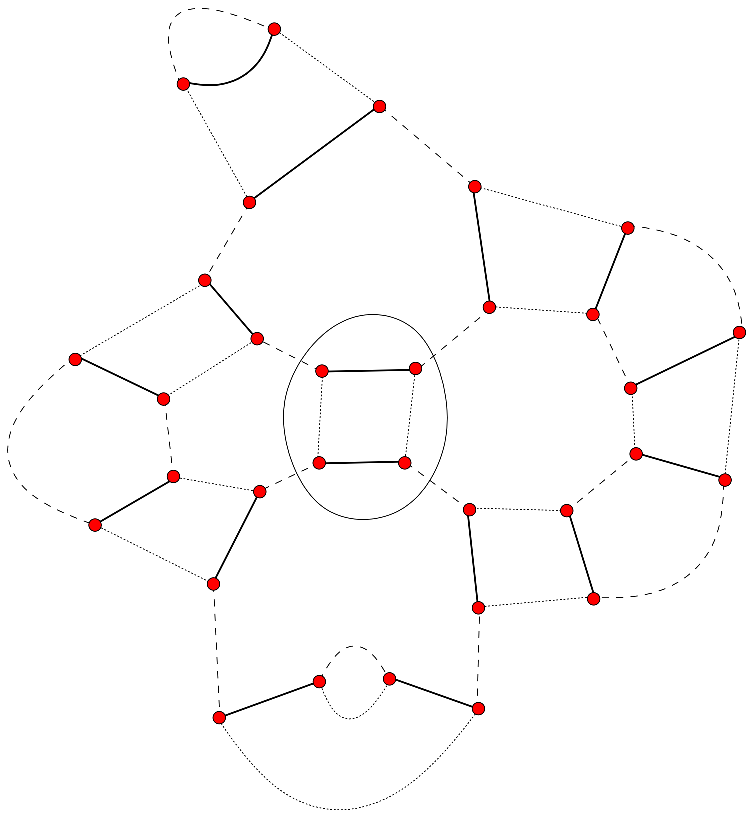
- L. J. Guibas and J. Stolfi 1985 [6].

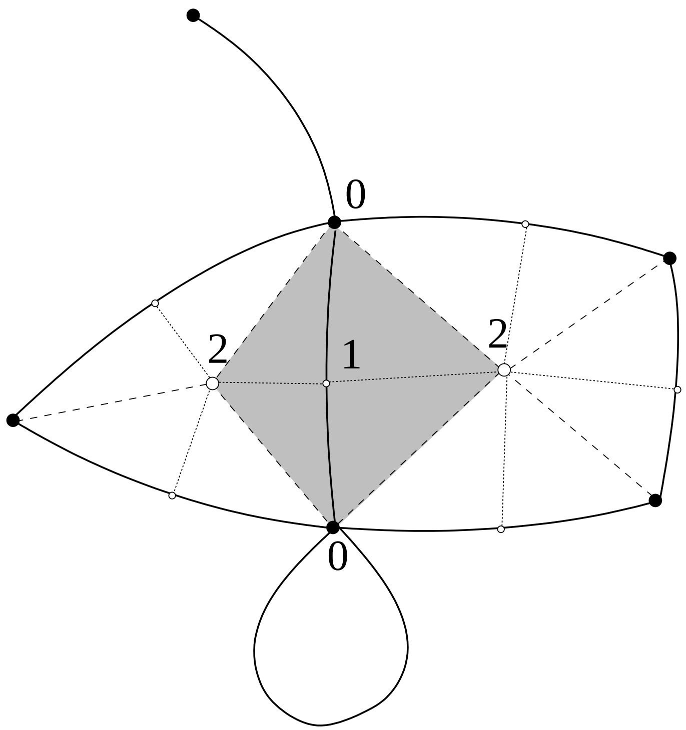




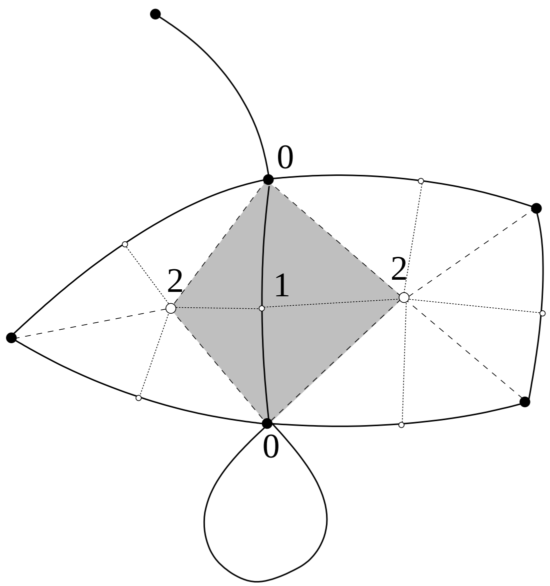
Relationship to quad-edge structure (2)

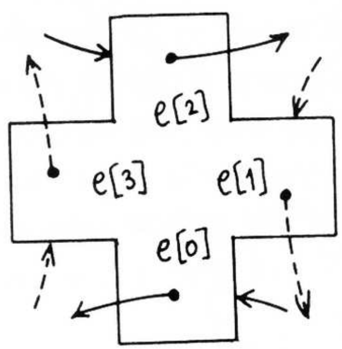
The barycentric gem partitions into $(0, 2)$-colored squares:

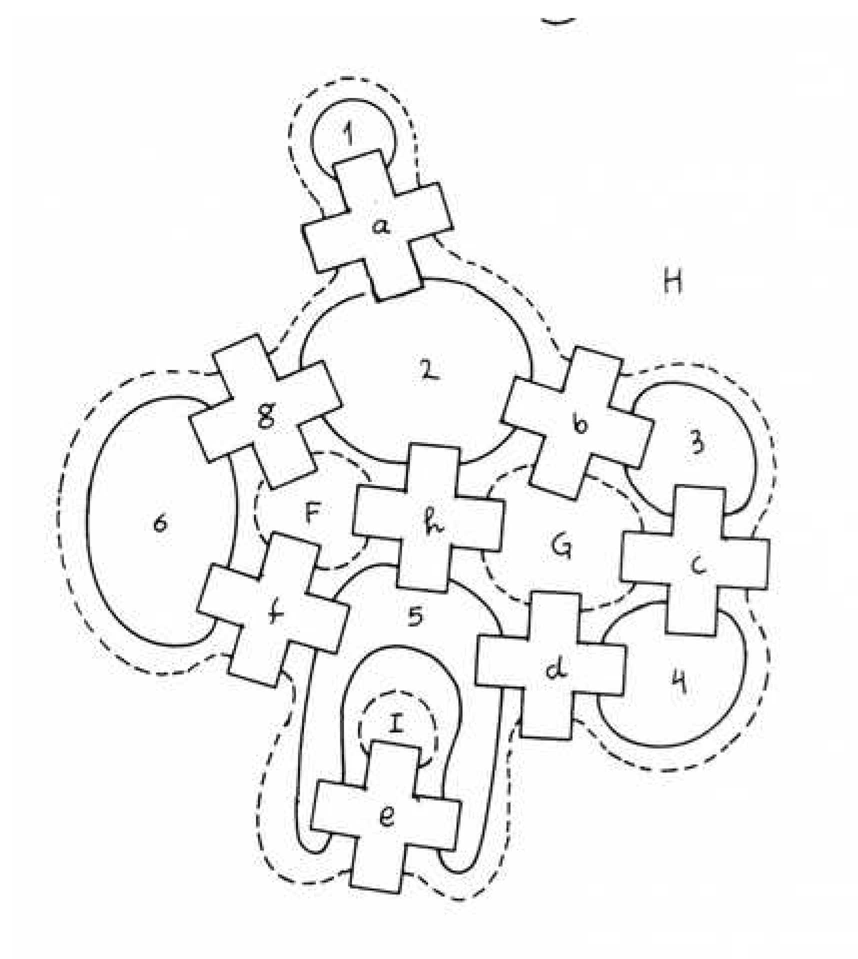




Relationship to quad-edge structure (3)



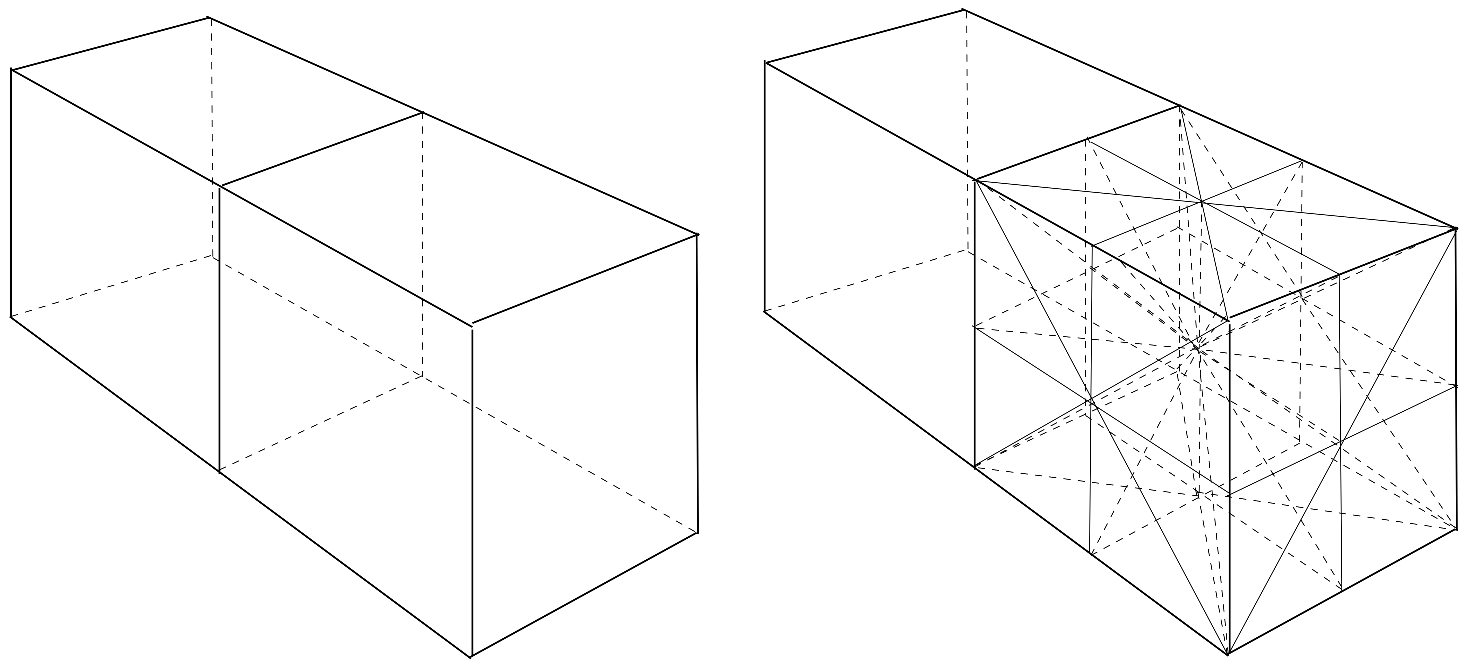




Relationship to facet-edge structure (1)

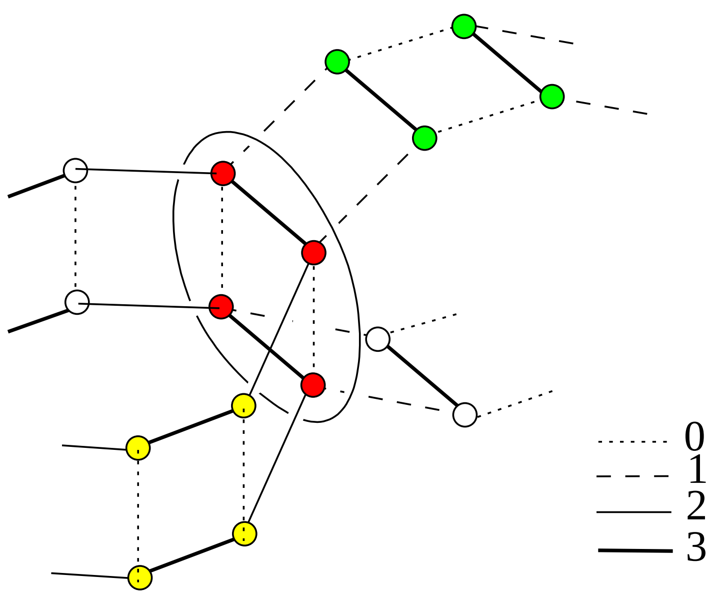
Facet-edge data structure for 3D maps

- D. P. Dobkin and M. J. Laszlo 1987 [4].

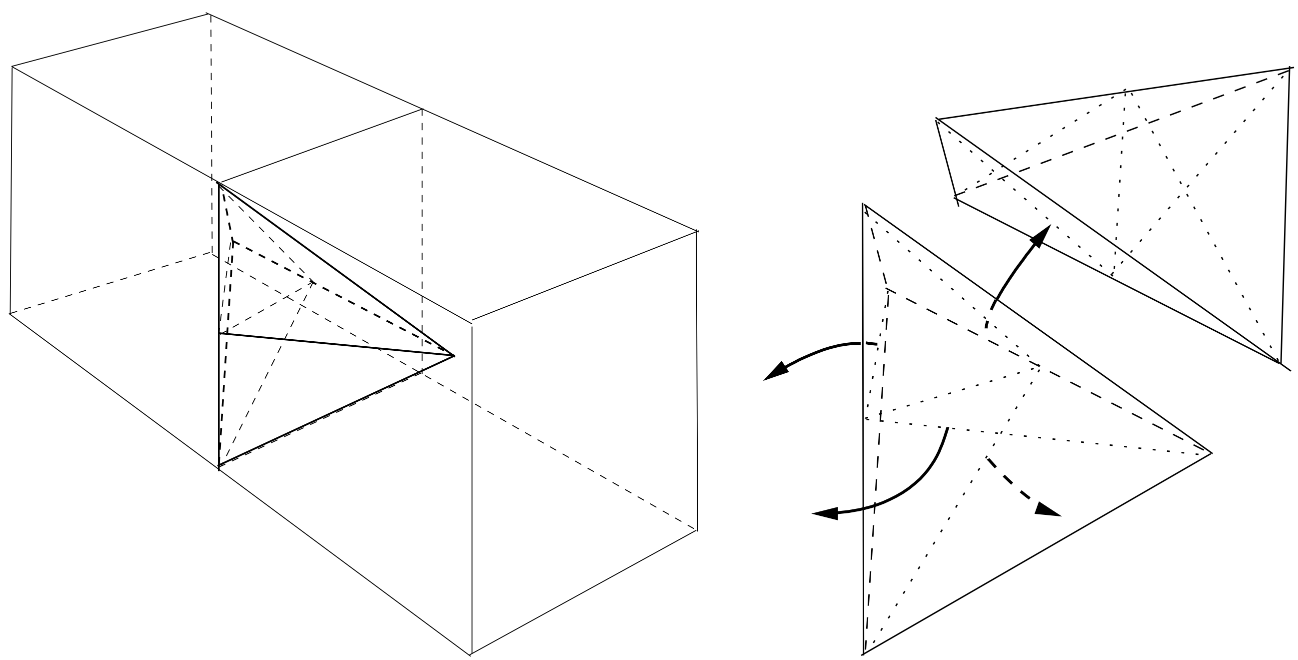


Relationship to facet-edge structure (2)

The barycentric gem partitions into $(0,3)$-colored squares:



Relationship to facet-edge structure (3)



Generalizing quad-edge/facet-edge

Barycentric gem property
(Lienhardt's n-G-map axiom 2 [8]):

$$\phi_i \phi_j \ = \ \phi_j \phi_i \qquad \text{if } |i-j| \geq 2$$

Generalizes quad-edge/facet-edge for d dimensions!

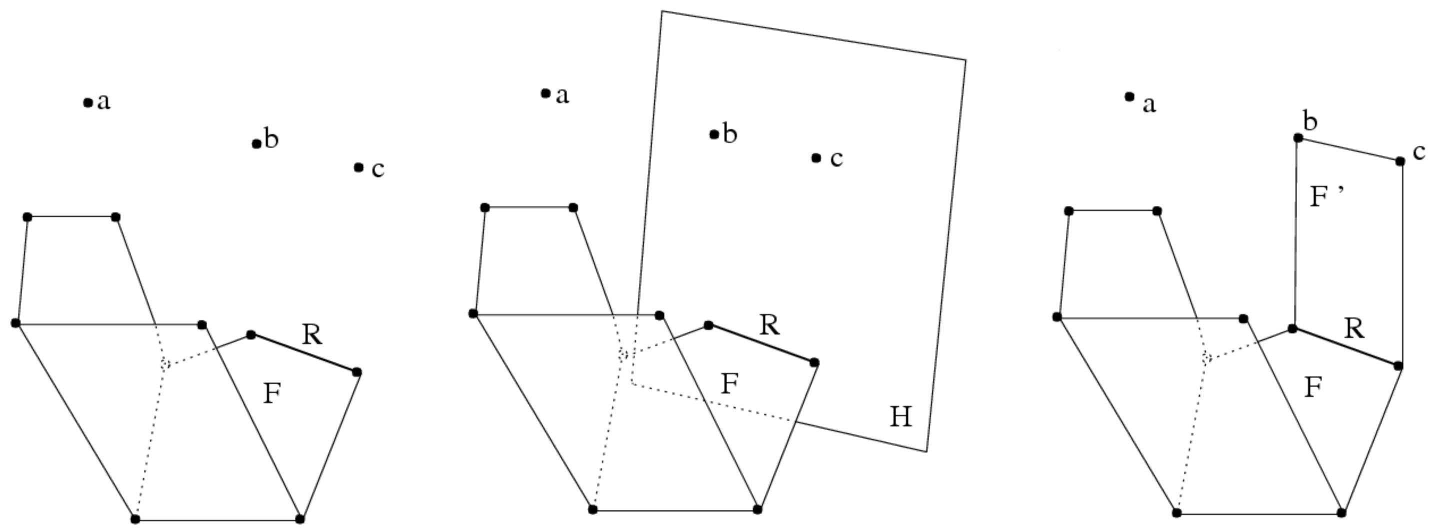
Example for $d = 7$:

- edges colored $0, 2, 5, 7$ comprise disjoint 4-cubes.
- store 16 nodes as 16 parts of same record.
- add 4 bits per pointer to identify which part.
- $\phi_0, \phi_2, \phi_5, \phi_7$ need no pointers.
- save more pointers using $\phi_5 = \phi_2 \phi_5 \phi_2$.
- structure supports duality.

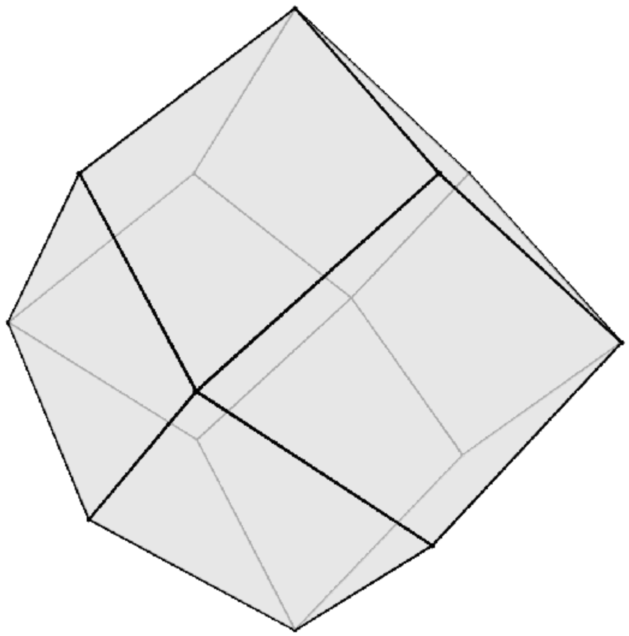
Applications: True convex hull (1)

Application of barycentric gems (n-G-maps, cell-tuple): True exact convex hull, with non-simplicial facets.

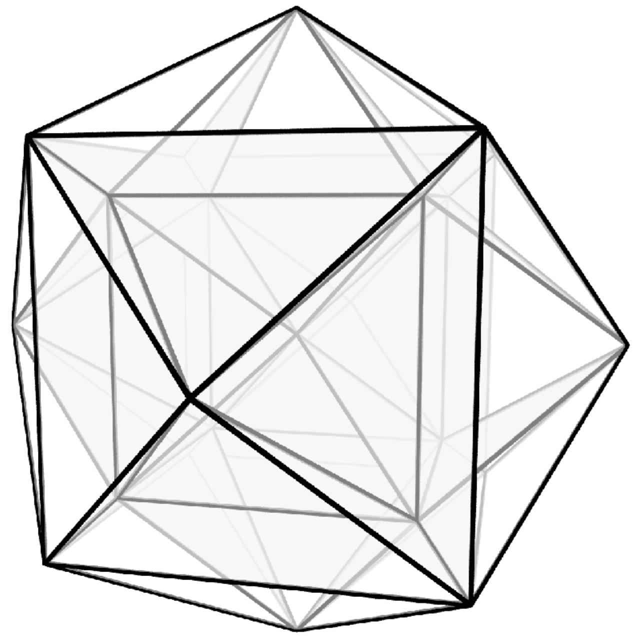
Gift-wrapping algorithm (D. R. Chand & S. S. Kapur 1970 [3]).



Application: True convex hull (2)



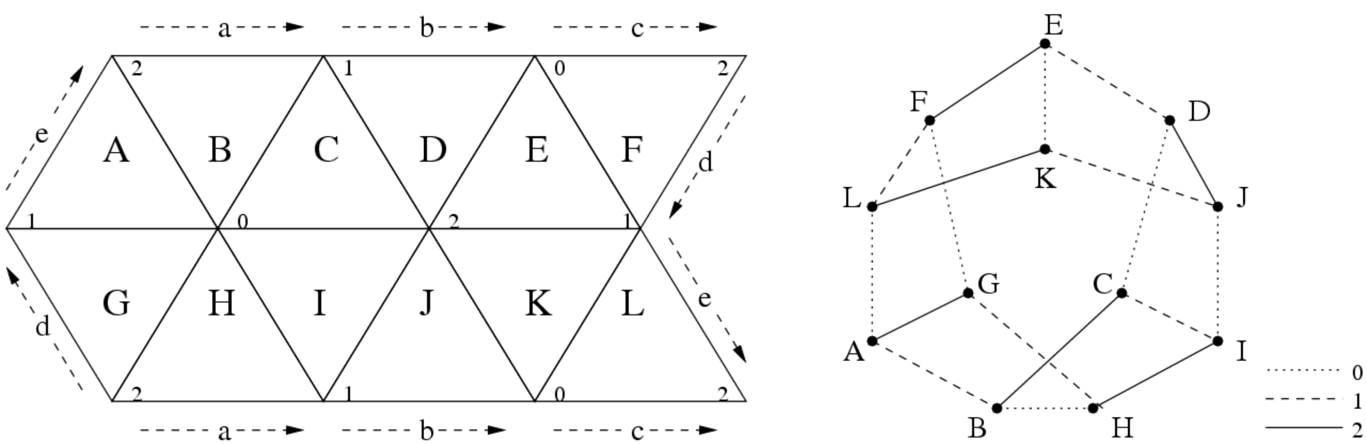
Rhombic dodecahedron (3D)



Regular 24-cell (4D)

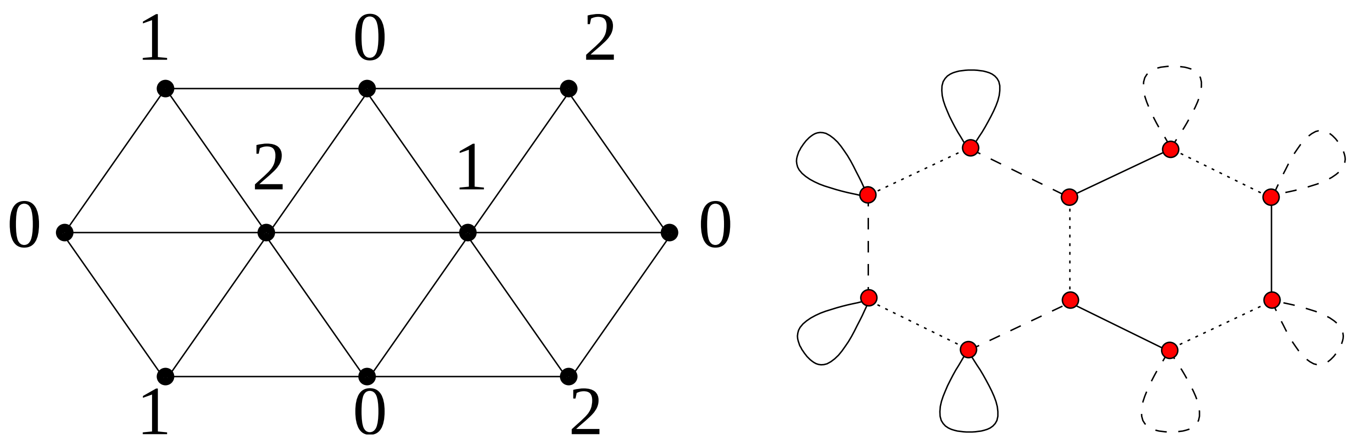
Non-barycentric gems (1)

Gems need not be barycentric subdivisions:



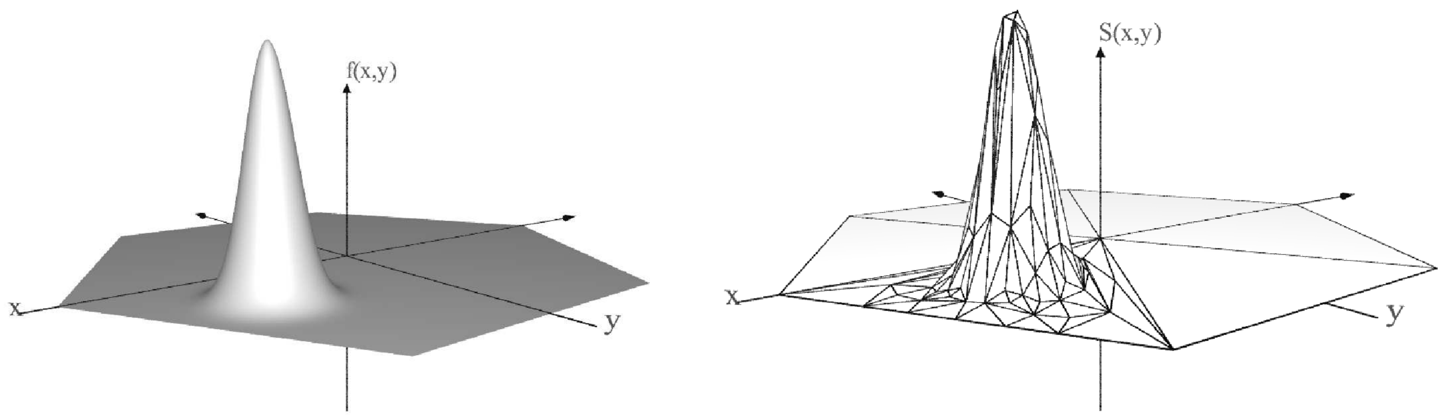
Non-barycentric gems (2)

The free border of a gem need not be of color d:



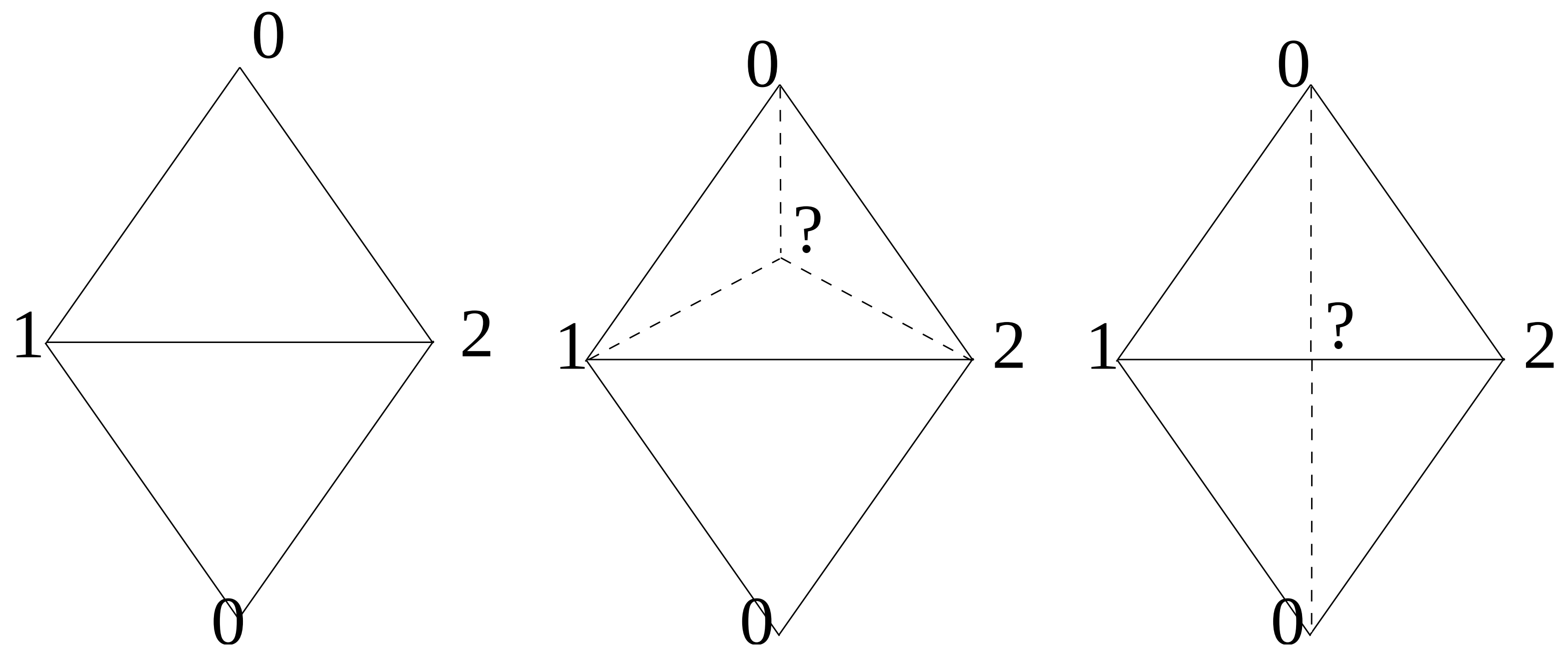
Applications: Adaptive subdivision (1)

Application of non-barycentric gems:
Approximation by adaptive triangular mesh.



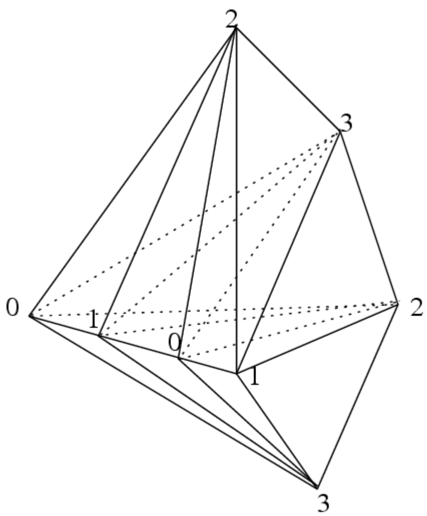
Applications: Adaptive subdivision (2)

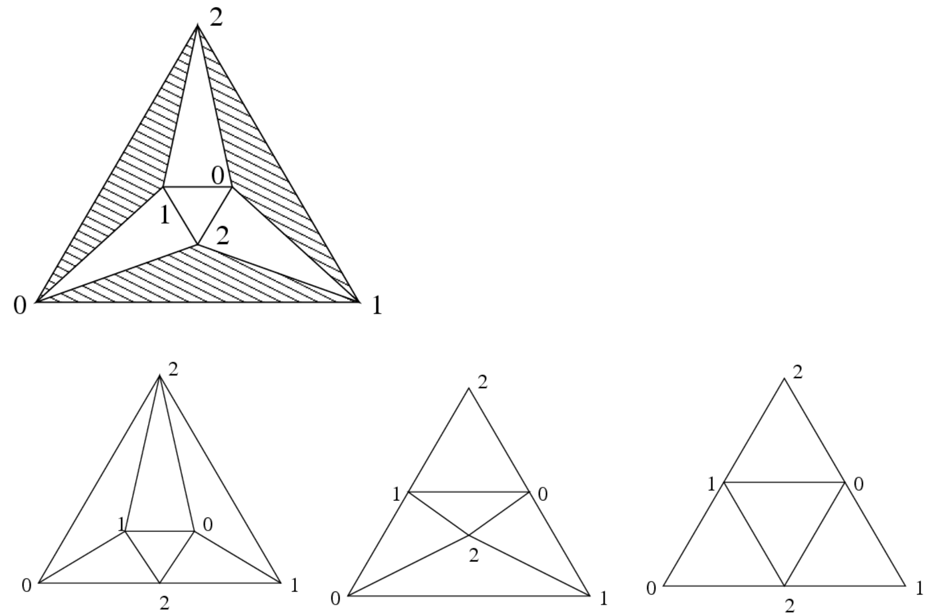
Most popular subdivision schemes don't work:



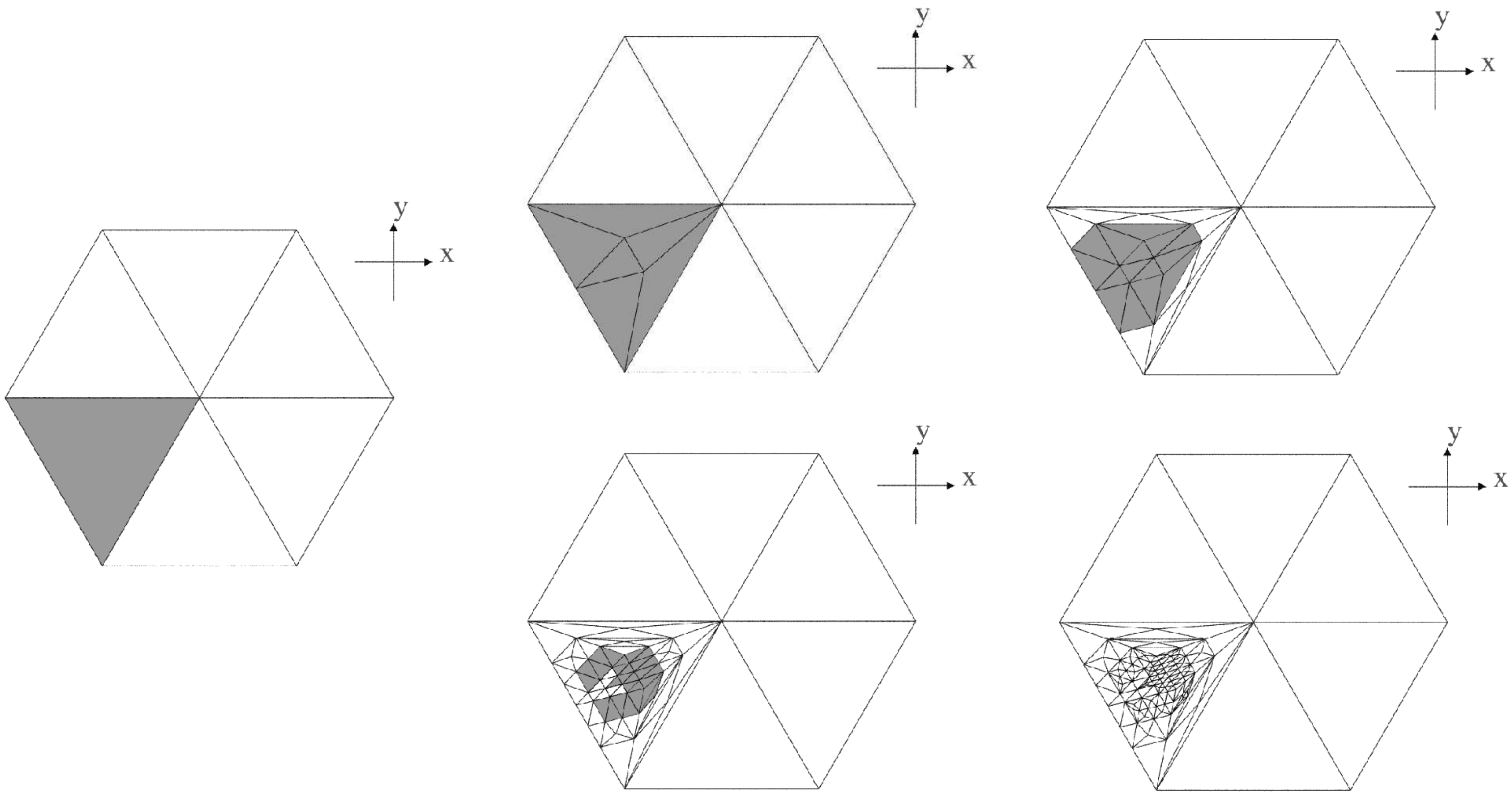
Applications: Adaptive subdivision (3)

Local colored refinement schemes do exist:



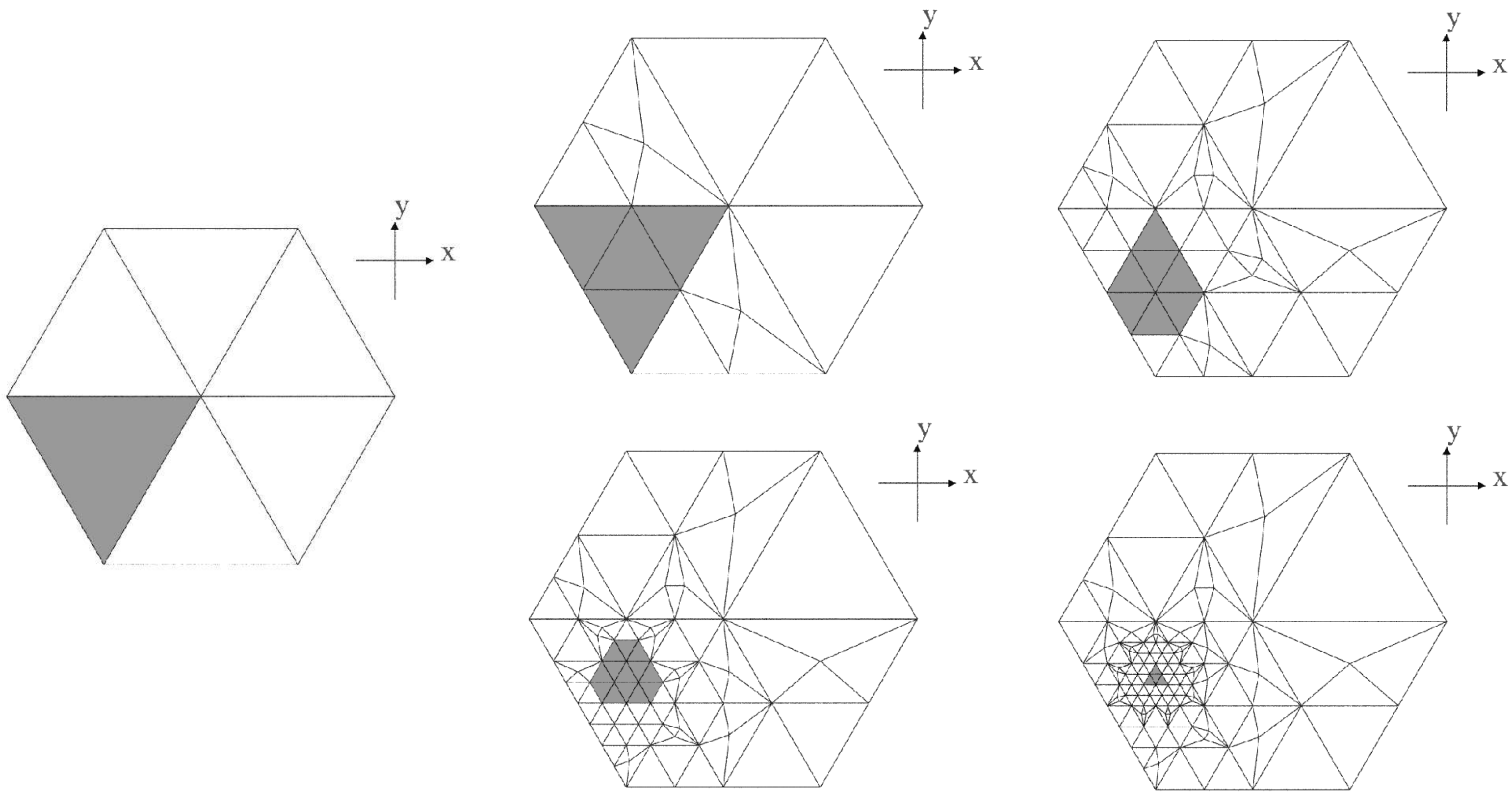


Applications: Adaptive subdivision (4)



Applications: Adaptive subdivision (5)

Can be done with minimum-angle guarantee:



Disadvantages

Disadvantages of gem data structure:

- Restricted triangulations (e.g. no Delaunay).
- Restricted operations (gluing, subdivision).
- More wasteful than quad-edge or facet-edge for maps.

Advantages

Advantages of gem data structure:

- Extends n-G-maps and cell-tuple:
  - Non-barycentric triangulations.
  - Arbitrary free borders.
- Very simple data structure.
- Very simple topological operators.
- Simplified connection to geometry.
- Residues are gems too.
- Poly-ality ($d!$ views) vs. duality (2 views).

Conclusions and extensions

Conclusions:

- $\triangle$s: barycentric $\subset$ colored $\subset$ general.
- Colored $\triangle$s are usable for modeling.
- Gem data structure for colored $\triangle$s.
- Gem data structure and operations are very simple.
- Generalized quad-edge/facet-edge structures.

Further work

Future work and open problems:

- Efficient adaptive subdivision in d dimensions.
- Colorizing general Δs by frugal splitting.

References

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