Gems: A general data structure for d-dimensional triangulations

Arnaldo J. Montagner and Jorge Stolfi
Institute of Computing, State University of Campinas,
13081-970 Campinas, SP.
arnaldom@liv.ic.unicamp.br, stolfi@ic.unicamp.br

Abstract: We describe in detail a novel data structure for d-dimensional triangulations. In an arbitrary d-dimensional triangulation, there are $d!$ ways in which a specific facet of an simplex can be glued to a specific facet of another simplex. Therefore, in data structures for general d-dimensional triangulations, this information must be encoded using $\lceil \log_2(d!) \rceil$ bits for each adjacent pair of simplices. We study a special class of triangulations, called the *colored triangulations*, in which there is a only one way two simplices can share a specific facet. The *gem data structure*, described here, makes use of this fact to greatly simplify the repertoire of elementary topological operators.

1 Introduction

We describe a data structure capable of representing simplicial meshes, or triangulations, with any dimension ≥ 1.

The standard way to represent d-dimensional triangulations is to represent each d-dimensional simplex by one data record, and use pointers between records to encode the adjacency relations between these simplices. In an arbitrary d-dimension triangulation, there are $d!$ ways in which a specific facet of a simplex can be glued to a specific facet of another simplex. Therefore, in data structures for general d-dimensional triangulations, one must use $\lceil \log_2(d!) \rceil$ bits for each adjacent pair, in order to encode this information. This approach is used in Shewchuk's Triangle code [9] and in the CGAL 2D and 3D triangulation data structures [2]. Another alternative is to recompute this information at each step when the structure is traversed [4].

In our work, we study a special class of triangulations, called the *colored triangulations*, in which there is only one way two simplices can share a specific facet. In these triangulations, there is no need to keep additional bits for the adjacency relations. The *gem data structure*, described here, makes use of this fact to greatly simplify the repertoire of elementary topological operators.

The gem data structure is based on the concept of a *gem* (acronym of *Graph Encoded Map*), an edge colored graph that provides a discrete representation of the topological structure of a colored triangulation. Gems were defined as mathematical devices by S. Lins in 1982 [6]. See in figure 1 examples of 2-dimensional (a) and 3-dimensional (c) triangulations, and their respective gems.

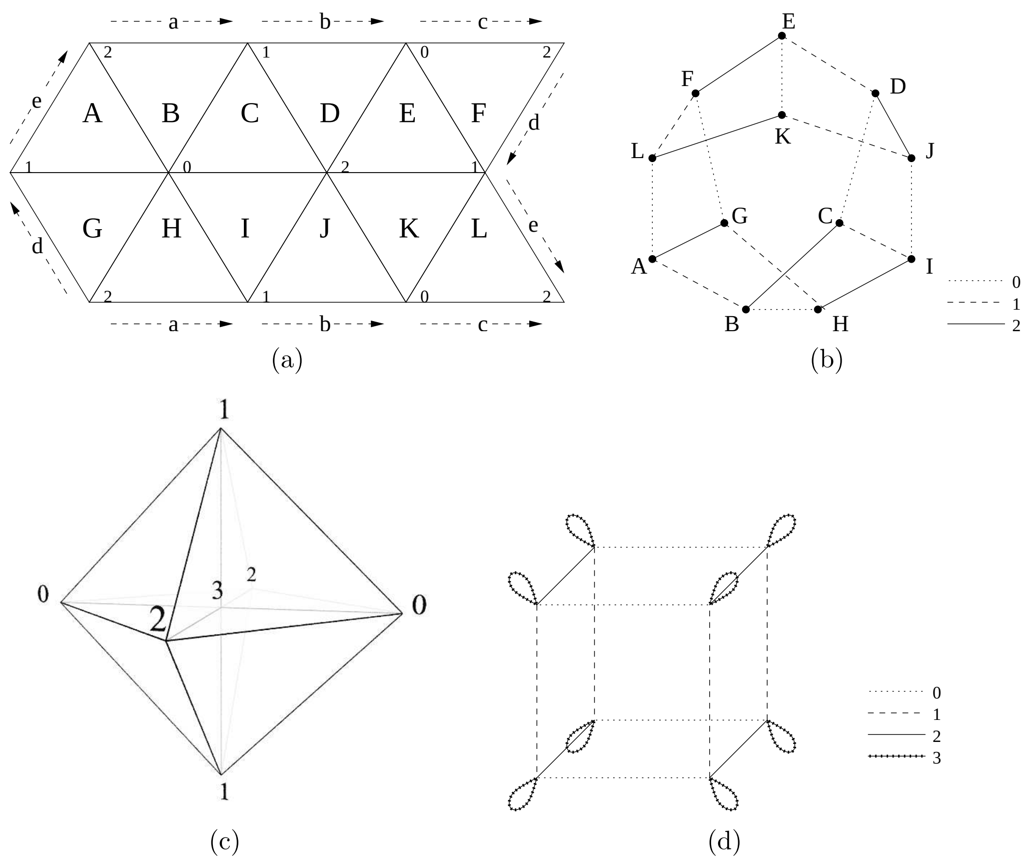

Figure 1: (a) A colored 2-triangulation on the Klein bottle and (b) the corresponding gem. (c) A colored 3-triangulation with border and (d) the corresponding gem.

The gem data structure is similar to Brisson's *cell-tuple* structure [3] and to Lienhardt's *n-G-maps* [5], and, like them, allows manifolds with borders and non-manifold (but triangulable) topological spaces. However, the gem data structure is interpreted in a different way (as a triangulation, rather than a map), and is more general — meaning that the valid cell-tuple and n-G-map structures are a proper subset of the valid gem structures.

2 Colored triangulations

We give an abstract definition of triangulation that generalizes most of the known triangulation data structures. The gem data structure can represent only a subclass of these triangulations.

Simplex Let $x_0, \ldots x_d$ be the unit vectors of the coordinate axes of $\mathbb{R}^{d+1}$. The *canonical d-simplex* $\mathcal{S}_d \subset \mathbb{R}^{d+1}$ is the convex hull of the points $x_0, \ldots x_d$. The convex hull of any $C \subseteq \{x_0, \ldots x_d\}$, with $|C| = k+1$, is a *k-face* of $\mathcal{S}_d$. A *d-simplex* a is a closed d-ball B with a specific homeomorphism f from $\mathcal{S}_d$ to B. The image of a k-face of $\mathcal{S}_d$ under f is a k-face of a.

Triangulation A *d-dimensional triangulation* (or *d-triangulation*) is a set T of d-simplices with disjoint interiors such that, for distinct a and $b \in T$, $a \cap b$ is a union of faces of a and b. Any k-face of a d-simplex $a \in T$ is called a *k-simplex* of T. The set of all simplices (of any dimension) of T will be denoted by $\hat{T}$.

The union of all simplices of a triangulation T, with the obvious topology, is a compact Hausdorff topological space, called the *underlying space* of T and denoted $|T|$.

Colored simplex Let $C \subset \mathbb{N}$ with $|C| = d+1$. A *C-colored simplex* (or *C-simplex*) is a d-simplex whose vertices are uniquely labeled with *colors*, which are the elements of C.

Note that for each $D \subseteq C$ there is a D-simplex that is a face of S. We call it the *D-face* of S. In particular, the C-face of S is S itself, and the $\emptyset$-face of S is the empty set.

Colored triangulation Let $C \subset \mathbb{N}$ with $|C| = d+1$. A *C-colored triangulation* (or *C-triangulation*) is a d-triangulation T composed of C-simplices such that for every vertex v of T the d-simplices containing v agree in color. See figure 1(a,c).

3 Gems

Let $C \subset \mathbb{N}$. A *C-gem* is a pair (V, ϕ) where V is a finite set of *gem nodes* and ϕ is a function that to each $i \in C$ associates an involution ϕ_i of V. If $|C| = d+1$, a C-gem is also called a d-gem.

We note that this definition is formally similar to Lienhardt's definition of n-G-maps [5], except that we only retain the first of his two axioms (ϕ_i is an involution for all i). As it turns out, Lienhardt's second axiom ($\phi_i\phi_j$ is an involution, whenever $j - i \geq 2$) is required only to allow interpretation of the data structure as the barycentric subdivision of a cell complex.

3.1 Gems as colored graphs

We can interpret a C-gem (V, ϕ) as a non-directed graph with C-colored edges, where V is the set of graph nodes and there is a i-colored edge between the nodes v and $w \in V$ if and only if $\phi_i(v) = w$. Take, for instance, a $\{0,1,2\}$-gem with node set $\{A, B, \ldots, K, L\}$ and with the involutions provided by figure 2. The graph of this gem is the one in figure 1(b).

Involutions	Nodes											
	A	B	C	D	E	F	G	H	I	J	K	L
ϕ_0	L	H	D	C	K	G	F	B	J	I	E	A
ϕ_1	B	A	I	E	D	L	H	G	C	K	J	F
ϕ_2	G	C	B	J	F	E	A	I	H	D	L	K

Figure 2: A gem specified as a set of involutions on a set of nodes.

This interpretation provides implicitly the concepts of *walk*, *path*, *connectedness*, etc. In particular, we say that two distinct gem nodes v and w are i*-adjacent* iff $\phi_i(v) = w$.

3.2 Gems as colored triangulations

Every d-gem can be interpreted as a d-dimensional colored triangulation, and completely determines its topology. Informally, the gem is the dual of the triangulation: each node represents a colored simplex, and two nodes are i-adjacent iff the corresponding simplices share the face opposite to vertex i, with matched vertex colors. See figure 1.

To formalize this interpretation, we need the following definitions:

Canonical triangulation of a gem If $G = (V, \phi)$ is a C-gem, with $|C| = d + 1$, its *canonical triangulation* $\mathcal{T}_G$ is obtained as follows. Let X be the topological space $V \times \mathcal{S}_d$, where $\mathcal{S}_d$ is the canonical d-simplex and V is taken with the discrete topology. Note that X consists of $|V|$ connected components, each isomorphic to $\mathcal{S}_d$. Let the vertices of $\mathcal{S}_d$ be colored with the set C, in such a way that vertex x_i of $\mathcal{S}_d$ gets the ith smallest color. Let $\simeq$ be a relation in X such that, for p_1 and $p_2 \in X$, $p_1 \simeq p_2$ iff $p_1 = (v_1, q)$ and $p_2 = (v_2, q)$, where v_1 and v_2 are i-adjacent nodes of V and q is a point of the $(C \setminus i)$-face of $\mathcal{S}_d$, for some $i \in C$. The transitive and reflexive closure $\approxeq$ of $\simeq$ is an equivalence relation. Let Y be the quotient space $X/\approxeq$, and let $[p]_{\approxeq} \in Y$ be the equivalence class of point $p \in X$. The canonical triangulation $\mathcal{T}_G$ is the set of simplices in Y corresponding to the components of X. More specifically, for each $v \in V$, $\mathcal{T}_G$ contains a simplex $\tau(v) = \{[p]_{\approxeq} : p \in \{v\} \times \mathcal{S}_d\}$ with the topology induced by Y.

Theorem 1 *If $G = (V, \phi)$ is a C-gem, $\mathcal{T}_G$ is a C-triangulation.*

Proof: We must prove that the interiors of the simplices of $\mathcal{T}_G$ are disjoint, and that the intersection of distinct simplices is the union of their common faces.

For the first part, we show that if p is a point belonging to the interior of some simplex of $\mathcal{T}_G$, then p is not contained in any other simplex of $\mathcal{T}_G$. By construction, two points of X are related by $\simeq$ only if they belong to facets of simplices. So, a point in the interior of a simplex is related by $\approxeq$ only to itself. This implies that p is an equivalence class containing only one point of X, and it cannot belong to any other simplex of $\mathcal{T}_G$.

For the second part, we show that if a simplex $\tau(v)$ of $\mathcal{T}_G$ contains a point p shared by a distinct simplex $\tau(v')$, and b is the smallest face of $\tau(v)$ containing p, then $\tau(v')$ contains the whole of b. Indeed, the point p is an equivalence class of $\approxeq$ containing the points (v, q) and (v', q) of X, where q is a point belonging to a facet of $\mathcal{S}_d$. Since $(v, q) \approxeq (v', q)$, there is a sequence $(w_1, \ldots, w_n)$ of nodes of V such that $w_1 = v$, $w_n = v'$ and, for $1 \leq i \leq n-1$, $(w_i, q) \simeq (w_{i+1}, q)$. By the construction of $\simeq$, if c is the smallest face of $\mathcal{S}_d$ containing q, then for every point q' of c and every $i \in \{1, \ldots, n-1\}$, $(w_i, q') \simeq (w_{i+1}, q')$. Thus, for every $q' \in c$, $(w_1, q') \approxeq (w_n, q')$. It follows that $\tau(v)$ and $\tau(v')$ share the set of equivalence classes containing the points $\{v, v'\} \times c$ of X, which is the face b.

Finally, note that the quotient from space X to Y respects the vertex colors, so we can say that the triangulation $\mathcal{T}_G$ is colored with C. $\square$

3.3 Residues of a gem

Besides encoding the adjacency of simplices through facets in the canonical triangulations, the gems also contain information about the lower dimensional simplices and their incidence relations. In order to describe how to obtain this information, we must define the concept of residue of a gem.

Residue Let $G = (V, \phi)$ be a C-gem, and let $D \subseteq C$. A *D-residue* of G is a D-gem that is a connected component of $(V, \phi|_D)$, where $\phi|_D$ is the restriction of ϕ to the color set D. We denote by $\hat{G}$ the set of all residues of G. We say that two nodes of a C-gem are *d-connected*, for some $D \subseteq C$, if they belong to the same D-residue; or, in other words, if they are connected by a path in the gem which uses only D-colored edges.

We extend the bijection $\tau : V \to \mathcal{T}_G$ to a bijection from $\hat{G}$ to $\hat{\mathcal{T}}_G$ as follows: for any D-residue $R = (V', \phi')$ of G, $\tau(R)$ is the $(C \setminus D)$-face of the C-simplex $\tau(v)$, for any $v \in V'$. See figure 3.

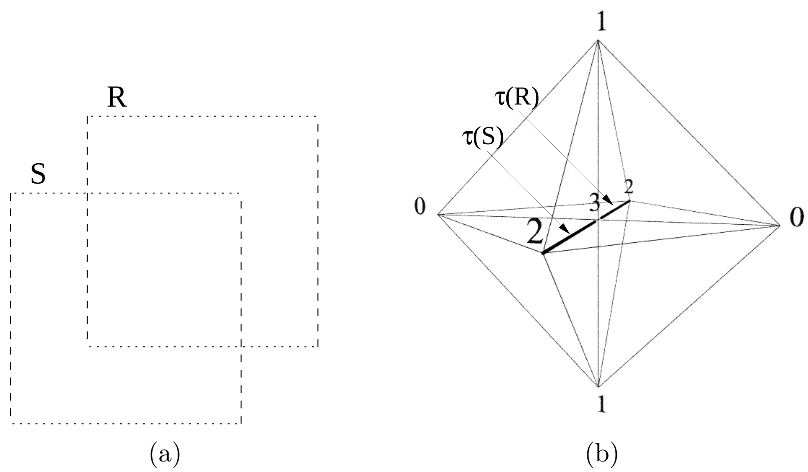

Figure 3: (a) The $\{0,1\}$-residues of the gem given in figure 1(d). (b) The $\{2,3\}$-colored edges of the triangulation that correspond to these residues.

To show that this definition is consistent, we must show that, for any pair of nodes v and w of a D-residue, $\tau(v)$ and $\tau(w)$ share the same $(C \setminus D)$-face, as stated in the following theorem. The proof of this theorem is given in the appendix.

Theorem 2 *If $G = (V, \phi)$ is a C-gem then for any v and $w \in V$ and any $D \subseteq C$, the simplices $\tau(v)$ and $\tau(w)$ of $\mathcal{T}_G$ share their $(C \setminus D)$-face iff v and w are D-connected in G.*

Notice that the extended bijection τ is consistent with the previous definition of this function, that is, every $\emptyset$-residue of G is a single node, which is mapped to the same C-simplex of $\mathcal{T}_G$ by both definitions.

Theorem 3 *If $G = (V, \phi)$ is a C-gem, the function $\tau : \hat{G} \to \hat{\mathcal{T}}_G$ is a bijection.*

Proof: First we show that for any D-simplex $a \in \hat{\mathcal{T}}_G$, with $D \subseteq C$, there is a residue R of G such that $\tau(R) = a$. Let b be a C-simplex of $\hat{\mathcal{T}}_G$ incident on a and let $v \in V$ such that $\tau(v) = b$. Then, if R is the $(C \setminus D)$-residue of G containing v, $\tau(R) = a$.

Now we prove that, for distinct R and $S \in \hat{G}$, $\tau(R) \neq \tau(S)$. Let R be a D-residue and S a E-residue of G, with D and $E \subseteq C$ and $R \neq S$. If $D \neq E$, $\tau(R)$ and $\tau(S)$ are clearly different, so let $D = E$. If v and w are nodes of R and S respectively, they are not D-connected, and thus $\tau(v)$ and $\tau(w)$ do not share their $(C \setminus D)$-face. So $\tau(R) \neq \tau(S)$. $\square$

Theorem 4 *If R and S are residues of a C-gem G, then $\tau(R)$ is a face of $\tau(S)$ iff S is a residue of R.*

Proof: Let R be a D-residue of G and S be a E-residue of G, with $D, E \subseteq C$.

First, suppose that S is a residue of R, which implies $E \subseteq D$. If v is a node of S, then it is also a node of R. So $\tau(S)$ is the $(C \setminus E)$-face of $\tau(v)$ and R is the $(C \setminus D)$-face of the same C-simplex $\tau(v)$. Since $(C \setminus E) \supseteq (C \setminus D)$, it is easy to see that $\tau(R)$ is a face of $\tau(S)$.

Now suppose that $\tau(R)$ is a face of $\tau(S)$, which implies $(C \setminus E) \supseteq (C \setminus D)$, i.e. $E \subseteq D$. Since every C-simplex incident on $\tau(S)$ is also incident on $\tau(R)$, every node of S is also a node of R. So, S is an E-residue of R. $\square$

3.4 Gem representing a triangulation

A gem encodes the topology not only of its canonical triangulation, but also of any triangulation that is topologically equivalent to the canonical. To formalize this, we need the following definition.

Isomorphism of triangulations Two C-triangulations T and T' are *isomorphic*, or *topologically equivalent*, iff there is a bijection $f : \hat{T} \to \hat{T}'$ mapping D-simplices of T into D-simplices of T', for $D \subseteq C$, such that, for any a and $b \in \hat{T}$, a is a face of b iff $f(a)$ is a face of $f(b)$.

We say that a C-gem G *represents* a C-triangulation T if T is isomorphic to $\mathcal{T}_G$. It follows that every gem represents some triangulation (in particular its canonical triangulation), and all the triangulations represented by some gem are isomorphic.

As a consequence of theorems 2, 3 and 4 it is possible to characterize the topology of triangulations represented by a gem using the incidence relations between simplices, as stated in the following corollary.

Corollary 5 *Let $G = (V, \phi)$ be a C-gem and T be a C-triangulation. The three statements below are equivalent.*

(i) The gem G represents T.

(ii) There is a bijection δ from V to T such that, for any pair v and $w \in V$, the simplices $\delta(v)$ and $\delta(w)$ share their $(C \setminus D)$-face, for $D \subseteq C$, iff v and w are D-connected.

(iii) There is a bijection δ from $\hat{G}$ to $\hat{T}$ such that, for any pair R and $S \in \hat{G}$, $\delta(R)$ is a face of $\delta(S)$ iff S is a residue of R.

Theorem 6 *Two gems are isomorphic if and only if their canonical triangulations are isomorphic.*

Proof: Let $G = (V, \phi)$ and $G' = (V', \phi')$ be C-gems. If $\mathcal{T}_G$ and $\mathcal{T}'_G$ are isomorphic, we can say that G' represents $\mathcal{T}_G$. In this case, there is a bijection δ from V' to $\mathcal{T}_G$ such that, for any pair v and $w \in V'$, the simplices $\delta(v)$ and $\delta(w)$ share their $(C \setminus D)$-face, for $D \subseteq C$, iff v and w are D-connected. So, the composition $\delta^{-1}\tau$ is an isomorphism from V to V'.

On the other hand, if G and G' are isomorphic, there is a bijection β from V to V' such that the nodes v and w from V are D-connected, for $D \subseteq C$, iff $\beta(v)$ and $\beta(w)$ are D-connected. So, the composition $\tau\beta$ is a bijection from V' to $\mathcal{T}_G$ such that for any v and $w \in V'$ and any $D \subseteq C$, the simplices $\tau\beta(v)$ and $\tau\beta(w)$ of $\mathcal{T}_G$ share their $(C \setminus D)$-face iff v and w are D-connected in G'. By corollary 5, G' represents $\mathcal{T}_G$, and, consequently, $\mathcal{T}_G$ and $\mathcal{T}'_G$ are isomorphic. $\square$

3.5 Colored triangulations as gems

As we saw, every gem represents some colored triangulation. On the other hand, not every colored triangulation is representable by a gem. We now characterize those that are. Intuitively, the triangulations must be the result of identifying pairs of facets of isolated simplices, so that each facet is identified at most once; and lower dimensional faces are identified only as a consequence of facet identification. To formalize this statement, we need the following definitions.

Star of a simplex Let a be a k-simplex of a d-triangulation T. The *star* of a, denoted $\mathrm{St}(a)$, is the union of the interiors of all simplices of $\hat{T}$ containing a.

Nice triangulation We say that a colored d-triangulation T is *nice* iff every $(d-1)$-simplex is face of at most two d-simplices, and for every k-simplex a, with $0 \leq k \leq d-2$, the star $\mathrm{St}(a) \setminus a$ is connected.

It is easy to see that every colored triangulation over a manifold, with or without border, is a nice triangulation. However, for $d \geq 3$, there are nice d-triangulations whose underlying spaces are not manifolds [7].

Theorem 7 *Every nice triangulation is represented by a gem.*

Theorem 8 *Every gem represents a nice triangulation.*

These theorems are proved in the appendix.

4 The gem data structure

In a colored triangulation, the constraints on vertex colors mean that two simplices can share a specific facet in only one way. The gem data structure makes use of this constraint to greatly simplify the repertoire of elementary topological operators.

The gem data structure represents a C-gem where $C \subseteq \{0, 1, \ldots, d\}$, for a given constant d. Its records correspond to the gem nodes. Each record r contains $d+1$ pointers to other records, representing the functions $\phi_0, \ldots, \phi_d$; with the constraint that, if pointer i of record r points to record r', then pointer i of r' points back to r. See figure 4.

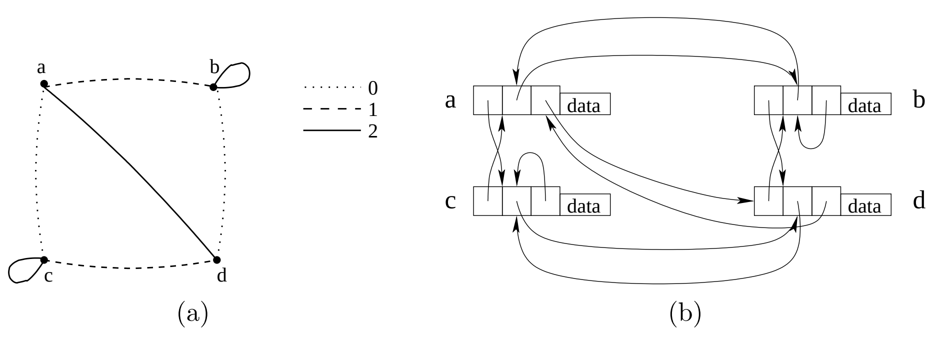

Figure 4: (a) A 2-gem given by a colored graph (b) the registers of the gem data structure.

We will write $\phi_i(r)$ to mean the record referenced by pointer i of record r. In specific applications, each record r may have additional fields, containing geometric or other data [8].

4.1 Restrictions

The fact that gems can represent only colored triangulations is a major limitation of this data structure. Take for instance a 1-triangulation consisting of a cycle of lines alternating with vertices. This triangulation is colorable if and only if the number of vertices is even. In the case of 2-triangulations, if the link of any vertex is an odd cycle, it cannot be colored. See figure 5.

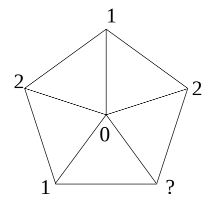

Figure 5: A 2-triangulation that is not colorable.

For this reason, gems cannot be used in problems that require *specific* triangulations, e.g. finding the Delaunay triangulation of a set of points. Still, colored triangulations are suited for many applications. Some of them will be discussed in section 5.

4.2 Traversing a gem structure

Let T be a C-triangulation represented by a gem G. In computer programs, any C-simplex of T is represented by a pointer to the corresponding gem node. More generally, a D-simplex a of $\hat{T}$, for any $D \subseteq C$, is referred by a pair $p = (v, D)$, where v is any node of the $(C \setminus D)$-residue corresponding to a.

Given this representation, we describe two operations to traverse the gem. The operation `GetFace(p,E)`, given a D-simplex a of T referred by $p = (v, D)$, and a set $E \subseteq D$, returns the E-face of a — that is simply the pair (v, E).

The operation `Enum(p,D)`, given an E-simplex b of T referred by $p = (w, E)$, and a set $D \supseteq E$, enumerates the D-simplices of T that have b as a face. This is the same as enumerating the $(C \setminus D)$-residues contained in the $(C \setminus E)$-residue corresponding to b. This enumeration can be performed by a straightforward depth first or breadth first search — assuming that there is some way to mark the nodes that have been visited.

4.3 Creating and modifying the gem structure

We introduce three basic operations to build and modify gem data structures.

`MakeNode()`: creates a new record v and makes $\phi_i(v) = v$ for $0 \leq i \leq d$.

`DeleteNode(`v`)`: is the inverse operation of `MakeNode`. It takes a record v, such that $\phi_i(v) = v$ for all $i \in \{0, \ldots, d\}$, and returns it to the storage pool.

`Swap(`v`,`w`,`i`)`: where v and w are records and $i \in \{0, \ldots, d\}$. This operation consists simply in exchanging the values of pointers $\phi_i(v)$ and $\phi_i(w)$.

Note that `Swap(`v`,`w`,`i`)` is its own inverse, and it is a no-op if $v = w$.

Theorem 9 *If* $\{\phi_i(v)\} \cup \{\phi_i(w)\} = \{v\} \cup \{w\}$*, the operation* `Swap(`$v$`,`$w$`,`$i$`)` *preserves the consistency of the gem.*

Proof: We need to show that, upon the application of `Swap`, for every node x and every color i of he gem, $\phi_i(\phi_i(x)) = x$.

Note that the only modification performed by this `Swap` occurs on the i-colored pointers of v and w. If $\{\phi_i(v)\} \cup \{\phi_i(w)\} = \{v\} \cup \{w\}$, thus either $\phi_i(v) = v$ and $\phi_i(w) = w$, or $\phi_i(v) = w$ and $\phi_i(w) = v$. In the first case, `Swap` performs $\phi_i(v) \leftarrow w$ and $\phi_i(w) \leftarrow v$; in the second case, `Swap` performs $\phi_i(v) \leftarrow v$ and $\phi_i(w) \leftarrow w$.

In either case, we have $\phi_i(phi_i(v)) = v$ and $\phi_i(phi_i(w)) = w$ after `Swap`, so the consistency of the gem is preserved. □

We say that a `Swap` call is *valid* if it satisfies the precondition stated in the theorem above. In practice, this operation is used to bind or separate a pair of gem nodes — that is, to glue or unglue two simplices of the triangulation.

Theorem 10 *Any gem data structure can be built by a sequence of* `MakeNodes` *and valid* `Swaps`*.*

Proof: Let G be any gem. If we perform `Swap(`v`,`w`,`i`)` for every pair of distinct nodes v and w of G that are i-adjacent, we get a collection of isolated nodes. Then if we perform `DeleteNode(`v`)` for every node v, we get an empty gem data structure.

The sequence of operations needed to construct the gem G is exactly the inverse of the sequence used to destroy it, which is a sequence of `MakeNodes` followed by a sequence of valid `Swaps`. □

For an example of use of these operations, see our convex hull algorithm [8].

5 Applications

An obvious application of the gem data structure is the representation of barycentric subdivision of general maps, as studied by Brisson [3] and Lienhardt [5]. The gem representation was employed in our exact convex hull algorithm [8] to represent the barycentric subdivision of the convex hull of a set of points in $\mathbb{R}^d$, thus allowing the representation of hulls with non-simplicial faces.

Gems can also be used as adaptive triangulations, for example, to approximate surfaces by simplicial meshes with prescribed accuracy. Note that it doesn't matter if such a mesh is a colored triangulation, as long as it is a good approximation to the surface. Even though colored triangulations are not as easily subdivided as general ones, they do admit local k-simplex refinement schemes for any $k \in \{1, \ldots, d\}$.

When making a local refinement on a colored triangulation one must guarantee that the newly created vertices can be colored. In general, a local refinement on a colored triangulation requires more simplices than would be necessary on a general triangulation. More precisely, a local piecewise-linear refinement of a d-simplex that preserves its boundary requires subdivision into $2^{d+1} - 1$ simplices. In contrast, the same operation can be performed on an unrestricted triangulation using $d + 1$ simplices. See figure 6.

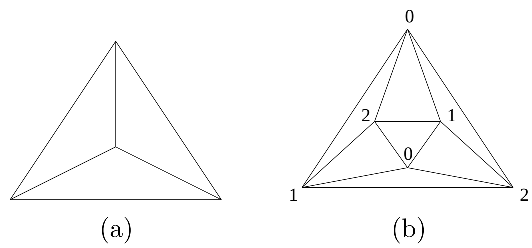

Figure 6: (a) The minimal piecewise-linear local refinement of a triangle on a general triangulation. (b) The minimal piecewise-linear local refinement of a triangle on a colored triangulation.

We can also adapt more sophisticated triangulation refinement algorithms for colored triangulations. In Bank et al.'s regular refinement algorithm for 2D triangulations [1], there are two ways a triangle t can be subdivided: if t is marked for subdivision or is adjacent to at least 2 triangles that are marked, it is *regularly subdivided* by connecting the midpoints of its edges, which produces four similar triangles; if t is not marked for subdivision but has one edge split by a regular subdivision on an adjacent triangle, t is *bisected* by connecting the midpoint of the subdivided edge to the opposite vertex.

It turns out that Bank's *regular* subdivision of a triangle is feasible in colored triangulations as well. Triangle *bisection*, though, must be replaced by subdivision into six triangles. See figures 7 and 8.

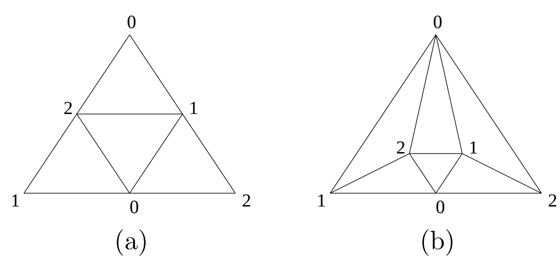

Figure 7: A colored triangle refined by (a) a regular subdivision and by (b) a single edge bisection.

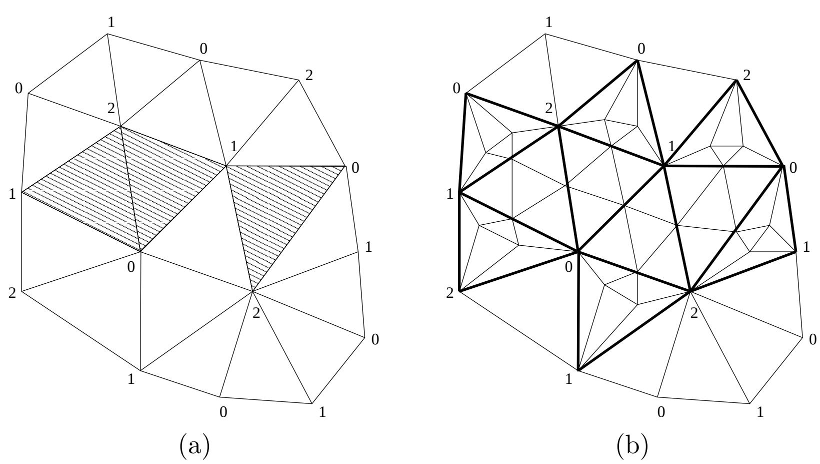

Figure 8: (a) A colored triangulation with some triangles marked for subdivision (hatched). (b) The refined triangulation, using the schemes of figure 7(a) on each marked triangle, and of figure 7(b) on adjacent triangles.

6 Conclusion

The gem data structure is a very simple and general way to represent the topology triangulations. This simplicity yields concise topological operators and elegant algorithms.

In spite of its restriction to colored triangulations, it is suitable for a range of applications, as the representation of barycentric subdivisions and the approximation of surfaces by affine triangulations.

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7 Appendix

Proof of theorem 2 If $G = (V, \phi)$ is a C-gem then for any v and $w \in V$ and any $D \subseteq C$, the simplices $\tau(v)$ and $\tau(w)$ of $\mathcal{T}_G$ share their $(C \setminus D)$-face iff v and w are D-connected in G.

Proof: In order to prove this statement, we must analyze the relation $\approxeq$ on space $X = V \times \mathcal{S}_d$ used in the construction of the canonical triangulation $\mathcal{T}_G$. If $p = (v, q)$ and $p' = (v', q')$ are points of X, $p \approxeq p'$ iff there is a sequence $(w_1, \ldots, w_n)$ of nodes of V such that $v = w_1$, $v' = w_n$ and, for $1 \leq i \leq n-1$, $(w_i, q_i) \simeq (w_{i+1}, q_{i+1})$. By construction, $(w', q') \simeq (w'', q'')$ iff there is a j in C such that w' and w'' are j-adjacent and q' and q'' are the same point belonging to the $(C \setminus \{j\})$-face of $\mathcal{S}_d$. So we can say that $p \approxeq p'$ iff there is $D \subseteq C$ such that v and v' are D-connected and q and q' are the same point belonging to the $(C \setminus D)$-face of $\mathcal{S}_d$.

Now we show that if $\tau(v)$ and $\tau(w)$ are simplices of $\mathcal{T}_G$ sharing a $(C \setminus D)$-face for some $D \subseteq C$, then the nodes v and w are D-connected. Let p be a point in the interior of this common face; it is an equivalence class containing the points (v, q) and (w, q) of X, where q is a point belonging to the interior of the $(C \setminus D)$-face of $\mathcal{S}_d$. This implies that there is a $D' \subseteq C$ such that v and w are D'-connected and q belongs to the $(C \setminus D')$-face of $\mathcal{S}_d$. Since q belongs to the interior of the $(C \setminus D)$-face of $\mathcal{S}_d$, $C \setminus D' \supseteq C \setminus D$ (or equivalently $D \supseteq D'$), and we can say that v and w are D-connected.

Finally, if v and w are D-connected nodes of V, then, for every point q' in the $(C \setminus D)$-face of $\mathcal{S}_d$, $(v, q') \approxeq (w, q')$, and consequently $\tau(v)$ and $\tau(w)$ share their $(C \setminus D)$-face. $\square$

Now we prove that the colored triangulations representable by gems are precisely the nice ones (theorems 7 and 8). We need some additional definitions:

The k-star of a simplex Let T be a d-triangulation and let a be a j-simplex of $\hat{T}$. The *k-star* of a, for $j \leq k \leq d$, is the union of the interiors of all simplices of $\hat{T}$ with dimension at least k and containing a. The k-star of a is denoted $\mathrm{St}_k(a)$. Note that $\mathrm{St}_k(a)$ is the union of the stars of all the k-simplices incident on a, and that, for $j < d$, $\mathrm{St}_{j+1}(a) = \mathrm{St}(a) \setminus a$.

k-star-sequence Let T be a d-triangulation and let a be a j-simplex of $\hat{T}$. A k-star-sequence of a, for $j \leq k \leq d$, is a finite sequence of k-simplices incident on a wherein the stars of consecutive k-simplices have nonempty intersection.

Lemma 1 *Let T be a d-triangulation and a be a j-simplex of $\hat{T}$. If, for $j \leq k \leq d$, a has a k-star-sequence connecting every pair of k-simplices incident on a, then a has a k-star-sequence containing all the k-simplices incident on a.*

Proof: If $l = (b_1, \ldots, b_n)$ is a k-star-sequence of a and c is a k-simplex incident on a and not contained in l, then we only have to obtain a k-star-sequence from b_n to c and concatenate it to l. By an induction on the number of k-simplices incident on a, we obtain the k-star-sequence containing all the k-simplices incident on a. $\square$

Lemma 2 *Let T be a d-triangulation and a be a j-simplex of $\hat{T}$. The k-star of a, for $j \leq k \leq d$, is connected iff, for any pair of k-simplices b and c incident on a, there is k-star-sequence of a from b to c.*

Proof: Let a be a j-simplex and let b and c be a pair of k-simplices with no k-star-sequence connecting them, for $0 \leq j \leq k \leq d$. Let R be a binary relation on the set of k-simplices incident on a such that bRc iff $\mathrm{St}(b) \cap \mathrm{St}(c) \neq \emptyset$. The closure of R, denoted $R^\star$, is an equivalence relation.

Since there is no k-star-sequence connecting b and c, then R defines more than one equivalence class. Hence, $H = \bigcup_{bRe} \mathrm{St}(e)$ and $\mathrm{St}_k(a) \setminus H$ are two disjoint nonempty open sets partitioning $\mathrm{St}_k(a)$, and thus $\mathrm{St}_k(a)$ is disconnected.

Now, let a be a j-simplex such that for any pair of k-simplices b and c incident on a, for $k \geq j$, there is a k-star-sequence of a from b to c. Let then l be the k-star-sequence of a given by lemma 1. Since the star of any k-simplex in l is connected, the union of all these stars, which is $\mathrm{St}_k(a)$, is connected. □

Corollary 11 *Let T be a d-triangulation and a be a simplex of $\hat{T}$. The $(d-1)$-star of a is connected iff, for any pair of d-simplices b and c containing a, there is a sequence of d-simplices from b to c wherein every pair of consecutive d-simplices share a $(d-1)$-face incident on a.*

Proof: It follows from the fact that every such sequence of d-simplices can be obtained from a $(d-1)$-star-sequence of a, and vice-versa. □

Lemma 3 *If T is a d-triangulation where every k-simplex with $0 \leq k < d-1$ has a connected $(k+1)$-star, then such simplices also have a connected $(d-1)$-star.*

Proof: Let a be a k-simplex of T with $0 \leq k < d-1$. We prove by an induction that a has a connected j-star for every j from $k+1$ to $d-1$. The base case is that a has a connected $(k+1)$-star. Then, we suppose that, for some j between $k+1$ and $d-2$ (limits included), $\mathrm{St}_j(a)$ is connected. Now we must prove that $\mathrm{St}_{j+1}(a)$ is connected.

Let $(b_1, \ldots, b_n)$ be a j-star-sequence containing all the j-simplices incident on a. Since, for any $1 \leq i \leq n$, $\mathrm{St}_{j+1}(b_i)$ is connected, the set $\bigcup_{1 \leq i \leq n} b_i$, which is $\mathrm{St}_{j+1}(a)$, is connected. □

D-walk A D-walk of a C-gem, for $D \subseteq C$, is a walk on the gem whose edge colors belong to D. Note that there is a D-walk between two nodes iff they are D-connected.

Proof of theorem 7 Let T be a nice C-triangulation. We will construct a C-gem $G = (V, \phi)$ that represents T. Let V be the set of C-simplices T, and, for all $i \in C$, let ϕ_i be an involution of V such that $\phi_i(a) = b$ for a and $b \in V$ iff (i) a and b are distinct C-simplices sharing their $(C \setminus \{i\})$-face or (ii) a and b are the same C-simplex whose $(C \setminus \{i\})$-face is contained in only one C-simplex. This definition is consistent, since every $(d-1)$-simplex of T is a face of one or two d-simplices.

Now let the bijection δ between V and T be the identity. We must show that for distinct v and $w \in V$, $\delta(v)$ and $\delta(w)$ share their D-face, for any $D \subset C$, iff v and w are $(C \setminus D)$-connected in G.

Let c be a D-face of $\delta(v)$ and $\delta(w)$. If $|D| = d-1$, $\mathrm{St}_{d-1}(c)$ is $\mathrm{St}(c)$, which is connected; if $|D| < d-1$, lemma 3 states that $\mathrm{St}_{d-1}(c)$ is connected. So, by corollary 11, there is a sequence of C-simplices containing c, from $\delta(v)$ to $\delta(w)$, wherein every pair of consecutive C-simplices share a $(d-1)$-face containing c. This sequence provides a $(C \setminus D)$-walk between v and w in G.

If there is a $(C \setminus D)$-walk between v and w in G, for every pair x and y of consecutive nodes in this walk, $\delta(x)$ and $\delta(y)$ share their D-face, so $\delta(v)$ and $\delta(w)$ also share their D-face. □

Proof of theorem 8 Let G be a C-gem and let T be a triangulation represented by G.

First we show that the $(d-1)$-star of any k-simplex a of T, for $0 \leq k \leq d-2$, is connected, and consequently, $\mathrm{St}(a) \setminus a$ is also connected. Let a be a D-simplex of T with $D \subset C$ and $|D| \leq |C| - 2$, and let R be the $(C \setminus D)$-residue of G corresponding to a. The residue R is connected, so there is a $(C \setminus D)$-walk between any pair of nodes in R. This walks provide the sequences of d-simplices required by corollary 11 to guarantee that $\mathrm{St}_{d-1}(a)$ is connected.

Now we prove that every $(d-1)$-simplex is face of at most two d-simplices. Every $(C \setminus \{i\})$-simplex c of T, for any $i \in C$, is face of a C-simplex $\delta(v)$, for some node v of G. If $\phi_i(v) = v$ then c is face of only one C-simplex, else c is face of two C-simplices. $\square$