Gems: A general data structure for colored d-dimensional triangulations

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Abstract: We describe in detail a novel data structure to represent the topology of d-dimensional triangulations. In an arbitrary d-dimensional triangulation, there are $d!$ ways in which a specific facet of an simplex can be glued to a specific facet of another simplex. Therefore, in data structures for general d-dimensional triangulations, this information must be either encoded using $\lceil \log_2(d!) \rceil$ bits for each adjacent pair of simplices, or recomputed at each step by comparing the two sets of $d+1$ vertices. We consider a special class of triangulations, called the *colored triangulations*, in which there is a only one way two simplices can share a specific facet. The *gem data structure*, described here, makes use of this fact to greatly simplify the repertoire of elementary topological operators. The gem data structure is similar to previous data structures for n-dimensional maps based on barycentric subdivision, but is generalized to a much wider class of triangulations. Although colored triangulations are a proper subset of all triangulations, the gem data structure is adequate for many applications, such as adaptive mesh construction.

1 Introduction

We describe a new data structure to represent the topology of simplicial meshes, or triangulations, with any dimension $d \geq 1$. Informally, a d-dimensional triangulation is a partition of some d-dimensional set of points into cells that are topologically equivalent to d-dimensional simplices (triangles, for $d = 2$, tetrahedra, for $d = 3$, etc).

The standard way to represent the topology of such triangulations is to represent each d-dimensional simplex by one data record, and use pointers between records to encode the adjacency relations between the corresponding simplices. However, in an arbitrary d-dimension triangulation, there are $(d+1)!$ ways in which a specific facet of a simplex can be shared with some facet of another simplex. Therefore, in data structures for general d-dimensional triangulations, one must store $\lceil \log_2((d+1)!) \rceil$ additional bits for each adjacent pair, in order to encode this information. This approach is used, for example, in Shewchuk's Triangle code [15]. Another alternative is to recompute this information at each step when the structure is traversed, as in Lee and Schachter's Delaunay triangulation algorithm [7] and in the CGAL 2D and 3D triangulation data structures [**?**].

We consider here a special class of d-dimensional triangulations, the *(vertex-)colored triangulations*, where each vertex is labeled with one of $d+1$ distinct colors in such a way that each simplex has exactly one vertex of each color. In such triangulations, a specific facet of one simplex can be be shared by another simplex in only one way; so each adjacency relation can be represented by a simple pointer, without the additional bits. The *gem data structure*, described here, makes use of this fact to greatly simplify the repertoire of elementary topological operators.

The idea of representing topological spaces by edge-colored graphs that are dual to colored simplicial maps was used by M. Pezzana in 1974 [13, 14], and developed in subsequent years by M. Ferri, C. Gagliardi, and others [4, 5]. The name *gem* (acronym of *Graph Encoded Map*) for such graphs was coined by S. Lins in 1982 [9]. In these works, gems were mathematical devices for theoretical studies of topological spaces, notably the characterization and identification of manifolds. In particular, Lins reserved the name for colored graphs whose underlying space was a compact manifold without border. Loop edges were therefore not allowed, and connectivity was often assumed in the definition.

The gem data structure described here is similar to Brisson's *cell-tuple* structure [2] and to Lienhardt's *n-G-maps* [8]. Like them, it allows manifolds with borders and non-manifold (but triangulable) topological spaces. However, the gem data structure is interpreted in a different way (as a triangulation, rather than a map), and is more general — meaning that the valid cell-tuple and n-G-map structures are a proper subset of the valid gem structures.

2 Triangulations

We give an abstract definition of triangulation that generalizes most of the known triangulation data structures. The gem data structure can represent only a subclass of these triangulations.

2.1 General triangulations

Canonical simplex Let $x_0, \ldots x_d$ be the unit vectors of the coordinate axes of $\mathbb{R}^{d+1}$. The *canonical d-simplex* $\mathcal{S}_d \subset \mathbb{R}^{d+1}$ is the convex hull of the points $x_0, \ldots x_d$. Note that $\mathcal{S}_d$, even though it is a subset of $\mathbb{R}^{d+1}$, is homeomorphic to the closed unit ball $\mathbb{B}^d$ of $\mathbb{R}^d$.

The convex hull of any subset of $\{x_0, \ldots x_d\}$ with $k+1$ elements is a *k-dimensional face*, or *k-face*, of $\mathcal{S}_d$. In particular, the empty set is the only (-1)-face and $\mathcal{S}_d$ is the only d-face. A k-face is said to be *proper* if $k < d$. Faces with dimension 0, 1, and 2 are called *vertices*, *edges*, and *walls*, respectively. For any k in $\{0, .. n\}$, the *k-skeleton* of $\mathcal{S}_d$ is the set of all its j-faces with $j \leq k$. ⋆[*skeleton* not used!]

A $(d-1)$-face is also called a *facet* of $\mathcal{S}_d$. We define the *border* of a k-face f of $\mathcal{S}_d$, denoted here by ∂f, as the union of all the j-faces properly contained in f; and the *core* of f, denoted by κf as the set $f \setminus \partial f$. The core of a k-face is called an *open k-face* (even though it is not, in general, an open subset of $\mathcal{S}_d$ in the topological sense). Note that a 0-face (vertex) has an empty border and a core consisting of a single point, and is therefore also an open 0-face. If $k > 0$, on the other hand, the border is non-empty, and the core is a proper subset of the k-face.

Simplicial morphisms A *simplicial morphism* is a continuous function s from some face f of the canonical simplex $\mathcal{S}_d$ into some topological space, such that the restriction of s to any open face of f is a homeomorphism. Note that the restriction of a simplicial morphism s to any closed face of its domain f is itself a simplicial morphism.

Compatible morphisms Two simplicial morphisms r, s defined on k-faces f, g of $\mathcal{S}_d$ are said to be *compatible* if they have disjoint ranges, or the composition rs^{-1} is a *linear* map of f to g. (Note that the map must be one-to-one, and it must take the vertices of f to those of g. There are only $\binom{d+1}{k+1}$ distinct linear maps with this property.)

Topological simplex More generally, a (*topological*) *d-simplex* s is a simplicial morphism with domain $\mathcal{S}_d$, such that the restrictions of s to any two open faces of s either are compatible or have disjoint ranges.

The restriction of s to a k-face of $\mathcal{S}_d$ will be called, by extension, a *k-face* of s. In the same way we extend the concepts of *vertex*, *edge*, *facet*, *border*, *core*, and *open face* to topological simplices.

We will denote the topological space that is the range of a homeomorphism s by $\langle s \rangle$. Note that, according to the definitions above, a topological simplex always has 2^{d+1} distinct faces, even if two or more faces have the same range.

⋆ [Verificar e consertar a definição!]

Triangulation A *d-dimensional triangulation* (or *d-triangulation*) is a set T of d-dimensional topological simplices such that, for any two simplices $a, b \in T$, and any open proper faces f of a and g of b, the subspaces $\langle \kappa\, a \rangle$ and $\langle \kappa\, b \rangle$ are disjoint, and the subspaces $\langle f \rangle$ and $\langle g \rangle$ are either disjoint or identical. See figure 1.

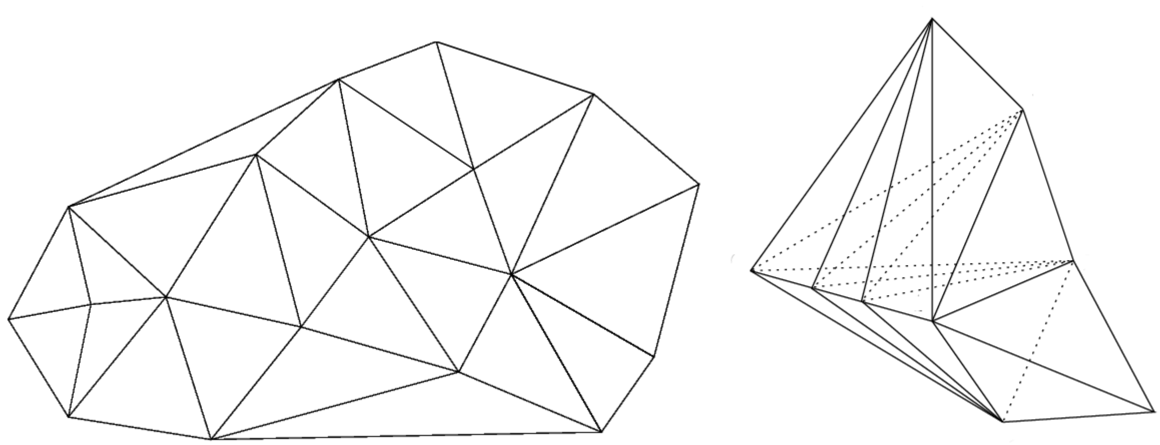


Figure 1: Examples of 2- and 3-dimensional triangulations.

By extension, any k-face (resp. open k-face) of a d-simplex $a \in T$ is called a *k-face* (resp. open k-face) of T. We will denote by $\hat{T}$ the set of all faces of T, of any dimension. We will also define the *space of* T as $\langle T \rangle = \cup_{s \in T} \langle s \rangle$; which is always a compact Hausdorff topological space.

2.2 Colored triangulations

⋆ [Agora todo simplexo é colorido! Basta dizer que as cores têm que casar. Ou seja o mapa rs^{-1}, onde for definido, deve ser a identidade.]

Colored simplex We define a *color set* as any finite subset of $\mathbb{N}$. If C is a color set, a *C-colored simplex* (or *C-simplex*) is a topological simplex with dimension $d = |C| - 1$, whose vertices are labeled with pairwise distinct elements of C (the *colors* of those vertices).

Note that, for any C-colored simplex s, and any $D \subseteq C$, there is a unique $(|D| - 1)$-face of s whose vertices are colored with the colors in D. We call it the *D-face* of s. In particular, the C-face of S is S itself, and the $\emptyset$-face of S is the empty set.

Colored triangulation If C is a color set, we define a *C-colored triangulation* (or *C-triangulation*) as a $(|C| - 1)$-triangulation whose vertices are labeled with colors of C, in such a way that the restriction of the labeling to each simplex of T makes it into a C-simplex. See figure 2.

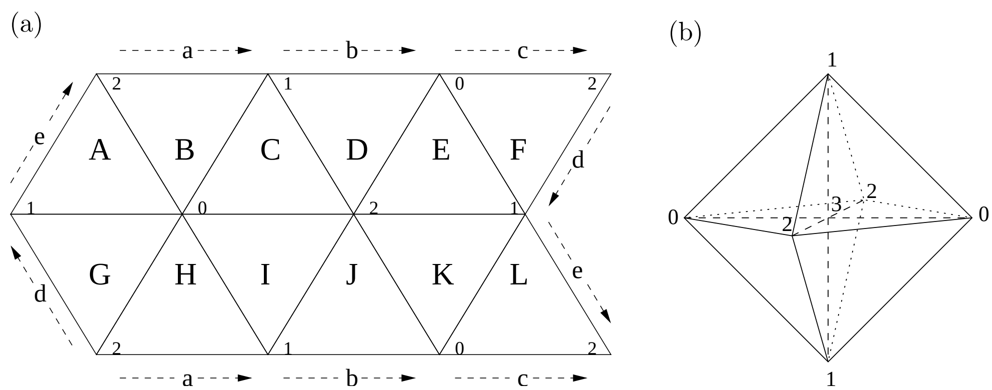


Figure 2: Examples of 2- and 3-dimensional colored triangulations.

3 Gems

Let C be a color set. A *C-gem* is a pair (V, ϕ) where V is a finite set of *gem nodes* and ϕ is a function that to each $i \in C$ associates an involution of V (a permutation of V which is its own inverse), denoted by ϕ_i. If $|C| = d + 1$, we also call this object a *d-gem.*

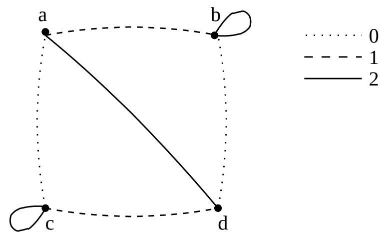
We note that this definition is formally similar to Lienhardt's definition of n-G-maps [8], except that we only retain the first of his two axioms (ϕ_i is an involution for all i). As it turns out, Lienhardt's second axiom ($\phi_i \phi_j$ is an involution, whenever $j - i \geq 2$) is required only to allow interpretation of the data structure as the barycentric subdivision of a cell complex.

3.1 Gems as colored graphs

We can interpret a C-gem (V, ϕ) as a non-directed graph with C-colored edges, where V is the set of graph nodes and there is a i-colored edge between the nodes v and $w \in V$ if and only if $\phi_i(v) = w$. In particular, if $\phi_i(v) = v$, there is a i-colored loop edge on vertex v. For example, figure 3 shows a $\{0,1,2\}$-gem with node set $\{A,B,\ldots,K,L\}$ and the involutions listed in the table (a), and the corresponding edge-colored graph 3(b).

Node	ϕ_0	ϕ_1	ϕ_2
a	c	b	d
b	d	a	b
c	a	d	c
d	b	c	a

(a)



(b)

Figure 3: A gem specified as a set of involutions (a) and as a graph (b).

The graph interpretation of gems provides implicitly the concepts of *walk*, *path*, *connectedness*, etc. In particular, we say that two distinct gem nodes v and w are *i-adjacent* iff $\phi_i(v) = w$.

3.2 Gems as colored triangulations

⋆ [Consertar!]

Every d-gem can be interpreted as a d-dimensional colored triangulation, and completely determines its topology. Informally, the triangulation is the dual of the gem: for each gem node there is a simplex in the triangulation, and two nodes are i-adjacent iff the corresponding simplices share the facet opposite to vertex i, with matched vertex colors. For example, the gem of figure 3 can be interpreted as the colored triangulation shows in figure 2.

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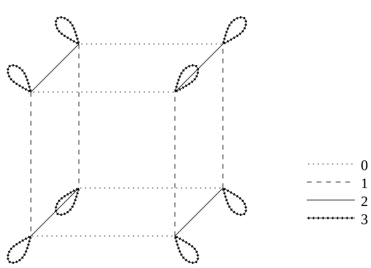
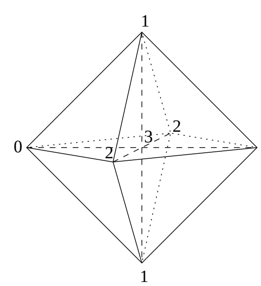
Figure 4: The colored 2-triangulation represented by the gem of figure 3.

Figure 5 shows a three-dimensional example with free border.



(a)



(b)

Figure 5: A 3-gem (a) and the corresponding triangulation (b).

⋆[Mais exemplos!!] To formalize this interpretation, we need the following definitions:

Canonical triangulation of a gem If $G = (V, \phi)$ is a C-gem, with $|C| = d + 1$, its *canonical triangulation* $\mathcal{T}_G$ is obtained as follows. Let X be the topological space $V \times \mathcal{S}_d$, where $\mathcal{S}_d$ is the canonical d-simplex and V is taken with the discrete topology. Note that X consists of $|V|$ connected components, each isomorphic to $\mathcal{S}_d$. Let the vertices of $\mathcal{S}_d$ be colored with the set C, in such a way that vertex x_i of $\mathcal{S}_d$ gets the ith smallest color. Let $\simeq$ be a relation in X such that, for p_1 and $p_2 \in X$, $p_1 \simeq p_2$ iff $p_1 = (v_1, q)$ and $p_2 = (v_2, q)$, where v_1 and v_2 are i-adjacent nodes of V and q is a point of the $(C \setminus i)$-face of $\mathcal{S}_d$, for some $i \in C$. The transitive and reflexive closure $\approxeq$ of $\simeq$ is an equivalence relation. Let Y be the quotient space $X/\approxeq$, and let $[p]_\approxeq \in Y$ be the equivalence class of point $p \in X$. For each $v \in V$, let $\tau(v)$ be the topological simplex such that $\tau(v)(q) = [(v, q)]_{\approxeq}$ for all $p \in \mathcal{S}_d$. The canonical triangulation $\mathcal{T}_G$ is, by definition, the set of topological simplices $\{\tau(v) \,:\, v \in V\}$.

Theorem 1 *If $G = (V, \phi)$ is a C-gem, $\mathcal{T}_G$ is a C-triangulation.*

Proof: We must prove that (1) the simplices of $\mathcal{T}_G$ have pairwise disjoint cores, and (2) any two open faces of T are either disjoint or identical.

Let v', v'' be two distinct nodes of V, and $s' = \tau(v')$, $s'' = \tau(v'')$ be the corresponding simplices in $\mathcal{T}_G$. For the first part, suppose $\langle \kappa\, r\rangle \cap \langle \kappa\, s\rangle \neq \{\}$. Then there are points q', q'' in $\kappa\, \mathcal{S}_d$ such that $(v', q') \simeq (v'', q'')$. However, by construction, these two pairs are related by $\simeq$ only if $v' = v''$, or if $q' == q''$ and both belong to a facet of $\mathcal{S}_d$ — a contradiction.

For the second part, suppose $\langle f'\rangle \cap \langle f''\rangle \neq \{\}$, where f', f'' are open k-faces of s' and s'', respectively, for some $k < d$. $\mathcal{T}_G$ contains a point p shared by a distinct simplex $\tau(v')$, and b is the smallest face of $\tau(v)$ containing p, then $\tau(v')$ contains the whole of b. Indeed, the point p is an equivalence class of $\approxeq$ containing the points (v, q) and (v', q) of X, where q is a point belonging to a facet of $\mathcal{S}_d$. Since $(v, q) \approxeq (v', q)$, there is a sequence $(w_1, \ldots, w_n)$ of nodes of V such that $w_1 = v$, $w_n = v'$ and, for $1 \leq i \leq n-1$, $(w_i, q) \simeq (w_{i+1}, q)$. By the construction of $\simeq$, if c is the smallest face of $\mathcal{S}_d$ containing q, then for every point q' of c and every $i \in \{1, \ldots, n-1\}$, $(w_i, q') \simeq (w_{i+1}, q')$. Thus, for every $q' \in c$, $(w_1, q') \approxeq (w_n, q')$. It follows that $\tau(v)$ and $\tau(v')$ share the set of equivalence classes containing the points $\{v, v'\} \times c$ of X, which is the face b.

Finally, note that the quotient from space X to Y respects the vertex colors, so we can say that the triangulation $\mathcal{T}_G$ is colored with C. $\square$

3.3 Residues of a gem

Besides encoding the adjacency of simplices through facets in the canonical triangulations, the gems also contain information about the lower dimensional simplices and their incidence relations. In order to describe how to obtain this information, we must define the concept of residue of a gem.

Residue Let $G = (V, \phi)$ be a C-gem, and let $D \subseteq C$. A *D-residue* of G is a D-gem that is a connected component of $(V, \phi|_D)$, where $\phi|_D$ is the restriction of ϕ to the color set D. We denote by $\hat{G}$ the set of all residues of G. We say that two nodes of a C-gem are *d-connected*, for some $D \subseteq C$, if they belong to the same D-residue; or, in other words, if they are connected by a path in the gem which uses only D-colored edges.

We extend the bijection $\tau : V \to \mathcal{T}_G$ to a bijection from $\hat{G}$ to $\hat{\mathcal{T}}_G$ as follows: for any D-residue $R = (V', \phi')$ of G, $\tau(R)$ is the $(C \setminus D)$-face of the C-face $\tau(v)$, for any $v \in V'$. See figure 6.

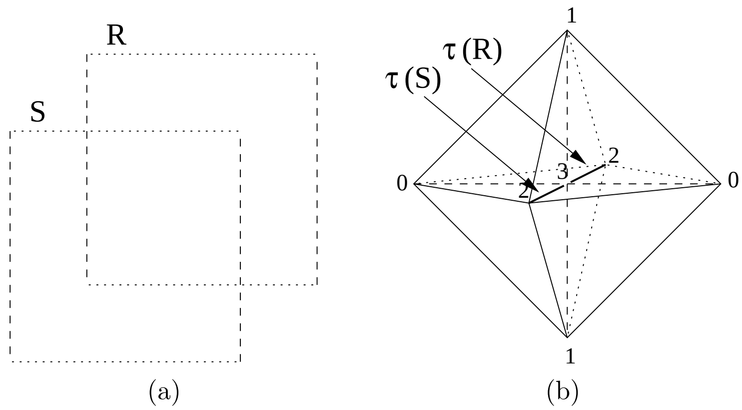


Figure 6: (a) The $\{0,1\}$-residues of the gem given in figure 2(d). (b) The $\{2,3\}$-colored edges of the triangulation that correspond to these residues.

To show that this definition is consistent, we must show that, for any pair of nodes v and w of a D-residue, $\tau(v)$ and $\tau(w)$ share the same $(C \setminus D)$-face, as stated in the following theorem. The proof of this theorem is given in the appendix.

Theorem 2 *If $G = (V, \phi)$ is a C-gem then for any v and $w \in V$ and any $D \subseteq C$, the simplices $\tau(v)$ and $\tau(w)$ of $\mathcal{T}_G$ share their $(C \setminus D)$-face iff v and w are D-connected in G.*

Notice that the extended bijection τ is consistent with the previous definition of this function, that is, every $\emptyset$-residue of G is a single node, which is mapped to the same C-face of $\mathcal{T}_G$ by both definitions.

Theorem 3 *If $G = (V, \phi)$ is a C-gem, the function $\tau : \hat{G} \to \hat{\mathcal{T}}_G$ is a bijection.*

Proof: First we show that for any D-face $a \in \hat{\mathcal{T}}_G$, with $D \subseteq C$, there is a residue R of G such that $\tau(R) = a$. Let b be a C-face of $\hat{\mathcal{T}}_G$ incident on a and let $v \in V$ such that $\tau(v) = b$. Then, if R is the $(C \setminus D)$-residue of G containing v, $\tau(R) = a$.

Now we prove that, for distinct R and $S \in \hat{G}$, $\tau(R) \neq \tau(S)$. Let R be a D-residue and S a E-residue of G, with D and $E \subseteq C$ and $R \neq S$. If $D \neq E$, $\tau(R)$ and $\tau(S)$ are clearly different, so let $D = E$. If v and w are nodes of R and S respectively, they are not D-connected, and thus $\tau(v)$ and $\tau(w)$ do not share their $(C \setminus D)$-face. So $\tau(R) \neq \tau(S)$. $\Box$

Theorem 4 *If R and S are residues of a C-gem G, then $\tau(R)$ is a face of $\tau(S)$ iff S is a residue of R.*

Proof: Let R be a D-residue of G and S be a E-residue of G, with $D, E \subseteq C$.

First, suppose that S is a residue of R, which implies $E \subseteq D$. If v is a node of S, then it is also a node of R. So $\tau(S)$ is the $(C \setminus E)$-face of $\tau(v)$ and R is the $(C \setminus D)$-face of the same C-face $\tau(v)$. Since $(C \setminus E) \supseteq (C \setminus D)$, it is easy to see that $\tau(R)$ is a face of $\tau(S)$.

Now suppose that $\tau(R)$ is a face of $\tau(S)$, which implies $(C \setminus E) \supseteq (C \setminus D)$, i.e. $E \subseteq D$. Since every C-face incident on $\tau(S)$ is also incident on $\tau(R)$, every node of S is also a node of R. So, S is an E-residue of R. $\Box$

3.4 Gem representing a triangulation

A gem encodes the topology not only of its canonical triangulation, but also of any triangulation that is topologically equivalent to the canonical. To formalize this, we need the following definition.

Isomorphism of triangulations Two C-triangulations T and T' are *isomorphic*, or *topologically equivalent*, iff there is a bijection $f : \hat{T} \to \hat{T}'$ mapping D-faces of T into D-faces of T', for $D \subseteq C$, such that, for any a and $b \in \hat{T}$, a is a face of b iff $f(a)$ is a face of $f(b)$.

We say that a C-gem G *represents* a C-triangulation T if T is isomorphic to $\mathcal{T}_G$. It follows that every gem represents some triangulation (in particular its canonical triangulation), and all the triangulations represented by some gem are isomorphic.

As a consequence of theorems 2, 3 and 4 it is possible to characterize the topology of triangulations represented by a gem using the incidence relations between simplices, as stated in the following corollary.

Corollary 5 *Let $G = (V, \phi)$ be a C-gem and T be a C-triangulation. The three statements below are equivalent.*

(i) The gem G represents T.

(ii) There is a bijection δ from V to T such that, for any pair v and $w \in V$, the simplices $\delta(v)$ and $\delta(w)$ share their $(C \setminus D)$-face, for $D \subseteq C$, iff v and w are D-connected.

(iii) There is a bijection δ from $\hat{G}$ to $\hat{T}$ such that, for any pair R and $S \in \hat{G}$, $\delta(R)$ is a face of $\delta(S)$ iff S is a residue of R.

Theorem 6 *Two gems are isomorphic if and only if their canonical triangulations are isomorphic.*

Proof :

Let $G = (V, \phi)$ and $G' = (V', \phi')$ be C-gems. If $\mathcal{T}_G$ and $\mathcal{T}'_G$ are isomorphic, we can say that G' represents $\mathcal{T}_G$. In this case, there is a bijection δ from V' to $\mathcal{T}_G$ such that, for any pair v and $w \in V'$, the simplices $\delta(v)$ and $\delta(w)$ share their $(C \setminus D)$-face, for $D \subseteq C$, iff v and w are D-connected. So, the composition $\delta^{-1}\tau$ is an isomorphism from V to V'.

On the other hand, if G and G' are isomorphic, there is a bijection β from V to V' such that the nodes v and w from V are D-connected, for $D \subseteq C$, iff $\beta(v)$ and $\beta(w)$ are D-connected. So, the composition $\tau\beta$ is a bijection from V' to $\mathcal{T}_G$ such that for any v and $w \in V'$ and any $D \subseteq C$, the simplices $\tau\beta(v)$ and $\tau\beta(w)$ of $\mathcal{T}_G$ share their $(C \setminus D)$-face iff v and w are D-connected in G'. By corollary 5, G' represents $\mathcal{T}_G$, and, consequently, $\mathcal{T}_G$ and $\mathcal{T}'_G$ are isomorphic.

□

3.5 Colored triangulations as gems

As we saw, every gem represents some colored triangulation, that is unique up to isomorphism. Conversely, every triangulation that satisfies a natural "niceness" condition can be completely represented by a gem. Informally, the condition says that the triangulation must be the result of identifying pairs of facets of isolated simplices, so that (1) each facet is identified at most once, and (2) proper faces which are not facets are identified only as a consequence of facet identification. To formalize this statement, we need the following definitions.

Star of a face Let a be a k-face of a d-triangulation T. The *star* of a, denoted $\mathrm{St}(a)$, is the union of the cores of all faces of $\hat{T}$ containing a.

Nice triangulation We say that a colored d-triangulation T is *nice* iff every $(d-1)$-face is face of at most two d-faces, and for every k-face a, with $0 \leq k \leq d-2$, the star $\mathrm{St}(a) \setminus a$ is connected.

It is easy to see that every colored triangulation over a manifold, with or without border, is a nice triangulation. However, for $d \geq 3$, there are nice d-triangulations whose spaces are not manifolds [10].

Theorem 7 *Every nice triangulation is represented by a gem.*

Theorem 8 *Every gem represents a nice triangulation.*

These theorems are proved in the appendix.

4 The gem data structure

In a colored triangulation, the constraints on vertex colors mean that two simplices can share a specific facet in only one way. The *gem data structure* makes use of this constraint to greatly simplify the repertoire of elementary topological operators.

The gem data structure represents a C-gem where $C \subseteq \{0, 1, \ldots, d\}$, for a given constant d. Its records correspond to the gem nodes. Each node s of the gem (or each d-simplex of the triangulation) is represented by record $\hat{s}$ containing $d+1$ pointers to other records, representing the functions $\phi_0, \ldots, \phi_d$. That is, pointer i of record $\hat{s}$ points to record $\hat{r}$, and vice versa, iff the simplices s and r share the facet opposite to vertex i. It follows that if pointer i of record $\hat{s}$ points to record $\hat{r}$, then pointer i of $\hat{r}$ points back to $\hat{s}$. See figure 7.

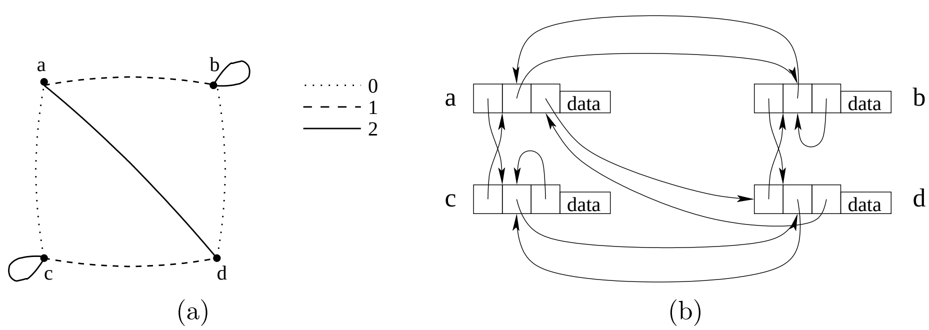


Figure 7: (a) A 2-gem given by a colored graph (b) the registers of the gem data structure.

We will write $\phi_i(r)$ to mean the record referenced by pointer i of record r. In specific applications, each record r may have additional fields, containing geometric or other data [**?**].

The gem data structure. A colored triangulation can be represented by a simple data structure.Since there is only one way two simplices can share a specific facet, the data structure does not require additional bits for the adjacency relations between simplices. For the same reason, the repertoire of topological operators of the gem structure is much simpler that that of arbitrary triangulation structures. Only three operators are sufficient: `MakeNode()` creates a new unattached node, `Swap(a,b,i)` exchanges the i-pointers of nodes a, b, and `Step(a,i)` follows the i-pointer of node a [11].

4.1 Restrictions

The fact that gems can represent only colored triangulations is a major limitation of this data structure. Take for instance a 1-triangulation consisting of a cycle of lines alternating with vertices. This triangulation is colorable if and only if the number of vertices is even. In the case of 2-triangulations, if the link of any vertex is an odd cycle, it cannot be colored. See figure 8.

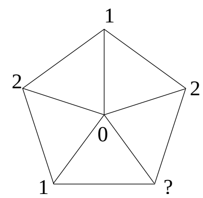


Figure 8: A 2-triangulation that is not colorable.

For this reason, gems cannot be used in problems that require *specific* triangulations, e.g. finding the Delaunay triangulation of a set of points. Still, colored triangulations are suited for many applications. Some of them will be discussed in section 5.

4.2 Traversing a gem structure

Let T be a C-triangulation represented by a gem G. In computer programs, any C-face of T is represented by a pointer to the corresponding gem node. More generally, a D-face a of $\hat{T}$, for any $D \subseteq C$, is referred by a pair $p = (v, D)$, where v is any node of the $(C \setminus D)$-residue corresponding to a.

Given this representation, we describe two operations to traverse the gem. The operation `GetFace(p,E)`, given a D-face a of T referred by $p = (v, D)$, and a set $E \subseteq D$, returns the E-face of a — that is simply the pair (v, E).

The operation `Enum(p,D)`, given an E-face b of T referred by $p = (w, E)$, and a set $D \supseteq E$, enumerates the D-faces of T that have b as a face. This is the same as enumerating the $(C \setminus D)$-residues contained in the $(C \setminus E)$-residue corresponding to b. This enumeration can be performed by a straightforward depth first or breadth first search — assuming that there is some way to mark the nodes that have been visited.

4.3 Creating and modifying the gem structure

We introduce three basic operations to build and modify gem data structures.

`MakeNode()`: creates a new record v and makes $\phi_i(v) = v$ for $0 \leq i \leq d$.

`DeleteNode(`v`)`: is the inverse operation of `MakeNode`. It takes a record v, such that $\phi_i(v) = v$ for all $i \in \{0, \ldots, d\}$, and returns it to the storage pool.

`Swap(`v`,`w`,`i`)`: where v and w are records and $i \in \{0, \ldots, d\}$. This operation consists simply in exchanging the values of pointers $\phi_i(v)$ and $\phi_i(w)$.

Note that `Swap(`v`,`w`,`i`)` is its own inverse, and it is a no-op if $v = w$.

Theorem 9 *If $\{\phi_i(v)\} \cup \{\phi_i(w)\} = \{v\} \cup \{w\}$, the operation `Swap(v,w,i)` preserves the consistency of the gem.*

Proof: We need to show that, upon the application of `Swap`, for every node x and every color i of he gem, $\phi_i(\phi_i(x)) = x$.

Note that the only modification performed by this `Swap` occurs on the i-colored pointers of v and w. If $\{\phi_i(v)\} \cup \{\phi_i(w)\} = \{v\} \cup \{w\}$, thus either $\phi_i(v) = v$ and $\phi_i(w) = w$, or $\phi_i(v) = w$ and $\phi_i(w) = v$. In the first case, `Swap` performs $\phi_i(v) \leftarrow w$ and $\phi_i(w) \leftarrow v$; in the second case, `Swap` performs $\phi_i(v) \leftarrow v$ and $\phi_i(w) \leftarrow w$.

In either case, we have $\phi_i(phi_i(v)) = v$ and $\phi_i(phi_i(w)) = w$ after `Swap`, so the consistency of the gem is preserved. □

We say that a `Swap` call is *valid* if it satisfies the precondition stated in the theorem above. In practice, this operation is used to bind or separate a pair of gem nodes — that is, to glue or unglue two simplices of the triangulation.

Theorem 10 *Any gem data structure can be built by a sequence of* `MakeNodes` *and valid* `Swaps`*.*

Proof: Let G be any gem. If we perform `Swap(`v`,`w`,`i`)` for every pair of distinct nodes v and w of G that are i-adjacent, we get a collection of isolated nodes. Then if we perform `DeleteNode(`v`)` for every node v, we get an empty gem data structure.

The sequence of operations needed to construct the gem G is exactly the inverse of the sequence used to destroy it, which is a sequence of `MakeNodes` followed by a sequence of valid `Swaps`. □

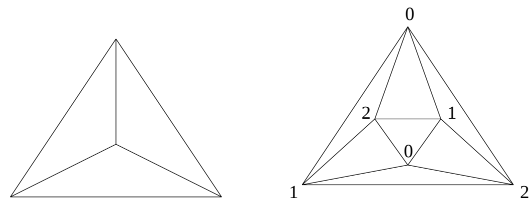
For an example of use of these operations, see our convex hull algorithm [**?**].

5 Applications

An obvious application of the gem data structure is the representation of barycentric subdivision of general maps, as studied by Brisson [2] and Lienhardt [8]. The gem representation was employed in our exact convex hull algorithm [**?**] to represent the barycentric subdivision of the convex hull of a set of points in $\mathbb{R}^d$, thus allowing the representation of hulls with non-simplicial faces.

Gems can also be used as adaptive triangulations, for example, to approximate surfaces by simplicial meshes with prescribed accuracy. Note that it doesn't matter if such a mesh is a colored triangulation, as long as it is a good approximation to the surface. Even though colored triangulations are not as easily subdivided as general ones, they do admit local k-face refinement schemes for any $k \in \{1, \dots, d\}$.

When making a local refinement on a colored triangulation one must guarantee that the newly created vertices can be colored. In general, a local refinement on a colored triangulation requires more simplices than would be necessary on a general triangulation. More precisely, a local piecewise-linear refinement of a d-face that preserves its border requires subdivision into $2^{d+1} - 1$ simplices. In contrast, the same operation can be performed on an unrestricted triangulation using $d + 1$ simplices. See figure 9.

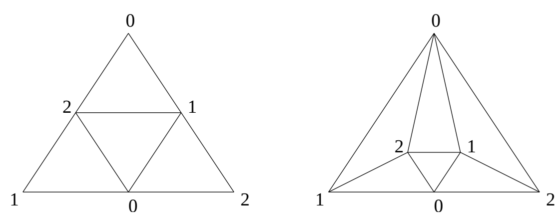


(a) (b)

Figure 9: (a) The minimal piecewise-linear local refinement of a triangle on a general triangulation. (b) The minimal piecewise-linear local refinement of a triangle on a colored triangulation.

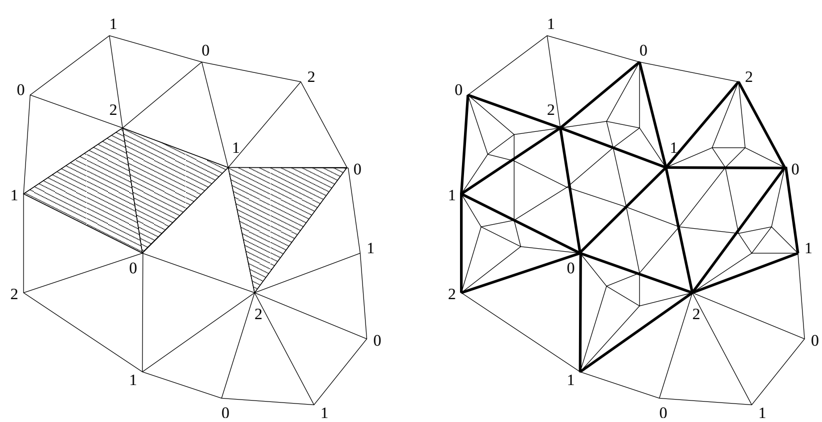
We can also adapt more sophisticated triangulation refinement algorithms for colored triangulations. In Bank et al.'s regular refinement algorithm for 2D triangulations [1], there are two ways a triangle t can be subdivided: if t is marked for subdivision or is adjacent to at least 2 triangles that are marked, it is *regularly subdivided* by connecting the midpoints of its edges, which produces four similar triangles; if t is not marked for subdivision but has one edge split by a regular subdivision on an adjacent triangle, t is *bisected* by connecting the midpoint of the subdivided edge to the opposite vertex.

It turns out that Bank's *regular* subdivision of a triangle is feasible in colored triangulations as well. Triangle *bisection*, though, must be replaced by subdivision into six triangles. See figures 10 and 11.



(a) (b)

Figure 10: A colored triangle refined by (a) a regular subdivision and by (b) a single edge bisection.



(a) (b)

Figure 11: (a) A colored triangulation with some triangles marked for subdivision (hatched). (b) The refined triangulation, using the schemes of figure 10(a) on each marked triangle, and of figure 10(b) on adjacent triangles.

6 Conclusions

The gem data structure is a very simple and general way to represent the topology triangulations. This simplicity yields concise topological operators and elegant algorithms.

In spite of its restriction to colored triangulations, it is suitable for a range of applications, as the representation of barycentric subdivisions and the approximation of surfaces by affine triangulations.

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7 Appendix

Proof of theorem 2 If $G = (V, \phi)$ is a C-gem then for any v and $w \in V$ and any $D \subseteq C$, the simplices $\tau(v)$ and $\tau(w)$ of $\mathcal{T}_G$ share their $(C \setminus D)$-face iff v and w are D-connected in G.

Proof: In order to prove this statement, we must analyze the relation $\approxeq$ on space $X = V \times \mathcal{S}_d$ used in the construction of the canonical triangulation $\mathcal{T}_G$. If $p = (v, q)$ and $p' = (v', q')$ are points of X, $p \approxeq p'$ iff there is a sequence $(w_1, \ldots, w_n)$ of nodes of V such that $v = w_1$, $v' = w_n$ and, for $1 \leq i \leq n-1$, $(w_i, q_i) \simeq (w_{i+1}, q_{i+1})$. By construction, $(w', q') \simeq (w'', q'')$ iff there is a j in C such that w' and w'' are j-adjacent and q' and q'' are the same point belonging to the $(C \setminus \{j\})$-face of $\mathcal{S}_d$. So we can say that $p \approxeq p'$ iff there is $D \subseteq C$ such that v and v' are D-connected and q and q' are the same point belonging to the $(C \setminus D)$-face of $\mathcal{S}_d$.

Now we show that if $\tau(v)$ and $\tau(w)$ are simplices of $\mathcal{T}_G$ sharing a $(C \setminus D)$-face for some $D \subseteq C$, then the nodes v and w are D-connected. Let p be a point in the core of this common face; it is an equivalence class containing the points (v, q) and (w, q) of X, where q is a point belonging to the core of the $(C \setminus D)$-face of $\mathcal{S}_d$. This implies that there is a $D' \subseteq C$ such that v and w are D'-connected and q belongs to the $(C \setminus D')$-face of $\mathcal{S}_d$. Since q belongs to the core of the $(C \setminus D)$-face of $\mathcal{S}_d$, $C \setminus D' \supseteq C \setminus D$ (or equivalently $D \supseteq D'$), and we can say that v and w are D-connected.

Finally, if v and w are D-connected nodes of V, then, for every point q' in the $(C \setminus D)$-face of $\mathcal{S}_d$, $(v, q') \approxeq (w, q')$, and consequently $\tau(v)$ and $\tau(w)$ share their $(C \setminus D)$-face. $\square$

Now we prove that the colored triangulations representable by gems are precisely the nice ones (theorems 7 and 8). We need some additional definitions:

The k-star of a face Let T be a d-triangulation and let a be a j-face of $\hat{T}$. The *k-star* of a, for $j \leq k \leq d$, is the union of the cores of all faces of $\hat{T}$ with dimension at least k and containing a. The k-star of a is denoted $\mathrm{St}_k(a)$. Note that $\mathrm{St}_k(a)$ is the union of the stars of all the k-faces incident on a, and that, for $j < d$, $\mathrm{St}_{j+1}(a) = \mathrm{St}(a) \setminus a$.

k-star-sequence Let T be a d-triangulation and let a be a j-face of $\hat{T}$. A k-star-sequence of a, for $j \leq k \leq d$, is a finite sequence of k-faces incident on a wherein the stars of consecutive k-faces have nonempty intersection.

Lemma 1 *Let T be a d-triangulation and a be a j-face of $\hat{T}$. If, for $j \leq k \leq d$, a has a k-star-sequence connecting every pair of k-faces incident on a, then a has a k-star-sequence containing all the k-faces incident on a.*

Proof: If $l = (b_1, \ldots, b_n)$ is a k-star-sequence of a and c is a k-face incident on a and not contained in l, then we only have to obtain a k-star-sequence from b_n to c and concatenate it to l. By an induction on the number of k-faces incident on a, we obtain the k-star-sequence containing all the k-faces incident on a. $\square$

Lemma 2 *Let T be a d-triangulation and a be a j-face of $\hat{T}$. The k-star of a, for $j \leq k \leq d$, is connected iff, for any pair of k-faces b and c incident on a, there is k-star-sequence of a from b to c.*

Proof: Let a be a j-face and let b and c be a pair of k-faces with no k-star-sequence connecting them, for $0 \leq j \leq k \leq d$. Let R be a binary relation on the set of k-faces incident on a such that bRc iff $\mathrm{St}(b) \cap \mathrm{St}(c) \neq \emptyset$. The closure of R, denoted $R^\star$, is an equivalence relation. Since there is no k-star-sequence connecting b and c, then R defines more than one equivalence class. Hence,

$H = \bigcup_{bRe} \mathrm{St}(e)$ and $\mathrm{St}_k(a) \setminus H$ are two disjoint nonempty open sets partitioning $\mathrm{St}_k(a)$, and thus $\mathrm{St}_k(a)$ is disconnected.

Now, let a be a j-face such that for any pair of k-faces b and c incident on a, for $k \geq j$, there is a k-star-sequence of a from b to c. Let then l be the k-star-sequence of a given by lemma 1. Since the star of any k-face in l is connected, the union of all these stars, which is $\mathrm{St}_k(a)$, is connected. $\square$

Corollary 11 *Let T be a d-triangulation and a be a face of $\hat{T}$. The $(d-1)$-star of a is connected iff, for any pair of d-faces b and c containing a, there is a sequence of d-faces from b to c wherein every pair of consecutive d-faces share a $(d-1)$-face incident on a.*

Proof: It follows from the fact that every such sequence of d-faces can be obtained from a $(d-1)$-star-sequence of a, and vice-versa. $\square$

Lemma 3 *If T is a d-triangulation where every k-face with $0 \leq k < d-1$ has a connected $(k+1)$-star, then such simplices also have a connected $(d-1)$-star.*

Proof: Let a be a k-face of T with $0 \leq k < d-1$. We prove by an induction that a has a connected j-star for every j from $k+1$ to $d-1$. The base case is that a has a connected $(k+1)$-star. Then, we suppose that, for some j between $k+1$ and $d-2$ (limits included), $\mathrm{St}_j(a)$ is connected. Now we must prove that $\mathrm{St}_{j+1}(a)$ is connected.

Let $(b_1, \ldots, b_n)$ be a j-star-sequence containing all the j-faces incident on a. Since, for any $1 \leq i \leq n$, $\mathrm{St}_{j+1}(b_i)$ is connected, the set $\bigcup_{1 \leq i \leq n} b_i$, which is $\mathrm{St}_{j+1}(a)$, is connected. $\square$

***D*-walk** A D-walk of a C-gem, for $D \subseteq C$, is a walk on the gem whose edge colors belong to D. Note that there is a D-walk between two nodes iff they are D-connected.

Proof of theorem 7 Let T be a nice C-triangulation. We will construct a C-gem $G = (V, \phi)$ that represents T. Let V be the set of C-faces T, and, for all $i \in C$, let ϕ_i be an involution of V such that $\phi_i(a) = b$ for a and $b \in V$ iff (i) a and b are distinct C-faces sharing their $(C \setminus \{i\})$-face or (ii) a and b are the same C-face whose $(C \setminus \{i\})$-face is contained in only one C-face. This definition is consistent, since every $(d-1)$-face of T is a face of one or two d-faces.

Now let the bijection δ between V and T be the identity. We must show that for distinct v and $w \in V$, $\delta(v)$ and $\delta(w)$ share their D-face, for any $D \subset C$, iff v and w are $(C \setminus D)$-connected in G.

Let c be a D-face of $\delta(v)$ and $\delta(w)$. If $|D| = d-1$, $\mathrm{St}_{d-1}(c)$ is $\mathrm{St}(c)$, which is connected; if $|D| < d-1$, lemma 3 states that $\mathrm{St}_{d-1}(c)$ is connected. So, by corollary **??**, there is a sequence of C-faces containing c, from $\delta(v)$ to $\delta(w)$, wherein every pair of consecutive C-faces share a $(d-1)$-face containing c. This sequence provides a $(C \setminus D)$-walk between v and w in G.

If there is a $(C \setminus D)$-walk between v and w in G, for every pair x and y of consecutive nodes in this walk, $\delta(x)$ and $\delta(y)$ share their D-face, so $\delta(v)$ and $\delta(w)$ also share their D-face. $\square$

Proof of theorem 8 Let G be a C-gem and let T be a triangulation represented by G.

First we show that the $(d-1)$-star of any k-face a of T, for $0 \leq k \leq d-2$, is connected, and consequently, $\mathrm{St}(a) \setminus a$ is also connected. Let a be a D-face of T with $D \subset C$ and $|D| \leq |C| - 2$, and let R be the $(C \setminus D)$-residue of G corresponding to a. The residue R is connected, so there is a $(C \setminus D)$-walk between any pair of nodes in R. This walks provide the sequences of d-faces required by corollary 11 to guarantee that $\mathrm{St}_{d-1}(a)$ is connected.

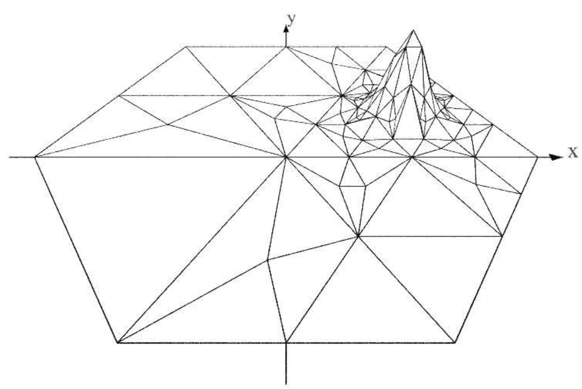
Now we prove that every $(d-1)$-face is face of at most two d-faces. Every $(C \setminus \{i\})$-face c of T, for any $i \in C$, is face of a C-face $\delta(v)$, for some node v of G. If $\phi_i(v) = v$ then c is face of only one C-face, else c is face of two C-faces. □

8 @@@ FROM KYOTOCGGTT2007 ABSTRACT @@@

Relationship to other data structures. The gem data structure is the common denominator of several other structures for representing general maps (cellular complexes) on manifolds. The connection is established by the barycentric subdivision of the map (see figure) which is always a colored triangulation. The n-G-maps of Lienhardt [8] and the cell-tuple structure of Brisson [2] are equivalent to the subset of gem structures representing barycentric subdivisions. The quad-edge structure of Guibas and Stolfi [6] and the facet-edge structure of Dobkin and Laszlo [3] are more specific versions of the structure, optimized for dimensions 2 and 3. Both structures exploit a peculiarity of the barycentric subdivision, namely that the edges with colors i and j, when $|i-j| \geq 2$, are arranged into cycles of length 4 (this is one of n-G-maps' axioms [8]). Therefore, one can merge the four records of each cycle into a single record, and replace the i and j pointers by two additional bits in each remaining pointer, indicating one of the four parts of the pointed record.

It is important to notice that gems are significantly more general than all those structures. In particular, any gem that violates the 4-cycle property is not the barycentric subdivision of any map. Moreover, in a barycentric subdivision, the free border (if any) is constrained to faces of a specific color, whereas in arbitrary colored triangulations there is no such requirement. For these reasons, algorithms have significantly more freedom to manipulate the gem structure than n-G-maps.

Applications. The gem data structure can be used to represent barycentric subdivisions, as the n-G-maps, in all the applications of the later, such as n-dimensional convex hulls [12]. However, colored triangulations that are not barycentric subdivisions have several applications of their own. One example is the approximation of functions and surfaces by adaptive meshes (see figure). We describe here some original algorithms for adaptive subdivision of colored triangulations that exploits this freedom.



9 @@@ FROM KYOTOCGGT TALK SLIDES @@@

Triangulations: A d-dimensional triangulation is a map whose cells are d-dimensional simplices (triangles, for $d = 2$, tetrahedra, for $d = 3$, etc). The standard way to represent d-dimensional triangulations is to represent each d-dimensional simplex by one data record, and use pointers between records to encode the adjacency relations between these simplices.

Colored triangulations: We consider here d-dimensional triangulations which are "colored", in the sense that the vertices are labeld with the integers $\{0, 1, \ldots, d\}$, in such a way that each simplex has one vertex of each color. See below (a) a colored 2-triangulation on the Klein bottle; and (b) a colored 3-triangulation with free border.

Gems: The dual of a colored d-dimensional triangulation T is a d-dimensional *gem* (acronym of *Graph Encoded Map*), as defined by M. Ferri [4] and S. Lins [9]: a graph G whose edges are labeled with the integers $\{0, 1, \ldots, d\}$ so that each node has exactly one incident edge of each color. Namely, there is an i-labeled edge between nodes u and v of G iff the cells of T corresponding to u and v share the face opposite to the i-labeled vertex. As a special case, if a facet of a simplex s is part of the triangulation's free border, the corresponding edge of the gem is a loop. See below the gems of the triangulations on figures (a) and (b).

It can be shown that every gem can be interpreted as a colored triangulation that is unique up to homeomorphism. Conversely, every colored triangulation that satisfies a natural "niceness" condition is completely described by its gem. (Informally, the niceness condition says that the d-triangulation must be the result of gluing isolated d-simplices by their $(d-1)$-faces.)

The gem data structure.: A colored triangulation can be represented by a simple data structure. Each d-simplex s is represented by a record $\hat{s}$ with $d+1$ pointers; pointer i of record $\hat{s}$ points to record $\hat{r}$ and vice versa iff the simplices s and r share the face opposite to vertex i. Since there is only one way two simplices can share a specific facet, the data structure does not require additional bits for the adjacency relations between simplices.

Basic topological operators: The repertoire of topological operators of the gem structure is much simpler that that of arbitrary triangulation structures. Only three operators are sufficient: `MakeNode()` creates a new unattached node, `Swap(a,b,i)` exchanges the i-pointers of nodes a, b, and `Step(a,i)` follows the i-pointer of node a [11].

Relationship to other data structures (1): The gem data structure is the common denominator of several other structures for representing general maps (cellular complexes) on manifolds. The connection is established by the barycentric subdivision of the map (see figure) which is always a colored triangulation.

The n-G-maps of P. Lienhardt [8] and the cell-tuple structure of E. Brisson [2] are equivalent to the subset of gem structures representing barycentric subdivisions.

Relationship to quad-edge data structure: The quad-edge structure of L. J. Guibas and J. Stolfi [6] can be seen as a specific version of the gem (or n-G map, or cell-tuple) structure, optimized for dimensions 2. It exploits a peculiarity of the barycentric subdivision of 2D maps, namely that the edges with colors 0 and 2 are arranged into cycles of length 4. Therefore, one can merge the four records of each cycle into a single record, and replace the 0 and 2 pointers by two additional bits in each remaining pointer, indicating one of the four parts of the pointed record.

Relationship to facet-edge data structure: The facet-edge structure of D. P. Dobkin and M. J. Laszlo [3] can also be seen as a specific version of the gem (or n-G-map, or cell-tuple) structure, optimized for dimension 3. The edges colored 0 and 3 are a set of disjoint 4-cycles, so each cycle is represented as a single record. (One could also use the color pair $(0, 2)$, or $(1, 3)$; but $(0, 3)$ allows a dual view of the structure.)

Generalized quad-edge/facet-edge: The quad-edge and facet-edge structures can be generalized to higher dimensions. For an d-dimensional barycentric gem the edges with colors i and j, when $|i-j| \geq 2$, are arranged into cycles of length 4 (this is one of n-G-maps' axioms [8]). So, if K is any set of k colors which differ by at least 2, then the edges with those colors comprise a set of disjoint subgraphs, each isomorphic to a colored $k = \lceil d/2 \rceil$-dimensional hypercube. We can represent each subgraph by a single record with 2^k sub-record, each with $d - k = \lfloor d/2 \rfloor$ pointers; using k bits to select the right sub-record. If K is invariant under d-complement, the structure still allows duality. This trick saves about half of the pointers of the gem structure, and still allows duality. The same trick could be used with the odd colors, but the savings would be

Applications (1): Convex hulls: The gem data structure can be used to represent barycentric subdivisions, as the n-G-maps, in all the applications of the later, such as d-dimensional convex hulls [12].

Non-barycentric gems: The gem data structure is significantly more general than all those map data structures. In particular, any gem that violates the 4-cycle property is not the barycentric subdivision of any map.

Arbitrary free border: Moreover, in a barycentric subdivision, the free border (if any) of the map is constrained to faces of a specific color; whereas in arbitrary colored triangulations there is no such requirement. For these reasons, algorithms have significantly more freedom to manipulate the gem structure than n-G-maps.

Applications: Adaptive subdivision (1): Colored triangulations that are not barycentric subdivisions have several applications of their own. One example is the approximation of functions and surfaces by adaptive meshes (see figure). We describe here some original algorithms for adaptive subdivision of colored triangulations that exploits this freedom.

Applications: Adaptive subdivision (3): Subdivision schemes must produce *colored* triangulations. Some popular schemes do not work, e.g. bisecting an edge, or splitting a triangle into three triangles.

Applications: Adaptive subdivision (3): If one must split a single element ant its star, with piecewise-linear simplices, then one must split an edge into *three* parts, or a triangle into *seven* parts, in general an d-dimensional simplex into $2^{d+1} - 1$ parts. However, more economical schemes exist for splitting several adjacent elements together.

Applications: Adaptive subdivision (4): One can do adaptive local subdivision of 2D colored triangulations. One can even do that while avoiding thin triangles.

Last edited 2007-08-13 16:09:30 by stolfi