Gems: A general data structure for d-dimensional triangulations

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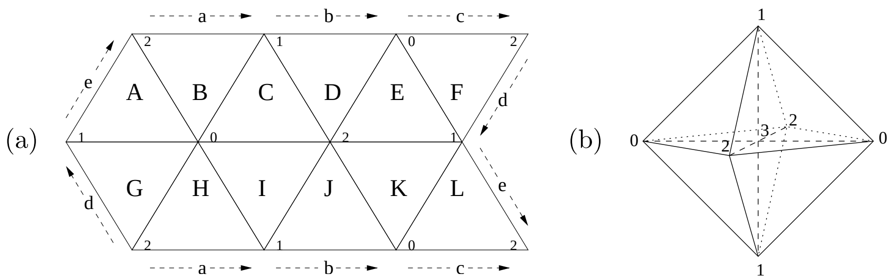
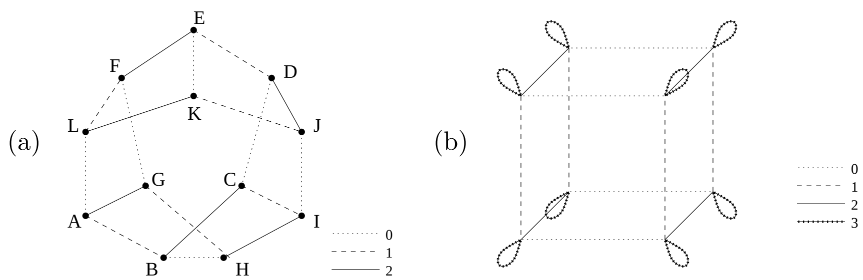
Triangulation data structures. A d-dimensional triangulation is a map whose cells are d-dimensional simplices (triangles, for $d = 2$, tetrahedra, for $d = 3$, etc). The standard way to represent d-dimensional triangulations is to represent each d-dimensional simplex by one data record, and use pointers between records to encode the adjacency relations between these simplices.

In an arbitrary d-dimension triangulation, there are $d!$ ways in which a specific facet of a simplex can be glued to a specific facet of another simplex. Therefore, in data structures for general d-dimensional triangulations, one must use $\lceil \log_2(d!) \rceil$ bits for each adjacent pair, in order to encode this information. This approach is used in Shewchuk's Triangle code [10] and in the CGAL 2D and 3D triangulation data structures [1]. Another alternative is to recompute this information at each step when the structure is traversed [5].

Colored triangulations and gems. We consider here d-dimensional triangulations which are "colored", in the sense that the vertices are labeld with the integers $\{0, 1, \ldots, d\}$, in such a way that each simplex has one vertex of each color. See below (a) a colored 2-triangulation on the Klein bottle; and (b) a colored 3-triangulation with free border.



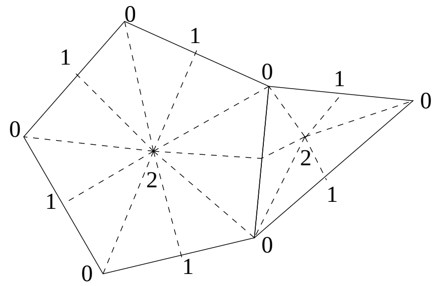
The dual of a colored d-dimensional triangulation T is a d-dimensional *gem* (acronym of *Graph Encoded Map*), as defined by Lins [7]: a graph G whose edges are labeled with the integers $\{0, 1, \ldots, d\}$ so that each node has exactly one incident edge of each color. Namely, there is an i-labeled edge between nodes u and v of G iff the cells of T corresponding to u and v share the face opposite to the i-labeled vertex. As a special case, if a facet of a simplex s is part of the triangulation's free border, the corresponding edge of the gem is a loop. See below the gems of the triangulations on figures (a) and (b).



It can be shown that every gem can be interpreted as a colored triangulation that is unique up to homeomorphism. Conversely, every colored triangulation that satisfies a natural "niceness" condition is completely described by its gem. (Informally, the niceness condition says that the d-triangulation must be the result of gluing isolated d-simplices by their $(d-1)$-faces.)

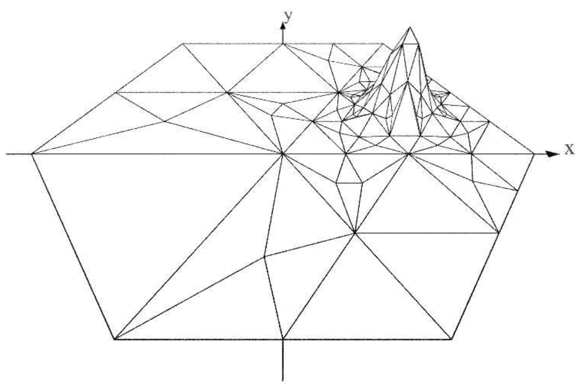
The gem data structure. A colored triangulation can be represented by a simple data structure. Each d-simplex s is represented by a record $\hat{s}$ with $d+1$ pointers; pointer i of record $\hat{s}$ points to record $\hat{r}$ and vice versa iff the simplices s and r share the face opposite to vertex i. Since there is only one way two simplices can share a specific facet, the data structure does not require additional bits for the adjacency relations between simplices. For the same reason, the repertoire of topological operators of the gem structure is much simpler that that of arbitrary triangulation structures. Only three operators are sufficient: `MakeNode()` creates a new unattached node, `Swap(a,b,i)` exchanges the i-pointers of nodes a, b, and `Step(a,i)` follows the i-pointer of node a [8].

Relationship to other data structures. The gem data structure is the common denominator of several other structures for representing general maps (cellular complexes) on manifolds. The connection is established by the barycentric subdivision of the map (see figure) which is always a colored triangulation. The n-G-maps of Lienhardt [6] and the cell-tuple structure of Brisson [2] are equivalent to the subset of gem structures representing barycentric subdivisions. The quad-edge structure of Guibas and Stolfi [4] and the facet-edge structure of Dobkin and Laszlo [3] are more specific versions of the structure, optimized for dimensions 2 and 3. Both structures exploit a peculiarity of the barycentric subdivision, namely that the edges with colors i and j, when $|i-j| \geq 2$, are arranged into cycles of length 4 (this is one of n-G-maps' axioms [6]). Therefore, one can merge the four records of each cycle into a single record, and replace the i and j pointers by two additional bits in each remaining pointer, indicating one of the four parts of the pointed record.



It is important to notice that gems are significantly more general than all those structures. In particular, any gem that violates the 4-cycle property is not the barycentric subdivision of any map. Moreover, in a barycentric subdivision, the free border (if any) is constrained to faces of a specific color, whereas in arbitrary colored triangulations there is no such requirement. For these reasons, algorithms have significantly more freedom to manipulate the gem structure than n-G-maps.

Applications. The gem data structure can be used to represent barycentric subdivisions, as the n-G-maps, in all the applications of the later, such as n-dimensional convex hulls [9]. However, colored triangulations that are not barycentric subdivisions have several applications of their own. One example is the approximation of functions and surfaces by adaptive meshes (see figure). We describe here some original algorithms for adaptive subdivision of colored triangulations that exploits this freedom.



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