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A PTAS for the metric case of the minimum sum-requirement communication spanning tree problem☆

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ABSTRACT

This work considers the metric case of the minimum sum-requirement communication spanning tree problem (SROCT), which is an NP-hard particular case of the minimum communication spanning tree problem (OCT). Given an undirected graph $G = (V, E)$ with non-negative lengths $\omega(e)$ associated to the edges satisfying the triangular inequality and non-negative routing weights $r(u)$ associated to nodes $u \in V$, the objective is to find a spanning tree T of G, that minimizes: $\frac{1}{2}\sum_{u\in V}\sum_{v\in V}(r(u)+r(v))\,d(T,u,v)$, where $d(H, x, y)$ is the minimum distance between nodes x and y in a graph $H \subseteq G$. We present a polynomial approximation scheme for the metric case of the SROCT improving the until now best existing approximation algorithm for this problem.

1. Introduction

In this work we consider a particular case of the minimum communication spanning tree problem (OCT). The OCT was introduced by Hu in 1974. The problem receives an undirected graph $G = (V, E)$ with non-negative length $\omega(e)$ associated to each edge $e \in E$ and non-negative requirement $\psi(u, v)$ between each pair of nodes $u, v \in V$. The objective is to find a spanning tree T of G which minimizes the total communication cost: $C(T) = \sum_{u\in V}\sum_{v\in V}\psi(u, v)d(T, u, v)$, where $d(H, x, y)$ denotes the minimum distance between nodes x and y in the sub-graph H of G [3,9].

A particular case of OCT is the minimum routing cost spanning tree problem (MRCT), in which the requirement between all pair of nodes is equal to one ($\psi(u, v) = 1$ for all $u, v \in V$). In [4] it was proved that MRCT is NP-hard (by a reduction from the 3-exact cover problem (3-EC)). In [12] a PTAS for the MRCT was given. The authors presented a reduction from the general to the metric case, which implies that MRCT with edge-lengths that satisfy the triangular inequality is also NP-hard. Also, in [12] an $O(\log^2(n))$-approximation was given for OCT applying a result from [1] which was later improved to a $O(\log(n))$-approximation by [7].

In [10], the minimum product-requirement communication spanning tree problem (PROCT) and the minimum sum-requirement communication spanning tree problem (SROCT) were introduced. In these problems each vertex $u \in V$ has a non-negative routing weight $r(u)$. For PROCT the requirement is defined as $\psi(u, v) = \frac{1}{2}r(u)r(v)$, and for SROCT $\psi(u, v) = \frac{1}{2}(r(u) + r(v))$. Both problems are NP-hard. In [10] a 1.577-approximation algorithm for PROCT and a 2-approximation for SROCT are given.

The approximation ratio for PROCT was improved in [11] where a PTAS was given. A particular case of SROCT is the weighted p-MRCT, where given an integer p, only p nodes of the graph may have a positive routing weight (i.e. the remaining

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nodes have zero weight). The particular case in which the p nodes have routing weight 1 is called p-MRCT. In [2] it was proved that 2-MRCT is NP-hard, also proved in [8], where the authors gave PTASs for 2-MRCT and the metric case of weighted 2-MRCT.

Recently in [6] was introduced the weighted source destination communication spanning tree problem (WSDOCT). This problem is the particular case of OCT where each vertex $u \in V$ has non-negative sending and receiving weights, $r_s(u)$ and $r_r(u)$ respectively, and the requirement is defined as $\psi(u, v) = \frac{1}{2}(r_s(u)r_r(v) + r_r(u)r_s(v))$. Observe that when $r_r(u) = 1$ for each $u \in V$ we have the SROCT problem and when $r_s(u) = r_r(u)$ for each $u \in V$ we have the PROCT problem. So WSDOCT is a generalization of both problems PROCT and SROCT. Also, in [6] PTASs where given for the metric cases of p-MRCT and fixed parameter of p-WSDOCT, which is the particular case of WSDOCT where only p nodes may have positive sending weight.

To the best of our knowledge, there are no results improving the 2-approximation ratio for SROCT which is also the best known ratio for the metric case of SROCT (denoted by m-SROCT). Observe that this problem is also NP-hard, since MRCT is a particular case in which $r(u) = 1$ for all $u \in V$.

In this work we give a PTAS for m-SROCT improving the best previous known result for this problem. The idea of our algorithm was inspired in the previous PTASs for related problems such as MRCT and PROCT. This paper is organized as follows. In the next section we present some notation. In Section 3 we show how to obtain an optimal k-star for SROCT in polynomial time for a fixed integer k. In Section 4 we present a PTAS for the m-SROCT. Finally, in Section 5 the conclusions and future work are given.

2. Definitions

Unless specified we consider all graphs as undirected graphs. Given a graph G we denote the set of its nodes by V_G and the set of its edges by E_G (when G is implicit by context we use V as V_G and E as E_G).

Definition 2.1. Given a graph G with non-negative lengths associated to its edges, the **length of a path** in G is defined as the sum of the lengths of its edges (a path with no edges has length zero). The **distance** between node x and node y in H sub-graph of G is the length of a path with minimum length between x and y in H and is denoted by $d(H, x, y)$.

Now we can define WSDOCT as:

Problem 2.1. WSDOCT—Weighted Source Destination Communication Spanning Tree problem

Input: $\langle G, \omega, r_s, r_r\rangle$. A graph G, a non-negative length function over the edges of G, $\omega : E \to \mathbb{Q}_+$, a non-negative sending weight function over the nodes of G, $r_s : V \to \mathbb{Q}_+$, and a non-negative receiving weight function over the nodes of G, $r_r : V \to \mathbb{Q}_+$.

Output: A spanning tree T of G which minimizes the total weighted routing cost:

$$\begin{aligned} C(T) &= \sum_{u\in V}\sum_{v\in V}\frac{1}{2}(r_s(u)r_r(v) + r_r(u)r_s(v))d(T, u, v) \\ &= \sum_{u\in V}\sum_{v\in V} r_s(u)r_r(v)d(T, u, v). \end{aligned}$$

Also we define the SROCT as:

Problem 2.2. SROCT—Sum-Requirement Communication Spanning Tree problem

Input: $\langle G, \omega, r_s\rangle$. A graph G, a non-negative length function over the edges of G, $\omega : E \to \mathbb{Q}_+$ and a non-negative routing weight function over the nodes of G, $r_s : V \to \mathbb{Q}_+$.

Output: A spanning tree T of G which minimizes the total weighted routing cost:

$$\begin{aligned} C(T) &= \sum_{u\in V}\sum_{v\in V}\frac{1}{2}(r_s(u) + r_s(v))d(T, u, v) \\ &= \sum_{u\in V}\sum_{v\in V} r_s(u)d(T, u, v). \end{aligned}$$

Observe that SROCT is the particular case of WSDOCT where $r_r(u) = 1$ for each $u \in V$.

Definition 2.2. Given a graph G, a non-negative sending weight function over the nodes of G, $r_s : V \to \mathbb{Q}_+$ and a non-negative receiving weight function over the nodes of G, $r_r : V \to \mathbb{Q}_+$, we denote $r_s(G) = \sum_{u\in V} r_s(u)$, $r_r(G) = \sum_{u\in V} r_r(u)$ and $n(G) = |V_G|$. When G is implicit by the context we use R_s to denote $r_s(G)$, R_r to denote $r_r(G)$ and n to denote $n(G)$.

This paper considers the m-SROCT and m-WSDOCT, the metric cases of SROCT and WSDOCT respectively, which are the particular cases of the problems where the graph G is complete and the length function over the edges satisfies the triangular inequality. In order to approximate an optimal solution of m-SROCT or m-WSDOCT we introduce the concept of a k-star[1]:

Definition 2.3. Given a graph G and a positive integer k, a k-**star** of G is a spanning tree of G with no more than k internal nodes (that is, at least $n-k$ leaves). A **core** of a k-star T of G is a tree resulting by eliminating $n-k$ leaves from T.

Note that a k-star T can be represented by (τ, S), where τ is a core of T and $S = \{S_{u_1}, \ldots, S_{u_k}\}$ is a vector indexed by the nodes in τ where S_{u_i} is the set of leaves adjacent in T to $u_i \in V_\tau$ $(1 \le i \le k)$.

The problem of finding an optimal k-star for m-SROCT can be defined as:

Problem 2.3. Optimum k-star for m-SROCT

Input: $\langle G, \omega, r_s, k\rangle$. A positive integer k and an instance of m-SROCT: a complete graph G, a non-negative length function over the edges of G which satisfies the triangular inequality, $\omega : E \to \mathbb{Q}_+$ and a non-negative routing weight function over the nodes of G, $r_s : V \to \mathbb{Q}_+$.

Output: A k-star T of G which minimizes the total weighted routing cost:
$C(T) = \sum_{u\in V} \sum_{v\in V} r_s(u) d(T, u, v)$.

The next section shows an efficient algorithm to find an optimal k-star for m-SROCT.

3. Optimal k-star for m-SROCT

First we introduce the notion of configuration of a k-star:

Definition 3.1. Given a k-star $T = (\tau, S)$ a **configuration** of T is (τ, L) where $L = \{l_{u_1}, \ldots, l_{u_k}\}$ is a vector of integers being $l_{u_i} = |S_{u_i}|$ $(1 \le i \le k)$. A configuration (τ, L) **is over** (k, G), where k is a positive integer and G is a graph, if τ is a tree of G with k nodes (that is, $\tau \subseteq G$ and $|V_\tau| = k$) and $\sum_{u\in V_\tau} l_u = n - k$.

In [12] it was observed that given a complete graph G and a fixed positive integer k, the number of configurations over (k, G) is polynomial in n, resulting $O(k^k n^{2k-1})$. Then, given an instance $\langle G, \omega, r, k\rangle$ of the optimum k-star for m-SROCT, our proposal is to enumerate all possible configurations over (k, G), finding an optimal k-star of each configuration, and finally select the best k-star among them.

We find an optimal k-star for an instance $\langle G, \omega, r_s, k\rangle$ of the optimum k-star for m-SROCT and a configuration (τ, L) over (k, G), by reducing the problem to an uncapacitated minimum cost flow problem (UMCF).

Problem 3.1. UMCF—Uncapacitated Minimum Cost Flow problem

Input: $\langle G, \omega, r\rangle$. A directed graph G, a cost function over the arcs $\omega : E \to \mathbb{Q}_+$ and a demand function over the nodes $r : V \to \mathbb{Z}$.

Output: An integer vector indexed by the arcs $X = (x_e)_{e\in E}$ which minimizes $C(X) = \sum_{e\in E} \omega(e) x_e$ and guarantees for each node $u \in V$:

$$\sum_{e\in\delta^+(u)} x_e - \sum_{e\in\delta^-(u)} x_e = r(u),$$

where $e \in \delta^+(w)$ and $e \in \delta^-(v)$ iff $e = \langle v, w\rangle$ $(\forall e \in E, v, w \in V)$.

Proposition 3.1. *Given an instance $I = \langle G, \omega, r_s, k\rangle$ of the optimum k-star for m-*SROCT *and a configuration $c = (\tau, L)$ over (k, G), the problem of finding an optimal k-star with configuration c for I can be reduced in polynomial time to the* UMCF *with instance $I' = \langle G', \omega', r'\rangle$, where:*

- $V_{G'} = V_G$;
- $E_{G'} = \{(u, v) | u \in V_{G-\tau} \wedge v \in \tau\}$;
- $\omega'(u, v) = R_s\omega(u, v) - 2r_s(u)\omega(u, v) + \sum_{w\in V_\tau} r_s(u) \left(d(\tau, v, w) + \omega(u, v)\right)(l_w + 1)$;
- *if $u \in V_{G-\tau}$ then $r'(u) = -1$, otherwise $r'(u) = l_u$.*

The graph G' is a complete bipartite graph on the same node set V_G of G. The bi-partition is given by the nodes in τ and outside this set. The cost of arc $\langle u, v\rangle$ is equivalent to the value of assigning u as adjacent of v in a k-star with the given configuration. We have to consider the cost of sending the routing weight from u to all nodes of τ assuming that each node $w \in V_\tau$ receives $(l_w + 1)$ times the value $r_s(u)$ (considering the transmission to the node w and the leaves adjacent to it); also, we add the cost of sending the routing weight of the entire graph $(R_s - r_s(u))$ to node u, which must pass by node v. Finally the demands r' are set to ensure assignment between nodes out of τ and nodes in τ.

[1] The definition of k-star used in this paper is the same used by [12,10,11], which is different from the usual definition of k-star in graph theory (a tree with k leaves linked to a single vertex of degree k).

Proof. Since demands are integers we know that in any feasible solution the values x_e will be either zero or one. Moreover, exactly $n-k$ arcs of G' will have value 1. This guarantees that every feasible solution S' of the flow problem represents an assignment of leaves outside τ to be adjacent to nodes in τ for a k-star T of G with configuration (τ, L). Also, it is easy to see that any k-star T with configuration (τ, L) provides a feasible solution to the flow problem: connect node $u \in \tau$ to the l_u leaves adjacent to it in T.

Consider for each $u \in V_{G-\tau}$, that $p(u)$ is the node in τ assigned to u in a solution S', then:

$$C(S') = \sum_{u\in V_{G-\tau}} \omega'(u, p(u))$$

$$C(S') = \sum_{u\in V_{G-\tau}} \left(\sum_{v\in V_G} r_s(v) - r_s(u)\right) \omega(u, p(u)) + \sum_{u\in V_{G-\tau}} \sum_{v\in V_\tau} r_s(u)\left(d(\tau, p(u), v) + \omega(u, p(u))\right)$$
$$+ \sum_{u\in V_{G-\tau}} \sum_{v\in V_{G-\tau}-u} r_s(u) d(\tau, p(u), p(v)) + \sum_{u\in V_{G-\tau}} \sum_{v\in V_{G-\tau}-u} r_s(u)\left(\omega(u, p(u)) + \omega(v, p(v))\right)$$

$$C(S') = \sum_{u\in V_\tau} \sum_{v\in V_\tau} r_s(v) d(\tau, u, v) - \sum_{u\in V_\tau} \sum_{v\in V_\tau} r_s(v) d(\tau, u, v) + \sum_{u\in V_{G-\tau}} \left(\sum_{v\in V_G} r_s(v) - r_s(u)\right) \omega(u, p(u))$$
$$+ \sum_{u\in V_{G-\tau}} \sum_{v\in V_\tau} r_s(v) d(\tau, p(u), v) - r_s(v) d(\tau, p(u), v) + \sum_{u\in V_{G-\tau}} \sum_{v\in V_\tau} r_s(u)\left(d(\tau, p(u), v) + \omega(u, p(u))\right)$$
$$+ \sum_{u\in V_{G-\tau}} \sum_{v\in V_{G-\tau}-u} r_s(u) d(\tau, p(u), p(v)) + \sum_{u\in V_{G-\tau}} \sum_{v\in V_{G-\tau}-u} r_s(u)\left(\omega(u, p(u)) + \omega(v, p(v))\right)$$

$$C(S') = \sum_{u\in V_G} \sum_{v\in V_G} r_s(u) d(T, u, v) - \sum_{u\in V_\tau} \sum_{v\in V_\tau} r_s(v) d(\tau, u, v) - \sum_{u\in V_{G-\tau}} \sum_{v\in V_\tau} r_s(v) d(\tau, p(u), v)$$

$$C(S') = C(T) - \sum_{u\in V_\tau} \sum_{v\in V_\tau} r_s(v) d(\tau, u, v) - \sum_{v\in V_\tau} \sum_{u\in V_\tau} r_s(v) d(\tau, u, v) l_u$$

$$C(S') = C(T) - \sum_{u\in V_\tau} \sum_{v\in V_\tau} r_s(v) d(\tau, u, v)(l_u + 1)$$

where $\sum_{u\in V_\tau} \sum_{v\in V_\tau} r_s(v) d(\tau, u, v)(l_u + 1)$ is the same for every solution with the same configuration. Then, an optimal solution to UMCF with instance I' is associated to an optimal k-star with configuration c of m-SROCT with instance I.

In order to obtain I' from I the cost of each arc in G' must be calculated. This can be done in $O((n-k)k^3)$. Defining the demands and the graph G' itself can be done in $O((n-k)k + n)$. Finally, obtaining the k-star T associated to a solution S' can be done in $O(n-k)$, while the complexity of calculating $C(T)$ would be $O(k^3)$. So, the reduction above can be done in $O(nk^3)$. □

It is well known that UMCF can be solved in $O(n\log(n)(nk + n\log(n))) = O(n^2\log^2(n))$ (e.g. [5]). Then, finding an optimal k-star for m-SROCT with fixed k can be done efficiently.

Lemma 3.1. *The optimum k-star for* m-SROCT *with fixed k can be solved in* $O(n^{2k+1}\log^2(n))$.

4. PTAS for *m*-SROCT

In this section we prove that for $0 < \delta \le \frac{1}{2}$ there exists a k-star, with k depending on δ, which is a $\frac{1}{1-\delta}$-approximation of m-WSDOCT and its particular case m-SROCT. For that, from now on, we will consider an instance I of m-WSDOCT. Remember that $n = n(G)$, $R_s = r_s(G)$ and $R_r = r_r(G)$.

The idea of the proof is similar to those presented in [12,10,11]. Given $0 < \delta \le \frac{1}{2}$ and a spanning tree T of G, we show the existence of a set Y of internally disjoint paths whose union results in a sub-tree S of T, such that the communication cost of each component $B \in T - S$ is at most a small fraction of the communication cost of T, which implies that most of the communication cost of T passes by S. Also, we prove that the size of Y is limited by a function of δ and we show how to construct a k-star from Y, where the value of k depends on the size of Y. The communication cost of the k-star approximates the communication cost of T by a factor of $\frac{1}{1-\delta}$.

4.1. Notation

First, in order to present the results of this section, we need some notation, which generalizes the notation given in [12,10,11]:

Definition 4.1. Given a spanning tree T of G, a set of edges H of T and a node u of T, $VB(T, H, u)$ is the set of nodes in the component of $T - H$ containing the vertex u.

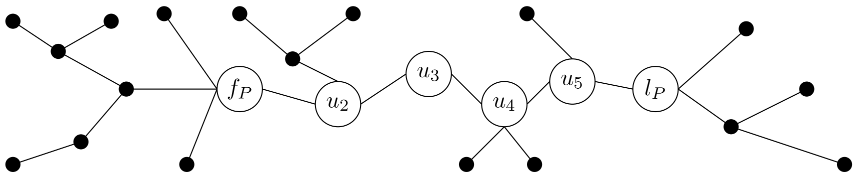

Fig. 1. Consider the above spanning tree T of a graph G with sending weight equals one for all the nodes, where all the edges have unitary weights and P is the path of T from node f_P to l_P. Observe that $VB(T, E_P, f_P)$ is the set of nodes to the left of f_P (including f_P), $VB(T, E_P, l_P)$ is the set of nodes to the right of l_P (including l_P), $VB(T, E_P, u_2)$ is the set of nodes containing u_2 and the three nodes above it, $VB(T, E_P, u_3) = \{u_3\}$, $VB(T, E_P, u_4)$ is the set of nodes containing u_4 and the two nodes below it, and $VB(T, E_P, u_5)$ is the set of nodes containing u_5 and the node above it. Then, $\sigma_P^f = 9$, $\sigma_P^l = 5$, $\sigma_P^m = 10$, and thus $\sigma_P^s = 9$ and $\sigma_P^l = 5$. Also, $R_s^f(P) = r_s(VB(T, E_P, u_2)) \times 1 + r_s(VB(T, E_P, u_3)) \times 2 + r_s(VB(T, E_P, u_4)) \times 3 + r_s(VB(T, E_P, u_5)) \times 4 = 4 \times 1 + 1 \times 2 + 3 \times 3 + 2 \times 4 = 23$ and $R_s^l(P) = r_s(VB(T, E_P, u_2)) \times 4 + r_s(VB(T, E_P, u_3)) \times 3 + r_s(VB(T, E_P, u_4)) \times 2 + r_s(VB(T, E_P, u_5)) \times 1 = 4 \times 4 + 1 \times 3 + 3 \times 2 + 2 \times 1 = 27$, which yields $R_s(p) = 23$.

Definition 4.2. Given a spanning tree T of G, a path $P = u_1, \ldots, u_h$ of T, we denote by f_P (or f, when P is clear by the context) the first node of P and l_P (or l) the last node. We will use σ to denote sending weights and ρ to denote receiving weights. We define:

For the sending weights (Fig. 1 gives an example of these notation):

- $\sigma_P^f = r_s\left(VB(T, E_P, f)\right)$ and $\sigma_P^l = r_s\left(VB(T, E_P, l)\right)$, are the sum of sending weights in the component of $T - E_P$ containing the first node of P and the component containing the last node of P,
- $\sigma_P^m = R_s - \sigma_P^f - \sigma_P^l$, is the sum of sending weights of all the nodes in the component of $T - E_P$ containing an internal node of P,
- $R_s(P, v) = \sum_{i=2}^{h-1} r_s\left(VB(T, E_P, u_i)\right) d(P, v, u_i)$, is the sum over each internal node u_i of P of sending weights in the component of $T - E_P$ containing u_i times the distance in P from u_i to node v,
- $R_s^f(P) = R_s(P, f)$, $R_s^l(P) = R_s(P, l)$, represent the sums over each internal node u_i of P of the sending weights in the component of $T - E_P$ containing u_i times the distance in P from u_i to the first node of P and to the last node of P,
- $\sigma_P^{\max} = \max\{\sigma_P^f, \sigma_P^l\}$, $\sigma_P^{\min} = \min\{\sigma_P^f, \sigma_P^l\}$,
- if $\sigma_P^f = \sigma_P^{\max}$ then $R_s(P) = R_s^f(P)$, else $R_s(P) = R_s^l(P)$.

Analogously, for the receiving weights:

- $\rho_P^f = r_r\left(VB(T, E_P, f)\right)$, $\rho_P^l = r_r\left(VB(T, E_P, l)\right)$, $\rho_P^m = R_r - \rho_P^f - \rho_P^l$,
- $R_r(P, v) = \sum_{i=2}^{h-1} r_r\left(VB(T, E_P, u_i)\right) d(P, v, u_i)$, $R_r^f(P) = R_r(P, f)$, $R_r^l(P) = R_r(P, l)$,
- $\rho_P^{\max} = \max\{\rho_P^f, \rho_P^l\}$, $\rho_P^{\min} = \min\{\rho_P^f, \rho_P^l\}$,
- if $\rho_P^f = \rho_P^{\max}$ then $R_r(P) = R_r^f(P)$, else $R_r(P) = R_r^l(P)$.

Now we introduce definitions for *separators*. A δ-separator is a sub-tree of a spanning tree T of G, whose deletion gives rise to components that are bounded (in the sending weight, receiving weight or both) by a factor δ of the total value (R_s or R_r). Formally:

Definition 4.3. Given $0 < \delta \leq \frac{1}{2}$ and a spanning tree T of G, a sub-tree S of T is a δ-σ-**separator** of T if every component B of $T - S$, satisfies $r_s(B) \leq \delta R_s$. If every component B of $T - S$, satisfies $r_r(B) \leq \delta R_r$, S is a δ-ρ-separator of T. If both conditions apply, S is a δ-$\sigma\rho$-**separator** of T.

Also, we define δ-$\sigma\rho$-path and δ-$\sigma\rho$-spine:

Definition 4.4. Given $0 < \delta \leq \frac{1}{2}$ and a spanning tree T of G, a path P of T is a δ-$\sigma\rho$-**path** of T if $\sigma_P^m \leq \delta \frac{R_s}{6}$ and $\rho_P^m \leq \delta \frac{R_r}{6}$.

Definition 4.5. Given $0 < \delta \leq \frac{1}{2}$ and a spanning tree T of G, a set $Y = \{P_1, P_2, \ldots, P_l\}$ of δ-$\sigma\rho$-paths internally-disjoint of T is a δ-$\sigma\rho$-spine, if $S = \bigcup_{i=1}^{l} P_i$ is a minimal δ-$\sigma\rho$-separator of T. $ext(Y)$ denotes the endpoints set of all paths in Y.

4.2. Approximation lemma

We prove that for any $0 < \delta \leq \frac{1}{2}$, any spanning tree T of G and any δ-$\sigma\rho$-spine Y of T, there exists a $|ext(Y)|$-star with communication cost bounded by $\frac{1}{1-\delta}C(T)$. This lemma, together with Lemma 4.3 is the basis of the main result of this work.

Lemma 4.1. *Given $0 < \delta \leq \frac{1}{2}$, a spanning tree T of G and a δ-$\sigma\rho$-spine Y of T, there exists a $|ext(Y)|$-star X of G satisfying $C(X) \leq \frac{1}{1-\delta}C(T)$.*

In order to conclude that lemma we prove some intermediary results. Given a δ-$\sigma\rho$-spine Y of a spanning tree T of G, we replace each path of Y by the edge connecting its endpoints and select the *best* endpoint to be adjacent of the nodes in the middle of the path. Since the paths of Y are δ-$\sigma\rho$-paths, the amount of communication requirement associated to its middle nodes is at most a $\frac{\delta}{6}$ part of the total requirement of the tree, also the graph is metric, so this modification increases the total communication cost of T in at most a δ fraction of its original value. Also, since the union of the paths in Y define a δ-$\sigma\rho$ separator S of T, the communication requirement of each component out of S is at must a δ part of the total communication requirement of T. Then, by removing the edges of each component out of S, and adding an edge between each node of the component to the nearest endpoint of Y, we are able to obtain a k-star of G such that its communication cost approximates the communication cost of T in a $(1+\epsilon)$ factor, with ϵ depending on δ and k equal to the number of endpoints of Y. The detailed proofs of these claims are given in next sections.

4.2.1. Upper bound for a k-star

First, given a δ-$\sigma\rho$-spine Y of a spanning tree of G, we show how to construct a $|ext(Y)|$-star of G and also, we give an upper bound for the communication cost of the star.

Proposition 4.1. *Given $0 < \delta \leq \frac{1}{2}$, a spanning tree T of G and a δ-$\sigma\rho$-spine Y of T, there exists a $|ext(Y)|$-star X of G which satisfies:*

$$\begin{aligned} C(X) \leq & \sum_{P\in Y}\left(\sigma_P^f\rho_P^l + \rho_P^f\sigma_P^l\right)\omega(P) + \min\left\{\Delta_{fl}(P), \Delta_{lf}(P)\right\} \\ & + R_r\sum_{u\in V_G} r_s(u)d(T,u,S) + R_s\sum_{u\in V_G} r_r(u)d(T,u,S) \end{aligned}$$

where $\Delta_{wz}(P) = \omega(P)\left(\sigma_P^m\rho_P^w + \rho_P^m\sigma_P^w\right) + R_rR_s^z(P) + R_sR_r^z(P)$, $w, z \in \{f, l\}$, and $S = \bigcup_{P\in Y} P$.

Proof. We are given $0 < \delta \leq \frac{1}{2}$ and a δ-$\sigma\rho$-spine Y of a spanning tree T of G. Let $S = \cup_{P\in Y} P$ and define a $|ext(Y)|$-star X as follows:

- The core τ of X has the set of nodes that are endpoints of the paths in Y ($ext(Y)$). Two nodes $u, v \in \tau$ are adjacent in τ if in Y there exists a path with endpoints u and v. Since the paths in Y are internally disjoint and their union results in the tree S, we conclude that τ is a tree over $ext(Y)$.
- For every node $u \in \tau$ and for every node $v \in VB(T, E_S, u) - \{u\}$, we also include an edge (u, v) in X.
- Observe that each node $u \in T$ not included in X by the previous steps belongs to $V - VB(T, E_S, f_P) - VB(T, E_S, l_P)$ for some path $P \in Y$. Then, we include edge (u, f_P) in X if $\Delta_{fl}(P) \leq \Delta_{lf}(P)$, otherwise we include edge (u, l_P) in X.

Formally X is defined:

- $V_X = V_G$
- $E_\tau = \{(u, v)|\exists P \in Y \text{ with endpoints } u \text{ and } v\}$
- $E_X = E_\tau \cup \{(u, v)|u \in V_\tau \wedge v \in VB(T, E_S, u)\}$
  $\cup\{(u, v)|\exists P \in Y : v \in V(T, P) \wedge g(P) = 1 \wedge u = f_P\}$
  $\cup\{(u, v)|\exists P \in Y : v \in V(T, P) \wedge g(P) = 0 \wedge u = l_P\}$
  where, if $\Delta_{fl}(P) \leq \Delta_{lf}(P)$ then $g(P) = 1$, otherwise $g(P) = 0$, and $V(T, P) = \bigcup_{u\in V_P-\{f,l\}} VB(T, E_P, u)$.

Our construction guarantees X to be a $|ext(Y)|$-star of G with core τ. Then, we only need to analyze its associated communication cost. For that, consider e_i and e_j the endpoints of edge $e \in E_X$. Also, notice that we can calculate the communication cost of a solution X by adding over each edge $e \in E_X$ the communication amount passing over e times the length of e:

$$C(X) = \sum_{e\in E_X} r_s\left(VB(X, e, e_i)\right) r_r\left(VB(X, e, e_j)\right)\omega(e) + \sum_{e\in E_X} r_s\left(VB(X, e, e_j)\right) r_r\left(VB(X, e, e_i)\right)\omega(e)$$

$$\begin{aligned} C(X) = & \sum_{e\in E_\tau} r_s\left(VB(X, e, e_i)\right) r_r\left(VB(X, e, e_j)\right)\omega(e) + \sum_{e\in E_\tau} r_s\left(VB(X, e, e_j)\right) r_r\left(VB(X, e, e_i)\right)\omega(e) \\ & + \sum_{e\in E_X-E_\tau} r_s\left(VB(X, e, e_i)\right) r_r\left(VB(X, e, e_j)\right)\omega(e) \\ & + \sum_{e\in E_X-E_\tau} r_s\left(VB(X, e, e_j)\right) r_r\left(VB(X, e, e_i)\right)\omega(e). \end{aligned}$$

Observe that for every edge $e \in E_X - E_\tau$ one of its endpoints is a leaf of X. If we denote by u the leaf endpoint of $e \in E_X - E_\tau$, then the communication amount over e results:

$$\begin{aligned} (R_s - r_s(u))\, r_r(u) + r_s(u)\,(R_r - r_r(u)) &= R_sr_r(u) + R_rr_s(u) - 2r_sr_r(u) \\ &\leq R_sr_r(u) + R_rr_s(u). \end{aligned}$$

Since every node $u \in V_{G-\tau}$ is a leaf endpoint of an edge $e \in E_X - E_\tau$ and $d(X, u, \tau) = \omega(e)$, then:

$$C(X) \le \sum_{e\in E_\tau} r_s(VB(X,e,e_i))\, r_r\left(VB(X,e,e_j)\right)\omega(e) + \sum_{e\in E_\tau} r_s\left(VB(X,e,e_j)\right) r_r(VB(X,e,e_i))\,\omega(e) + \sum_{u\in V_{G-\tau}} (R_r r_s(u) + R_s r_r(u))\, d(X,u,\tau).$$

If we define as $p(u)$ the node in τ adjacent in X to $u \in V_{X-\tau}$, then $d(X, u, \tau) = d(X, u, p(u))$ and:

$$C(X) \le \sum_{e\in E_\tau} r_s(VB(X,e,e_i))\, r_r\left(VB(X,e,e_j)\right)\omega(e) + \sum_{e\in E_\tau} r_s\left(VB(X,e,e_j)\right) r_r(VB(X,e,e_i))\,\omega(e) + \sum_{u\in V_{G-\tau}} (R_r r_s(u) + R_s r_r(u))\, d(X,u,p(u)).$$

If we define $\partial(T, S, u, v)$ as the sum of the lengths of the edges in S which also are in the path between u and v in T, then $d(T, u, v) = d(T, u, S) + \partial(T, S, u, v)$ and:

$$C(X) \le \sum_{e\in E_\tau} r_s(VB(X,e,e_i))\, r_r\left(VB(X,e,e_j)\right)\omega(e) + \sum_{e\in E_\tau} r_s\left(VB(X,e,e_j)\right) r_r(VB(X,e,e_i))\,\omega(e) + \sum_{u\in V_{G-\tau}} (R_r r_s(u) + R_s r_r(u))\, d(T,u,S) + \sum_{u\in V_{G-\tau}} (R_r r_s(u) + R_s r_r(u))\, \partial(T,S,u,p(u)).$$

Since $V(T, P) = \bigcup_{u\in V_P-\{f,l\}} VB(T, E_P, u)$ and for every node $u \in V_{G-\tau}$, $p(u) \in V_\tau \subseteq V_S$, then:

$$C(X) \le \sum_{e\in E_\tau} r_s(VB(X,e,e_i))\, r_r\left(VB(X,e,e_j)\right)\omega(e) + \sum_{e\in E_\tau} r_s\left(VB(X,e,e_j)\right) r_r(VB(X,e,e_i))\,\omega(e) + \sum_{u\in V_{G-\tau}} (R_r r_s(u) + R_s r_r(u))\, d(T,u,S) + \sum_{P\in Y}\sum_{u\in V(T,P)} (R_r r_s(u) + R_s r_r(u))\, \partial(T,S,u,p(u)).$$

Notice that for every $P \in Y$ if $\Delta_{fl}(P) \le \Delta_{lf}(P)$ then $g(P) = 1$ and for each $u \in V(T, P)$, $p(u) = f_P$, otherwise $g(P) = 0$ and for each $u \in V(T, P)$, $p(u) = l_P$. Then:

$$C(X) \le \sum_{e\in E_\tau} r_s(VB(X,e,e_i))\, r_r\left(VB(X,e,e_j)\right)\omega(e) + \sum_{e\in E_\tau} r_s\left(VB(X,e,e_j)\right) r_r(VB(X,e,e_i))\,\omega(e) + \sum_{u\in V_{G-\tau}} (R_r r_s(u) + R_s r_r(u))\, d(T,u,S) + \sum_{P\in Y}\sum_{u\in V(T,P)} (R_r r_s(u) + R_s r_r(u))\, g(P)\partial(T,S,u,f_P) + \sum_{P\in Y}\sum_{u\in V(T,P)} (R_r r_s(u) + R_s r_r(u))\,(1-g(P))\,\partial(T,S,u,l_P)$$

$$C(X) \le \sum_{e\in E_\tau} r_s(VB(X,e,e_i))\, r_r\left(VB(X,e,e_j)\right)\omega(e) + \sum_{e\in E_\tau} r_s\left(VB(X,e,e_j)\right) r_r(VB(X,e,e_i))\,\omega(e) + \sum_{u\in V_{G-\tau}} (R_r r_s(u) + R_s r_r(u))\, d(T,u,S) + R_r \sum_{P\in Y}\left(g(P)R_s^f(P) + (1-g(P))R_s^l(P)\right) + R_s \sum_{P\in Y}\left(g(P)R_r^f(P) + (1-g(P))R_r^l(P)\right).$$

Now we analyze the edges in τ. For that consider $e \in E_\tau$ and let P be the path with endpoints e_i and e_j in Y. Observe that $r_s(VB(X,e,e_i)) = \sigma_P^f + g(P)\sigma_P^m$, $r_s\left(VB(X,e,e_j)\right) = \sigma_P^l + (1-g(P))\,\sigma_P^m$, $r_r(VB(X,e,e_i)) = \rho_P^f + g(P)\rho_P^m$ and $r_r\left(VB(X,e,e_j)\right) = \rho_P^l + (1-g(P))\,\rho_P^m$. Then, by the triangular inequality:

$$\begin{aligned} &r_s(VB(X,e,e_i))\, r_r\left(VB(X,e,e_j)\right)\omega(e) + r_s\left(VB(X,e,e_j)\right) r_r(VB(X,e,e_i))\,\omega(e)\\ &\le \left(\sigma_P^f + g(P)\sigma_P^m\right)\left(\rho_P^l + (1-g(P))\,\rho_P^m\right)\omega(P) + \left(\sigma_P^l + (1-g(P))\,\sigma_P^m\right)\left(\rho_P^f + g(P)\rho_P^l\right)\omega(P)\\ &= \left(\sigma_P^f\rho_P^l + \sigma_P^l\rho_P^f\right)\omega(P) + \left((1-g(P))\left(\sigma_P^f\rho_P^m + \sigma_P^m\rho_P^f\right) + g(P)\left(\sigma_P^m\rho_P^l + \sigma_P^l\rho_P^m\right)\right)\omega(P). \end{aligned}$$

Finally, we obtain:

$$\begin{aligned} C(X) \leq & \sum_{P\in Y}\left(\sigma_P^f\rho_P^l+\sigma_P^l\rho_P^f\right)\omega(P)+\sum_{P\in Y}(1-g(P))\left(\sigma_P^f\rho_P^m+\sigma_P^m\rho_P^f\right)\omega(P) \\ & +\sum_{P\in Y}g(P)\left(\sigma_P^m\rho_P^l+\sigma_P^l\rho_P^m\right)\omega(P)+\sum_{u\in V_{G-\tau}}(R_r r_s(u)+R_s r_r(u))\,d(T,u,S) \\ & +R_r\sum_{P\in Y}\left(g(P)R_s^f(P)+(1-g(P))R_s^l(P)\right)+R_s\sum_{P\in Y}\left(g(P)R_r^f(P)+(1-g(P))R_r^l(P)\right) \\ C(X) \leq & \sum_{P\in Y}\left(\sigma_P^f\rho_P^l+\rho_P^f\sigma_P^l\right)\omega(P)+\min\left\{\Delta_{fl}(P),\Delta_{lf}(P)\right\} \\ & +R_r\sum_{u\in V_G}r_s(u)d(T,u,S)+R_s\sum_{u\in V_G}r_r(u)d(T,u,S). \quad \square \end{aligned}$$

4.2.2. Lower bound for the communication cost of a spanning tree

The previous proposition gave us an upper bounded k-star of G and now we show a lower bound for the communication cost of a spanning tree of G. Observe that the combination of these results will help us to obtain a relation between the communication cost of a spanning tree of G and the k-star we construct to approximate it.

Proposition 4.2. *Given $0<\delta\leq\frac{1}{2}$, a spanning tree T of G and a δ-$\sigma\rho$-spine Y of T, then:*

$$\begin{aligned} C(T) \geq & \sum_{P\in Y}\left(\rho_P^l\sigma_P^f+\rho_P^f\sigma_P^l+\rho_P^{\min}\sigma_P^m+\sigma_P^{\min}\rho_P^m\right)\omega(P)+\sum_{P\in Y}\left(\sigma_P^{\max}-\sigma_P^{\min}\right)R_r(P)+\left(\rho_P^{\max}-\rho_P^{\min}\right)R_s(P) \\ & +(1-\delta)\left(R_s\sum_{u\in V_G}r_r(u)d(T,u,S)+R_r\sum_{u\in V_G}r_s(u)d(T,u,S)\right) \end{aligned}$$

where $S=\bigcup_{P\in Y}P$.

Proof. We are given $0<\delta\leq\frac{1}{2}$ and a δ-$\sigma\rho$-spine Y of a spanning tree T of G, being $S=\cup_{P\in Y}P$. Then, the communication cost of T is:

$$\begin{aligned} C(T) = & \sum_{e\in E_T}r_s(VB(T,e,e_i))\,r_r\left(VB(T,e,e_j)\right)\omega(e)+\sum_{e\in E_T}r_s\left(VB(T,e,e_j)\right)r_r(VB(T,e,e_i))\,\omega(e) \\ C(T) = & \sum_{e\in E_S}r_s(VB(T,e,e_i))\,r_r\left(VB(T,e,e_j)\right)\omega(e)+\sum_{e\in E_S}r_s\left(VB(T,e,e_j)\right)r_r(VB(T,e,e_i))\,\omega(e) \\ & +\sum_{e\in E_T-E_S}r_s(VB(T,e,e_i))\,r_r\left(VB(T,e,e_j)\right)\omega(e) \\ & +\sum_{e\in E_T-E_S}r_s\left(VB(T,e,e_j)\right)r_r(VB(T,e,e_i))\,\omega(e). \end{aligned}$$

Observe that for $e\in E_{T-S}$ one of the endpoints, without loss of generality e_j, satisfies: $r_r\left(VB(T,e,e_j)\right)\leq\delta R_r$ and $r_s\left(VB(T,e,e_j)\right)\leq\delta R_s$, so: $r_r(VB(T,e,e_i))\geq(1-\delta)R_r$ and $r_s(VB(T,e,e_i))\geq(1-\delta)n$, then:

$$\begin{aligned} C(T) \geq & \sum_{e\in E_S}r_s(VB(T,e,e_i))\,r_r\left(VB(T,e,e_j)\right)\omega(e)+\sum_{e\in E_S}r_s\left(VB(T,e,e_j)\right)r_r(VB(T,e,e_i))\,\omega(e) \\ & +\sum_{e\in E_T-E_S}(1-\delta)R_s r_r\left(VB(T,e,e_j)\right)\omega(e)+\sum_{e\in E_T-E_S}(1-\delta)R_r r_s\left(VB(T,e,e_j)\right)\omega(e) \\ C(T) = & \sum_{e\in E_S}r_s(VB(T,e,e_i))\,r_r\left(VB(T,e,e_j)\right)\omega(e)+\sum_{e\in E_S}r_s\left(VB(T,e,e_j)\right)r_r(VB(T,e,e_i))\,\omega(e) \\ & +(1-\delta)\left(R_s\sum_{u\in V_G}r_r(u)d(T,u,S)+R_r\sum_{u\in V_G}r_s(u)d(T,u,S)\right). \end{aligned}$$

As every edge $e\in E_S$ is in exactly one path of Y, we have:

$$\begin{aligned} C(T) \geq & \sum_{P\in Y}\sum_{e\in P}r_s(VB(T,e,e_i))\,r_r\left(VB(T,e,e_j)\right)\omega(e) \\ & +\sum_{P\in Y}\sum_{e\in P}r_s\left(VB(T,e,e_j)\right)r_r(VB(T,e,e_i))\,\omega(e) \\ & +(1-\delta)\left(R_s\sum_{u\in V_G}r_r(u)d(T,u,S)+R_r\sum_{u\in V_G}r_s(u)d(T,u,S)\right). \end{aligned}$$

If $P \in Y$ and $P = v_1 \dots v_h$, then:

$$\sum_{e\in P} r_s\left(VB(T,e,e_i)\right) r_r\left(VB(T,e,e_j)\right)\omega(e) + \sum_{e\in P} r_s\left(VB(T,e,e_j)\right) r_r\left(VB(T,e,e_i)\right)\omega(e)$$
$$= \sum_{l=1}^{h-1} r_s\left(VB(T,(v_l,v_{l+1}),v_l)\right) r_r\left(VB(T,(v_l,v_{l+1}),v_{l+1})\right)\omega(v_l,v_{l+1})$$
$$+ \sum_{l=1}^{h-1} r_s\left(VB(T,(v_l,v_{l+1}),v_{l+1})\right) r_r\left(VB(T,(v_l,v_{l+1}),v_l)\right)\omega(v_l,v_{l+1}).$$

Notice that for each $l \in \{1,\dots,h-1\}$:

$$r_s\left(VB(T,(v_l,v_{l+1}),v_l)\right) = \sigma_P^f + \sum_{k=2}^{l} r_s\left(VB(T,E_P,v_k)\right),$$
$$r_s\left(VB(T,(v_l,v_{l+1}),v_{l+1})\right) = \sigma_P^l + \sum_{k=l+1}^{h-1} r_s\left(VB(T,E_P,v_k)\right),$$
$$r_r\left(VB(T,(v_l,v_{l+1}),v_l)\right) = \rho_P^f + \sum_{k=2}^{l} r_r\left(VB(T,E_P,v_k)\right),$$
$$r_r\left(VB(T,(v_l,v_{l+1}),v_{l+1})\right) = \rho_P^l + \sum_{k=l+1}^{h-1} r_r\left(VB(T,E_P,v_k)\right).$$

Then:

$$\sum_{e\in P} r_s\left(VB(T,e,e_i)\right) r_r\left(VB(T,e,e_j)\right)\omega(e) + \sum_{e\in P} r_s\left(VB(T,e,e_j)\right) r_r\left(VB(T,e,e_i)\right)\omega(e)$$
$$\geq \left(\sigma_P^f\rho_P^l + \rho_P^f\sigma_P^l\right)\sum_{l=1}^{h-1}\omega(v_l,v_{l+1}) + \sigma_P^f\sum_{l=1}^{h-1}\left(\sum_{k=l+1}^{h-1} r_r\left(VB(T,E_P,v_k)\right)\right)\omega(v_l,v_{l+1})$$
$$+\sigma_P^l\sum_{l=1}^{h-1}\left(\sum_{k=2}^{l} r_r\left(VB(T,E_P,v_k)\right)\right)\omega(v_l,v_{l+1}) + \rho_P^f\sum_{l=1}^{h-1}\left(\sum_{k=l+1}^{h-1} r_s\left(VB(T,E_P,v_k)\right)\right)\omega(v_l,v_{l+1})$$
$$+\rho_P^l\sum_{l=1}^{h-1}\left(\sum_{k=2}^{l} r_s\left(VB(T,E_P,v_k)\right)\right)\omega(v_l,v_{l+1})$$
$$= \left(\sigma_P^f\rho_P^l + \rho_P^f\sigma_P^l\right)\omega(P) + \sigma_P^f R_r^f(P) + \sigma_P^l R_r^l(P) + \rho_P^f R_s^f(P) + \rho_P^l R_s^l(P).$$

Since $\sigma_P^m\omega(P) = R_s^f(P) + R_s^l(P)$ and $\rho_P^m\omega(P) = R_r^f(P) + R_r^l(P)$, then we conclude:

$$\sum_{e\in P} r_s\left(VB(T,e,e_i)\right) r_r\left(VB(T,e,e_j)\right)\omega(e) + \sum_{e\in P} r_s\left(VB(T,e,e_j)\right) r_r\left(VB(T,e,e_i)\right)\omega(e)$$
$$\leq \left(\sigma_P^f\rho_P^l + \rho_P^f\sigma_P^l\right)\omega(P) + \sigma_P^{\max}R_r(P) + \sigma_P^{\min}\left(\rho_P^m\omega(P) - R_r(P)\right) + \rho_P^{\max}R_s(P) + \rho_P^{\min}\left(\sigma_P^m\omega(P) - R_s(P)\right)$$
$$= \left(\sigma_P^f\rho_P^l + \rho_P^f\sigma_P^l + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right)\omega(P) + \left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_r(P) + \left(\rho_P^{\max} - \rho_P^{\min}\right)R_s(P).$$

Finally, we obtain the lower bound:

$$C(T) \geq \sum_{P\in Y}\left(\rho_P^l\sigma_P^f + \rho_P^f\sigma_P^l + \rho_P^{\min}\sigma_P^m + \sigma_P^{\min}\rho_P^m\right)\omega(P) + \sum_{P\in Y}\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_r(P) + \left(\rho_P^{\max} - \rho_P^{\min}\right)R_s(P)$$
$$+ (1-\delta)\left(R_s\sum_{u\in V_G} r_r(u)d(T,u,S) + R_r\sum_{u\in V_G} r_s(u)d(T,u,S)\right). \quad \square$$

4.2.3. Property of δ-$\sigma\rho$-paths

In order to obtain a relation between the upper bound for a *k*-star and the lower bound for a spanning tree we need the following property over each δ-$\sigma\rho$-path.

Proposition 4.3. *Given* $0 < \delta \leq \frac{1}{2}$ *a* δ-$\sigma\rho$-*path* P *of a* δ-$\sigma\rho$-*spine of a spanning tree* T *of* G*, then:*

$$\begin{aligned}&\left(R_s + R_r - \sigma_P^m - \rho_P^m\right)\left(\sigma_P^f\rho_P^l + \sigma_P^l\rho_P^f\right)\omega(P) + \left(\sigma_P^l + \rho_P^l\right)\left(\omega(P)\left(\sigma_P^m\rho_P^f + \sigma_P^f\rho_P^m\right) + R_sR_r^l(P) + R_rR_s^l(P)\right)\\&+ \left(\sigma_P^f + \rho_P^f\right)\left(\omega(P)\left(\sigma_P^m\rho_P^l + \sigma_P^l\rho_P^m\right) + R_sR_r^f(P) + R_rR_s^f(P)\right)\\&\leq \frac{6+5\delta}{6}\left(R_s + R_r\right)\left(\sigma_P^f\rho_P^l + \sigma_P^l\rho_P^f + \sigma_P^{\min}\rho_P^m + \sigma_P^m\rho_P^{\min}\right)\omega(P)\\&+ \left(R_s + R_r\right)\left(\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_r(P) + \left(\rho_P^{\max} - \rho_P^{\min}\right)R_s(P)\right).\end{aligned}$$

Before proving that proposition, first we prove the fact that follows.

Fact 4.1. *Given a spanning tree* T *of* G *and a path* P *of* T*:*

$$\left(\sigma_P^{\min}\rho_P^{\max} + \sigma_P^{\max}\rho_P^{\min}\right) \leq \left(\sigma_P^f\rho_P^l + \sigma_P^l\rho_P^f\right) \leq \left(\sigma_P^{\max}\rho_P^{\max} + \sigma_P^{\min}\rho_P^{\min}\right).$$

Proof. By definition $\sigma_P^{\min} \leq \sigma_P^{\max}$, then $\sigma_P^{\max} = \sigma_P^{\min} + \xi$ with $\xi \geq 0$, then:

$$\sigma_P^{\min}\rho_P^{\max} + \sigma_P^{\max}\rho_P^{\min} = \sigma_P^{\max}\rho_P^{\max} + \sigma_P^{\min}\rho_P^{\min} - \xi(\rho_P^{\max} - \rho_P^{\min}).$$

By definition $\rho_P^{\min} \leq \rho_P^{\max}$, then: $\sigma_P^{\min}\rho_P^{\max} + \sigma_P^{\max}\rho_P^{\min} \leq \sigma_P^{\max}\rho_P^{\max} + \sigma_P^{\min}\rho_P^{\min}$.
Since $\sigma_P^f\rho_P^l + \sigma_P^l\rho_P^f \in \left\{\sigma_P^{\min}\rho_P^{\max} + \sigma_P^{\max}\rho_P^{\min}, \sigma_P^{\max}\rho_P^{\max} + \sigma_P^{\min}\rho_P^{\min}\right\}$, we conclude:

$$\left(\sigma_P^{\min}\rho_P^{\max} + \sigma_P^{\max}\rho_P^{\min}\right) \leq \left(\sigma_P^f\rho_P^l + \sigma_P^l\rho_P^f\right) \leq \left(\sigma_P^{\max}\rho_P^{\max} + \sigma_P^{\min}\rho_P^{\min}\right). \quad \square$$

Now, we present a proof for Proposition 4.3.

Proof. We are given $0 < \delta \leq \frac{1}{2}$, a δ-$\sigma\rho$-spine Y of a spanning tree T of G and a path $P \in Y$. Consider $S = \cup_{Q\in Y} Q$, define $\phi(P)$ as:

$$\begin{aligned}\phi(P) = &\left(R_s + R_r - \sigma_P^m - \rho_P^m\right)\left(\sigma_P^f\rho_P^l + \sigma_P^l\rho_P^f\right)\omega(P)\\&+ \left(\sigma_P^l + \rho_P^l\right)\left(\omega(P)\left(\sigma_P^m\rho_P^f + \sigma_P^f\rho_P^m\right) + R_sR_r^l(P) + R_rR_s^l(P)\right)\\&+ \left(\sigma_P^f + \rho_P^f\right)\left(\omega(P)\left(\sigma_P^m\rho_P^l + \sigma_P^l\rho_P^m\right) + R_sR_r^f(P) + R_rR_s^f(P)\right).\end{aligned}$$

Observe that, in the equation above, the first line is equal to the sum of lines (1) and (2), also the second line is equal to the sum of lines (3)–(5), and the third line is equal to the sum of lines (6)–(8).

$$\phi(P) = \left(R_s + R_r\right)\left(\sigma_P^f\rho_P^l + \sigma_P^l\rho_P^f\right)\omega(P) \tag{1}$$

$$+ \left(-\sigma_P^m - \rho_P^m\right)\left(\sigma_P^f\rho_P^l + \sigma_P^l\rho_P^f\right)\omega(P) \tag{2}$$

$$+ \left(\sigma_P^m\sigma_P^l\rho_P^f + \rho_P^m\sigma_P^f\rho_P^l\right)\omega(P) \tag{3}$$

$$+ \left(\sigma_P^m\rho_P^f\rho_P^l + \rho_P^m\sigma_P^f\sigma_P^l\right)\omega(P) \tag{4}$$

$$+ R_s\sigma_P^lR_r^l(P) + R_r\sigma_P^lR_s^l(P) + R_s\rho_P^lR_r^l(P) + R_r\rho_P^lR_s^l(P) \tag{5}$$

$$+ \left(\sigma_P^m\sigma_P^f\rho_P^l + \rho_P^m\sigma_P^l\rho_P^f\right)\omega(P) \tag{6}$$

$$+ \left(\sigma_P^m\rho_P^f\rho_P^l + \rho_P^m\sigma_P^f\sigma_P^l\right)\omega(P) \tag{7}$$

$$+ R_s\sigma_P^fR_r^f(P) + R_r\sigma_P^fR_s^f(P) + R_s\rho_P^fR_r^f(P) + R_r\rho_P^fR_s^f(P). \tag{8}$$

Notice that, the sum of lines (2), (3) and (6) results zero, and lines (4) and (7) are the same. Then:

$$\begin{aligned}
\phi(P) = {} & (R_s+R_r)\left(\sigma_P^f\rho_P^l+\sigma_P^l\rho_P^f\right)\omega(P) \\
& +2\left(\sigma_P^m\rho_P^f\rho_P^l+\rho_P^m\sigma_P^f\sigma_P^l\right)\omega(P) && (9)\\
& +R_s\sigma_P^l R_r^l(P)+R_r\sigma_P^l R_s^l(P) && (10)\\
& +R_s\rho_P^l R_r^l(P)+R_r\rho_P^l R_s^l(P) && (11)\\
& +R_s\sigma_P^f R_r^f(P)+R_r\sigma_P^f R_s^f(P) && (12)\\
& +R_s\rho_P^f R_r^f(P)+R_r\rho_P^f R_s^f(P). && (13)
\end{aligned}$$

Since $\{\sigma_P^{\max},\sigma_P^{\min}\}=\{\sigma_P^f,\sigma_P^l\}$, $\{\rho_P^{\max},\rho_P^{\min}\}=\{\rho_P^f,\rho_P^l\}$, $\sigma_P^m\omega(P)=R_s^f(P)+R_s^l(P)$ and $\rho_P^m\omega(P)=R_r^f(P)+R_r^l(P)$. Then, line (9) is equal to (14) and the sum of lines (10)–(13) is equal to the sum of lines (15)–(18) which is also equal to the sum of lines (20)–(22).

$$\begin{aligned}
\phi(P) = {} & (R_s+R_r)\left(\sigma_P^f\rho_P^l+\sigma_P^l\rho_P^f\right)\omega(P) \\
& +2\left(\sigma_P^m\rho_P^{\max}\rho_P^{\min}+\rho_P^m\sigma_P^{\max}\sigma_P^{\min}\right)\omega(P) && (14)\\
& +R_s\sigma_P^{\max}R_r(P)+R_r\sigma_P^{\max}R_s(P) && (15)\\
& +R_s\rho_P^{\max}R_r(P)+R_r\rho_P^{\max}R_s(P) && (16)\\
& +R_r\sigma_P^{\min}\left(\sigma_P^m\omega(P)-R_s(P)\right)+R_r\sigma_P^{\min}\left(\rho_P^m\omega(P)-R_s(P)\right) && (17)\\
& +R_r\rho_P^{\min}\left(\sigma_P^m\omega(P)-R_s(P)\right)+R_r\rho_P^{\min}\left(\rho_P^m\omega(P)-R_s(P)\right) && (18)\\
\phi(P) = {} & (R_s+R_r)\left(\sigma_P^f\rho_P^l+\sigma_P^l\rho_P^f\right)\omega(P) \\
& +2\left(\sigma_P^m\rho_P^{\max}\rho_P^{\min}+\rho_P^m\sigma_P^{\max}\sigma_P^{\min}\right)\omega(P) && (19)\\
& +R_s\left(\sigma_P^{\max}-\sigma_P^{\min}\right)R_r(P)+R_r\left(\rho_P^{\max}-\rho_P^{\min}\right)R_s(P) && (20)\\
& +R_r\left(\sigma_P^{\max}-\sigma_P^{\min}\right)R_s(P)+R_s\left(\rho_P^{\max}-\rho_P^{\min}\right)R_r(P) && (21)\\
& +\left(R_r\sigma_P^{\min}\sigma_P^m+R_s\sigma_P^{\min}\rho_P^m+R_r\rho_P^{\min}\sigma_P^m+R_s\rho_P^{\min}\rho_P^m\right)\omega(P). && (22)
\end{aligned}$$

Grouping the first and last terms of line (22), with line (19) we obtain:

$$\begin{aligned}
\phi(P) = {} & (R_s+R_r)\left(\sigma_P^f\rho_P^l+\sigma_P^l\rho_P^f\right)\omega(P) \\
& +\left(2\sigma_P^m\rho_P^{\max}\rho_P^{\min}+2\rho_P^m\sigma_P^{\max}\sigma_P^{\min}+R_r\sigma_P^{\min}\sigma_P^m+R_s\rho_P^{\min}\rho_P^m\right)\omega(P) \\
& +R_s\left(\sigma_P^{\max}-\sigma_P^{\max}\right)R_r(P)+R_r\left(\rho_P^{\max}-\rho_P^{\min}\right)R_s(P) \\
& +R_r\left(\sigma_P^{\max}-\sigma_P^{\min}\right)R_s(P)+R_s\left(\rho_P^{\max}-\rho_P^{\min}\right)R_r(P)+R_s\sigma_P^{\min}\rho_P^m\omega(P)+R_r\rho_P^{\min}\sigma_P^m\omega(P). && (23)
\end{aligned}$$

Then, by adding:

$$R_r\sigma_P^{\min}\rho_P^m\omega(P)+R_s\rho_P^{\min}\sigma_P^m\omega(P)-\left(R_r\sigma_P^{\min}\rho_P^m\omega(P)+R_s\rho_P^{\min}\sigma_P^m\omega(P)\right),$$

to the line (23) we obtain:

$$\begin{aligned}
\phi(P) = {} & (R_s+R_r)\left(\sigma_P^f\rho_P^l+\sigma_P^l\rho_P^f\right)\omega(P) \\
& +\left(2\sigma_P^m\rho_P^{\max}\rho_P^{\min}+2\rho_P^m\sigma_P^{\max}\sigma_P^{\min}+R_r\sigma_P^{\min}\sigma_P^m+R_s\rho_P^{\min}\rho_P^m\right)\omega(P) \\
& +R_s\left(\sigma_P^{\max}-\sigma_P^{\min}\right)R_r(P)+R_r\left(\rho_P^{\max}-\rho_P^{\min}\right)R_s(P) \\
& +R_r\left(\sigma_P^{\max}-\sigma_P^{\min}\right)R_s(P)+R_s\left(\rho_P^{\max}-\rho_P^{\min}\right)R_r(P)-\left(R_r\sigma_P^{\min}\rho_P^m+R_s\rho_P^{\min}\sigma_P^m\right)\omega(P) \\
& +(R_s+R_r)\left(\sigma_P^{\min}\rho_P^m+\rho_P^{\min}\sigma_P^m\right)\omega(P).
\end{aligned}$$

Grouping the first line of the above equation with the last line we obtain:

$$\begin{aligned}\phi(P) = {} & (R_s + R_r)\left(\sigma_P^f\rho_P^l + \sigma_P^l\rho_P^f + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right)\omega(P)\\ & + \left(2\sigma_P^m\rho_P^{\max}\rho_P^{\min} + 2\rho_P^m\sigma_P^{\max}\sigma_P^{\min} + R_r\sigma_P^{\min}\sigma_P^m + R_s\rho_P^{\min}\rho_P^m\right)\omega(P)\\ & + R_s\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_r(P) + R_r\left(\rho_P^{\max} - \rho_P^{\min}\right)R_s(P)\\ & + R_r\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_s(P) + R_s\left(\rho_P^{\max} - \rho_P^{\min}\right)R_r(P) - \left(R_r\sigma_P^{\min}\rho_P^m + R_s\rho_P^{\min}\sigma_P^m\right)\omega(P).\end{aligned}$$

Observe that $\sigma_P^m \leq \frac{\delta}{6}R_s$, $\rho_P^m \leq \frac{\delta}{6}R_r$, $R_r \geq \rho_P^{\max}$ and $R_s \geq \sigma_P^{\max}$, then:

$$\begin{aligned}& 2\sigma_P^m\rho_P^{\max}\rho_P^{\min} + 2\rho_P^m\sigma_P^{\max}\sigma_P^{\min} + R_r\sigma_P^{\min}\sigma_P^m + R_s\rho_P^{\min}\rho_P^m\\ & \leq \frac{\delta}{3}\left(R_s\rho_P^{\max}\rho_P^{\min} + R_r\sigma_P^{\max}\sigma_P^{\min}\right) + \frac{\delta}{6}\left(R_sR_r\sigma_P^{\min} + R_sR_r\rho_P^{\min}\right)\\ & \leq \frac{\delta}{3}\left(R_sR_r\rho_P^{\min} + R_sR_r\sigma_P^{\min}\right) + \frac{\delta}{6}\left(R_sR_r\sigma_P^{\min} + R_sR_r\rho_P^{\min}\right)\\ & \leq \frac{\delta}{2}\left(R_sR_r\rho_P^{\min} + R_sR_r\sigma_P^{\min}\right).\end{aligned}$$

Since $\rho_P^{\max} \geq \rho_P^{\min}$, $\sigma_P^{\max} \geq \sigma_P^{\min}$, $R_s = \sigma_P^{\max} + \sigma_P^{\min} + \sigma_P^m$ and $R_r = \rho_P^{\max} + \rho_P^{\min} + \rho_P^m$:

$$\begin{aligned}R_sR_r\sigma_P^{\min} + R_sR_r\rho_P^{\min} \leq {} & R_sR_r\sigma_P^{\min} + R_sR_r\rho_P^{\min} + R_s\left(\rho_P^{\min}(\sigma_P^{\max} - \sigma_P^{\min}) + \rho_P^{\min}\sigma_P^m\right)\\ & + R_r\left(\sigma_P^{\min}(\rho_P^{\max} - \rho_P^{\min}) + \sigma_P^{\min}\rho_P^m\right)\\ = {} & R_s\left(\rho_P^{\max} + \rho_P^{\min} + \rho_P^m\right)\sigma_P^{\min} + \left(\sigma_P^{\max} + \sigma_P^{\min} + \sigma_P^m\right)R_r\rho_P^i\\ & + \quad R_s\left(\rho_P^{\min}(\sigma_P^{\max} - \sigma_P^{\min}) + \rho_P^{\min}\sigma_P^m\right) + R_r\left(\sigma_P^{\min}(\rho_P^{\max} - \rho_P^{\min}) + \sigma_P^{\min}\rho_P^m\right)\\ = {} & R_s\left(\sigma_P^{\max}\rho_P^{\min} + \rho_P^{\max}\sigma_P^{\min} + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right)\\ & + R_r\left(\sigma_P^{\max}\rho_P^{\min} + \rho_P^{\max}\sigma_P^{\min} + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\right)\\ = {} & (R_s + R_r)\left(\sigma_P^{\max}\rho_P^{\min} + \rho_P^{\min}\sigma_P^{\max} + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right)\\ \leq {} & (R_s + R_r)\left(\sigma_P^f\rho_P^l + \rho_P^f\sigma_P^l + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right).\end{aligned}$$

The last inequality was given by Fact 4.1. Then:

$$\begin{aligned}\phi(P) \leq {} & (R_s + R_r)\left(\sigma_P^f\rho_P^l + \rho_P^f\sigma_P^l + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right)\omega(P) && (24)\\ & + \frac{\delta}{2}(R_s + R_r)\left(\sigma_P^f\rho_P^l + \rho_P^f\sigma_P^l + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right)\omega(P) && (25)\\ & + R_s\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_r(P) + R_r\left(\rho_P^{\max} - \rho_P^{\min}\right)R_s(P) && (26)\\ & + R_r\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_s(P) + R_s\left(\rho_P^{\max} - \rho_P^{\min}\right)R_r(P) - \left(R_r\sigma_P^{\min}\rho_P^m + R_s\rho_P^{\min}\sigma_P^m\right)\omega(P).\end{aligned}$$

Notice that, the sum of lines (24) and (25) results in the first line of next equation, and line (26) is equal to the sum of lines (27) and (28).

$$\begin{aligned}\phi(P) \leq {} & \frac{(2+\delta)}{2}(R_s + R_r)\left(\sigma_P^f\rho_P^l + \rho_P^f\sigma_P^l\sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right)\omega(P)\\ & + (R_s + R_r)\left(\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_r(P) + \left(\rho_P^{\max} - \rho_P^{\min}\right)R_s(P)\right) && (27)\\ & - R_r\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_r(P) - R_s\left(\rho_P^{\max} - \rho_P^{\min}\right)R_s(P) && (28)\\ & + R_r\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_s(P) + R_s\left(\rho_P^{\max} - \rho_P^{\min}\right)R_r(P) - \left(R_r\sigma_P^{\min}\rho_P^m + R_s\rho_P^{\min}\sigma_P^m\right)\omega(P).\end{aligned}$$

Since $\rho_P^m\omega(P) = R_r^f(P) + R_r^l(P)$ and $\sigma_P^m\omega(P) = R_s^f(P) + R_s^l(P)$, then $\rho_P^m\omega(P) \geq R_r(P)$ and $\sigma_P^m\omega(P) \geq R_s(P)$, so:

$$\begin{aligned}
\phi(P) \leq\ & \frac{(2+\delta)}{2}(R_s+R_r)\left(\sigma_P^f\rho_P^l + \rho_P^f\sigma_P^l + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right)\omega(P) \\
& + (R_s+R_r)\left(\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_r(P) + \left(\rho_P^{\max} - \rho_P^{\min}\right)R_s(P)\right) \\
& - R_r\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_r(P) - R_s\left(\rho_P^{\max} - \rho_P^{\min}\right)R_s(P) \\
& + R_r\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_s(P) + R_s\left(\rho_P^{\max} - \rho_P^{\min}\right)R_r(P) - \left(R_r\sigma_P^{\min}\rho_P^m + R_s\rho_P^{\min}\sigma_P^m\right)\omega(P) \\
\phi(P) \leq\ & \frac{(2+\delta)}{2}(R_s+R_r)\left(\sigma_P^f\rho_P^l + \rho_P^f\sigma_P^l + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right)\omega(P) \\
& + (R_s+R_r)\left(\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_r(P) + \left(\rho_P^{\max} - \rho_P^{\min}\right)R_s(P)\right) \\
& - R_r\sigma_P^{\max}R_r(P) - R_s\rho_P^{\max}R_s(P) \qquad (29) \\
& + R_r\sigma_P^{\min}R_r(P) + R_s\rho_P^{\min}R_s(P) \qquad (30) \\
& + R_r\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_s(P) + R_s\left(\rho_P^{\max} - \rho_P^{\min}\right)R_r(P) \\
& - R_r\sigma_P^{\min}R_r(P) - R_s\rho_P^{\min}R_s(P). \qquad (31)
\end{aligned}$$

Notice that the sum of line (30) with line (31) results zero, then:

$$\begin{aligned}
\phi(P) \leq\ & \frac{(2+\delta)}{2}(R_s+R_r)\left(\sigma_P^f\rho_P^l + \rho_P^f\sigma_P^l + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right)\omega(P) \\
& + (R_s+R_r)\left(\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_r(P) + \left(\rho_P^{\max} - \rho_P^{\min}\right)R_s(P)\right) - R_r\sigma_P^{\max}R_r(P) - R_s\rho_P^{\max}R_s(P) \\
& + R_r\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_s(P) + R_s\left(\rho_P^{\max} - \rho_P^{\min}\right)R_r(P) \\
\phi(P) \leq\ & \frac{(2+\delta)}{2}(R_s+R_r)\left(\sigma_P^f\rho_P^l + \rho_P^f\sigma_P^l + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right)\omega(P) \\
& + (R_s+R_s)\left(\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_r(P) + \left(\rho_P^{\max} - \rho_P^{\min}\right)R_s(P)\right) \\
& + \left(R_r\sigma_P^{\max} - R_r\sigma_P^{\min} - R_s\rho_P^{\max}\right)R_s(P) + \left(R_s\rho_P^{\max} - R_s\rho_P^{\min} - R_r\sigma_P^{\max}\right)R_r(P).
\end{aligned}$$

Now we have two symmetric possibilities $R_r\sigma_P^{\max} \geq R_s\rho_P^{\max}$ or $R_r\sigma_P^{\max} \leq R_s\rho_P^{\max}$. The analysis of both cases is similar, being just necessary to replace R_s, σ and $R_s(P)$, respectively by R_r, ρ and $R_r(P)$ (and vice-verse), to transform the result of one case to the other. Then without loss of generality suppose $R_r\sigma_P^{\max} \geq R_s\rho_P^{\max}$:

$$\begin{aligned}
& \left(R_r\sigma_P^{\max} - R_r\sigma_P^{\min} - R_s\rho_P^{\max}\right)R_s(P) + \left(R_r\rho_P^{\max} - R_s\rho_P^{\min} - R_r\sigma_P^{\max}\right)R_r(P) \\
& \quad \leq \left(R_r\sigma_P^{\max} - R_r\sigma_P^{\min} - R_s\rho_P^{\max}\right)R_s(P).
\end{aligned}$$

Since $\rho_P^{\max} = R_r - \rho_P^{\min} - \rho_P^m$, $\sigma_P^{\max} \leq R_s$ and $\sigma_P^m\omega(P) \geq R_s(P)$:

$$\begin{aligned}
& \left(R_r\sigma_P^{\max} - R_r\sigma_P^{\min} - R_s\rho_P^{\max}\right)R_s(P) \\
& \quad = \left(R_r\sigma_P^{\max} - R_r\sigma_P^{\min} - R_s\left(R_r - \rho_P^{\min} - \rho_P^m\right)\right)R_s(P) \\
& \quad = \left(R_r\left(\sigma_P^{\max} - R_s\right) - R_r\sigma_P^{\min} + R_s\rho_P^{\min} + R_s\rho_P^m\right)R_s(P) \\
& \quad \leq \left(R_s\rho_P^{\min} + R_s\rho_P^m\right)R_s(P) \leq 2R_s\max\{\rho_P^{\min}, \rho_P^m\}R_s(P) \\
& \quad \leq 2R_s\max\{\rho_P^{\min}, \rho_P^m\}\sigma_P^m\omega(P).
\end{aligned}$$

Now we analyze two cases:

- If $\rho_P^m > \rho_P^{\min}$, then since $\sigma_P^m \leq \frac{\delta}{6}R_s$ and $\rho_P^m \leq \frac{\delta}{6}R_r$:

$$\begin{aligned}
& \left(R_r\sigma_P^{\max} - R_r\sigma_P^{\min} - R_s\rho_P^{\max}\right)R_s(P) + \left(R_s\rho_P^{\max} - R_s\rho_P^{\min} - R_r\sigma_P^{\max}\right)R_r(P) \\
& \quad \leq \frac{\delta^2}{18}R_sR_sR_r\omega(P).
\end{aligned}$$

Without loss of generality suppose $\sigma_P^f = \sigma_P^{\max}$, then there are two possibilities:

– If $\rho_P^f = \rho_P^{\max}$. Since $\rho_P^m > \rho_P^{\min} = \rho_P^l$ and $\sigma_P^l = \sigma_P^{\min}$, if $\sigma_P^{\min} \leq \frac{\delta}{2}R_s$, then $\rho_P^m + \rho_P^l \leq 2\rho_P^m \leq \frac{\delta}{3}R_r < \delta R_r$ and $\sigma_P^m + \sigma_P^l \leq \frac{2\delta}{3}R_s < \delta R_s$, so the induced sub-tree of S on $VB(S, E_P, f_P)$ (i.e. $S - \bigcup_{v \in V_P - f_P} VB(S, E_P, v)$) is a δ-$\sigma\rho$-separator, implying that S is not minimal, which is a contradiction. Then, $\sigma_P^{\min} > \frac{\delta}{2}R_s$:

$$\begin{aligned}&\left(R_r\sigma_P^{\max} - R_r\sigma_P^{\min} - R_s\rho_P^{\max}\right)R_s(P) + \left(R_s\rho_P^{\max} - R_s\rho_P^{\min} - R_r\sigma_P^{\max}\right)R_r(P)\\ &\le \frac{\delta^2}{18}R_sR_sR_r\omega(P) \le \frac{\delta^2}{6}R_sR_sR_r\omega(P) = \frac{\delta}{3}\left(\frac{\delta}{2}R_s\right)R_sR_r\omega(P)\\ &\le \frac{\delta}{3}\sigma_P^{\min}R_sR_r\omega(P) = \frac{\delta}{3}R_s\sigma_P^{\min}\left(\rho_P^{\max} + \rho_P^{\min} + \rho_P^m\right)\omega(P)\\ &\le \frac{\delta}{3}R_s\left(\sigma_P^{\min}\rho_P^{\max} + \sigma_P^{\max}\rho_P^{\min} + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right)\omega(P)\\ &\le \frac{\delta}{3}R_s\left(\sigma_P^f\rho_P^l + \sigma_P^l\rho_P^f + \sigma_P^{\min}\rho_P^m + \rho_P^{\min}\sigma_P^m\right)\omega(P).\end{aligned}$$

– If $\rho_P^l = \rho_P^{\max}$, then, since $2\rho_P^{\max} \ge R_r - \rho_P^m \ge R_r - \frac{\delta}{6}R_r \ge \frac{11}{12}R_r$ and $2\sigma_P^{\max} \ge R_s - \sigma_P^m \ge R_s - \frac{\delta}{6}R_s \ge \frac{11}{12}R_s$:

$$\begin{aligned}&\left(R_r\sigma_P^{\max} - R_r\sigma_P^{\min} - R_s\rho_P^{\max}\right)R_s(P) + \left(R_s\rho_P^{\max} - R_s\rho_P^{\min} - R_r\sigma_P^{\max}\right)R_r(P)\\ &\le \frac{\delta^2}{18}R_sR_sR_r\omega(P) = \frac{\delta^2}{3}\frac{1}{6}R_sR_sR_r\omega(P)\\ &\le \frac{\delta^2}{3}\frac{121}{576}R_sR_sR_r\omega(P)\\ &= \frac{\delta^2}{3}\left(\frac{11}{24}R_s\right)\left(\frac{11}{24}R_r\right)R_s\omega(P) \le \frac{\delta^2}{3}\rho_P^{\max}\sigma_P^{\max}R_s\omega(P)\\ &\le \frac{\delta}{3}R_s\left(\rho_P^{\max}\sigma_P^{\max} + \rho_P^{\min}\sigma_P^{\min} + \rho_P^{\min}\sigma_P^m + \sigma_P^{\min}\rho_P^m\right)\omega(P)\\ &= \frac{\delta}{3}R_s\left(\rho_P^l\sigma_P^f + \rho_P^f\sigma_P^l + \rho_P^{\min}\sigma_P^m + \sigma_P^{\min}\rho_P^m\right)\omega(P).\end{aligned}$$

- If $\rho_P^m \le \rho_P^{\min}$:

$$\begin{aligned}&\left(R_r\sigma_P^{\max} - R_r\sigma_P^{\min} - R_s\rho_P^{\max}\right)R_s(P) + \left(R_s\rho_P^{\max} - R_s\rho_P^{\min} - R_r\sigma_P^{\max}\right)R_r(P)\\ &\le 2R_s\rho_P^{\min}\sigma_P^m\omega(P).\end{aligned}$$

  Since $\sigma_P^m \le \frac{\delta}{6}R_s$:

$$\begin{aligned}&\left(R_r\sigma_P^{\max} - R_r\sigma_P^{\min} - R_s\rho_P^{\max}\right)R_s(P) + \left(R_s\rho_P^{\max} - R_s\rho_P^{\min} - R_r\sigma_P^{\max}\right)R_r(P)\\ &\le \frac{\delta}{3}R_s\rho_P^{\min}R_s\omega(P) = \frac{\delta}{3}R_s\rho_P^{\min}\left(\sigma_P^{\max} + \sigma_P^{\min} + \sigma_P^m\right)\omega(P)\\ &\le \frac{\delta}{3}R_s\left(\rho_P^{\min}\sigma_P^{\max} + \rho_P^{\max}\sigma_P^{\min} + \rho_P^{\min}\sigma_P^m + \sigma_P^{\min}\rho_P^m\right)\omega(P)\\ &\le \frac{\delta}{3}R_s\left(\rho_P^f\sigma_P^l + \rho_P^l\sigma_P^f + \rho_P^{\min}\sigma_P^m + \sigma_P^{\min}\rho_P^m\right)\omega(P).\end{aligned}$$

So, in any case:

$$\begin{aligned}&\left(R_r\sigma_P^{\max} - R_r\sigma_P^{\min} - R_s\rho_P^{\max}\right)R_s(P) + \left(R_s\rho_P^{\max} - R_s\rho_P^{\min} - R_r\sigma_P^{\max}\right)R_r(P)\\ &\le \frac{\delta}{3}(R_s + R_r)\left(\rho_P^f\sigma_P^l + \rho_P^l\sigma_P^f + \rho_P^{\min}\sigma_P^m + \sigma_P^{\min}\rho_P^m\right)\omega(P).\end{aligned}$$

Concluding:

$$\begin{aligned}\phi(P) \le\ &\frac{6+5\delta}{6}(R_s + R_r)\left(\sigma_P^f\rho_P^l + \sigma_P^l\rho_P^f + \sigma_P^{\min}\rho_P^m + \sigma_P^m\rho_P^{\min}\right)\omega(P)\\ &+ (R_s + R_r)\left(\left(\sigma_P^{\max} - \sigma_P^{\min}\right)R_r(P) + \left(\rho_P^{\max} - \rho_P^{\min}\right)R_s(P)\right). \quad \square\end{aligned}$$

4.2.4. Proof of Lemma 4.1

Now we demonstrate Lemma 4.1, which states that if we are given a δ-$\sigma\rho$-spine Y of a spanning tree T, then we can construct a star whose core lies on $ext(Y)$, such that its communication cost is bounded by $\frac{1}{1-\delta}C(T)$.

Proof. Let $S = \bigcup_{P\in Y} P$ be the minimal δ-$\sigma\rho$-separator associated with Y and X the $|ext(Y)|$-star of G given by Proposition 4.1, then:

$$\begin{aligned}C(X) &\le \sum_{P\in Y}\left(\sigma_P^f\rho_P^l+\rho_P^f\sigma_P^l\right)\omega(P)+\sum_{P\in Y}\min\left\{\Delta_{fl}(P),\Delta_{lf}(P)\right\}\\ &\quad +R_r\sum_{u\in V_G} r_s(u)d(T,u,S)+R_s\sum_{u\in V_G} r_r(u)d(T,u,S)\\ C(X) &\le \sum_{P\in Y}\frac{R_s+R_r-\rho_P^m-\sigma_P^m}{R_s+R_r-\rho_P^m-\sigma_P^m}\left(\sigma_P^f\rho_P^l+\rho_P^f\sigma_P^l\right)\omega(P)+\min\left\{\Delta_{fl}(P),\Delta_{lf}(P)\right\}\\ &\quad +R_r\sum_{u\in V_G} r_s(u)d(T,u,S)+R_s\sum_{u\in V_G} r_r(u)d(T,u,S).\end{aligned}$$

Since the minimum between two numbers is less than or equal to their weighted median, we have:

$$\begin{aligned}C(X) &\le \sum_{P\in Y}\frac{R_s+R_r-\rho_P^m-\sigma_P^m}{R_s+R_r-\rho_P^m-\sigma_P^m}\left(\sigma_P^f\rho_P^l+\rho_P^f\sigma_P^l\right)\omega(P)\\ &\quad +\sum_{P\in Y}\frac{\sigma_P^l+\rho_P^l}{\sigma_P^l+\rho_P^l+\sigma_P^f+\rho_P^f}\Delta_{fl}(P)+\frac{\sigma_P^f+\rho_P^f}{\sigma_P^l+\rho_P^l+\sigma_P^f+\rho_P^f}\Delta_{lf}(P)\\ &\quad +R_r\sum_{u\in V_G} r_s(u)d(T,u,S)+R_s\sum_{u\in V_G} r_r(u)d(T,u,S).\end{aligned}$$

Since every path P satisfies $R_s+R_r-\sigma_P^m-\sigma_P^m=\sigma_P^f+\sigma_P^l+\rho_P^f+\rho_P^l$, and every $P\in Y$ is a δ-$\sigma\rho$-path, then by applying Proposition 4.3 we conclude:

$$\begin{aligned}C(X) &\le \sum_{P\in Y}\frac{\left(\frac{6+5\delta}{6}\right)(R_s+R_r)\omega(P)}{R_s+R_r-\rho_P^m-\sigma_P^m}\left(\sigma_P^f\rho_P^l+\rho_P^f\sigma_P^l+\sigma_P^{\min}\rho_P^m+\rho_P^{\min}\sigma_P^m\right)\\ &\quad +\sum_{P\in Y}\frac{R_s+R_r}{R_s+R_r-\rho_P^m-\sigma_P^m}\left(\sigma_P^{\max}-\sigma_P^{\min}\right)R_r(P)+\sum_{P\in Y}\frac{R_s+R_r}{R_s+R_r-\rho_P^m-\sigma_P^m}\left(\rho_P^{\max}-\rho_P^{\min}\right)R_s(P)\\ &\quad +R_r\sum_{u\in V_G} r_s(u)d(T,u,S)+R_s\sum_{u\in V_G} r_r(u)d(T,u,S).\end{aligned}$$

Notice that any δ-$\sigma\rho$-path satisfies: $\rho_P^m+\sigma_P^m\le\frac{\delta}{6}R_s+\frac{\delta}{6}R_r=\frac{\delta}{6}(R_s+R_r)$, then:

$$\begin{aligned}C(X) &\le \sum_{P\in Y}\frac{\left(\frac{6+5\delta}{6}\right)(R_s+R_r)\omega(P)}{R_s+R_r-\frac{\delta}{6}(R_s+R_r)}\left(\sigma_P^f\rho_P^l+\rho_P^f\sigma_P^l+\sigma_P^{\min}\rho_P^m+\rho_P^{\min}\sigma_P^m\right)\\ &\quad +\sum_{P\in Y}\frac{R_s+R_r}{R_s+R_r-\frac{\delta}{6}(R_s+R_r)}\left(\sigma_P^{\max}-\sigma_P^{\min}\right)R_r(P)+\sum_{P\in Y}\frac{R_s+R_r}{R_s+R_r-\frac{\delta}{6}(R_s+R_r)}\left(\rho_P^{\max}-\rho_P^{\min}\right)R_s(P)\\ &\quad +R_r\sum_{u\in V_G} r_s(u)d(T,u,S)+R_s\sum_{u\in V_G} r_r(u)d(T,u,S)\\ &= \frac{6+5\delta}{6-\delta}\sum_{P\in Y}\left(\sigma_P^f\rho_P^l+\rho_P^f\sigma_P^l+\sigma_P^{\min}\rho_P^m+\rho_P^{\min}\sigma_P^m\right)\omega(P)\\ &\quad +\frac{6}{6-\delta}\sum_{P\in Y}\left(\left(\sigma_P^{\max}-\sigma_P^{\min}\right)R_r(P)+\left(\rho_P^{\max}-\rho_P^{\min}\right)R_s(P)\right)\\ &\quad +\frac{(1-\delta)}{1-\delta}\left(R_r\sum_{u\in V_G} r_s(u)d(T,u,S)+R_s\sum_{u\in V_G} r_r(u)d(T,u,S)\right).\end{aligned}$$

Since $\frac{6+5\delta}{6-\delta}=\frac{(6+5\delta)(1-\delta)}{(6-\delta)(1-\delta)}=\frac{6-\delta-5\delta^2}{(6-\delta)(1-\delta)}<\frac{6-\delta}{(6-\delta)(1-\delta)}=\frac{1}{1-\delta}$ and $\frac{6}{6-\delta}<\frac{6+5\delta}{6-\delta}<\frac{1}{1-\delta}$, by applying Proposition 4.2 we obtain: $C(X)\le\frac{1}{1-\delta}C(T)$. □

4.3. *Existence of bounded δ-$\sigma\rho$-spine*

In the next lemma we show that there exists a δ-$\sigma\rho$-spine Y of T such that $|ext(Y)|$ is bounded by a function of δ.

Lemma 4.2. *Given $0 < \delta \leq \frac{1}{2}$ and a spanning tree T of G, there exists a δ-$\sigma\rho$-spine Y of T satisfying $|ext(Y)| \leq 3\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right)$.*

Proof. Consider a minimal δ-ρ-separator S_ρ of T and a minimal δ-σ-separator S_σ of T. If S_ρ and S_σ have at least one node in common, then define $S' = S_\rho \cup S_\sigma$ and obviously S' is a δ-$\sigma\rho$-separator. If S_ρ and S_σ have no nodes in common, then since both are trees, S_σ must be included in a component of $T - S_\rho$. Since S_ρ is a δ-ρ-separator of T, every component of $T - S_\rho$ has weight bounded by δR_r, so the path P in T connecting S_ρ to S_σ satisfies $\rho_P^m < \delta R_r$. Analogously, P also satisfies $\sigma_P^m < \delta R_s$. Then, P can be divided into 6 paths each one with sending weight bounded by $\frac{\delta R_s}{6}$ and another 6 paths each one with receiving weight bounded by $\frac{\delta R_r}{6}$. Since each division uses 5 internal nodes, in the worst case, using 10 internal nodes we obtain a division of P in δ-$\sigma\rho$-paths, and $S' = S_\rho \cup S_\sigma \cup P$ is a δ-$\sigma\rho$-separator.

Next section shows a modification of a proof of [12,10,11], such that it guarantees the existence of Y'_ρ and Y'_σ sets of internally-disjoint δ-$\sigma\rho$-paths, satisfying $\cup_{P\in Y'_\rho} P = S_\rho$, $\cup_{P\in Y'_\sigma} P = S_\sigma$ and $\left|ext(Y'_\rho)\right|, \left|ext(Y'_\sigma)\right| \leq \left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1$.

For each path $P \in Y'_\rho$ if P contains internal nodes in $ext(Y'_\sigma)$, divide P on those nodes to create new internally-disjoint δ-$\sigma\rho$-paths and put those new paths in Y_ρ, otherwise add P to Y_ρ. Analogously, define Y_σ from Y'_σ and $ext(Y'_\rho)$. Observe that no path of $Y_\sigma \cup Y_\rho$ has an internal node in $ext(Y_\sigma) \cup ext(Y_\rho)$, also Y_σ and Y_ρ are sets of internally-disjoint δ-$\sigma\rho$-paths such that $\cup_{P\in Y_\rho} P = S_\rho$, $\cup_{P\in Y_\sigma} P = S_\sigma$ and:

$$\left|ext(Y_\rho \cup Y_\sigma)\right| \leq 2\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right).$$

Notice that, since $S_\sigma \cup S_\rho$ is acyclic, each path of Y_ρ internally-intersects at most one path in Y_σ and vice-versa.

If there are two paths $P_\sigma \in Y_\sigma$ and $P_\rho \in Y_\rho$ whose internal-intersection is not empty and their end-points do not belong to their intersection, then no other path of Y_σ intersects any path of Y_ρ, and by removing from P_σ the internal nodes of the intersection we add at most two new extremal points (the end-points of the intersection). Then $Y' = (Y_\sigma - P_\sigma) \cup Y_\rho \cup \left(P_\sigma - \left(P_\sigma \cap P_\rho\right)\right)$ is a set of internally-disjoint δ-$\sigma\rho$-paths which satisfies $\cup_{P\in Y'} P = S'$ and:

$$\left|ext(Y')\right| \leq 2\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right) + 2.$$

Otherwise, if no path of Y_σ intersects any path of Y_ρ then, as seen before, there exists a path P connecting S_σ to S_ρ that can be divided in at most 11 δ-$\sigma\rho$-paths, and the union of those paths with Y_σ and Y_ρ results in a set Y' of internally-disjoint δ-$\sigma\rho$-paths such that $\cup_{P\in Y'} P = S'$ and:

$$\left|ext(Y')\right| \leq 2\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right) + 10.$$

The last possibility is that at least one path of Y_σ internally-intersects a path of Y_ρ and each not-empty intersection between a path of Y_σ and a path of Y_ρ contains at least one endpoint. Then, remove from each path in Y_σ the internal nodes of the intersection with each path in Y_ρ (notice that a path of Y_σ at most internally-intersects one path in Y_ρ). In this case the number of new extremal points will be at most $|Y'_\sigma|$ and the set Y' defined by the union of Y_ρ with the modified Y_σ is a set of internally-disjoint δ-$\sigma\rho$-paths that satisfies $\cup_{P\in Y'} P = S'$ and:

$$\left|ext(Y')\right| \leq 3\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right).$$

Since, for $0 < \delta \leq \frac{1}{2}$: $\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right) \geq 12^2 - 11(12) + 1 = 13 > 10$, then we always can obtain a set Y' of internally-disjoint δ-$\sigma\rho$-paths which satisfies $\cup_{P\in Y'} P = S'$ and:

$$\left|ext(Y')\right| \leq 3\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right).$$

If S' is a minimal δ-$\sigma\rho$-separator, then $Y = Y'$ is a δ-$\sigma\rho$-spine. Otherwise, exists a minimal δ-$\sigma\rho$-separator $S \subset S'$ and by deleting from each path in Y' the elements that are not contained in S we obtain a δ-$\sigma\rho$-spine Y of T satisfying:

$$|ext(Y)| \leq 3\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right). \quad \square$$

4.3.1. On the existence of bounded δ-σ-spine and δ-ρ-spine

First we prove a result obtained in [12,10,11]:

Fact 4.2. *Given $0 < \delta \leq \frac{1}{2}$ and a minimal δ-ρ-separator (δ-σ-separator) S of a spanning tree T of G, if u is a leaf of S, then $r_r\left(VB(T, E_S, u)\right) > \delta R_r$ $\left(r_s\left(VB(T, E_S, u)\right) > \delta R_s\right)$.*

Proof. If $n(S) = 1$, then u is the only element in S and trivially we have: $r_r\left(VB(T, E_S, u)\right) = R_r > \frac{1}{2}R_r \geq \delta R_r$ $\left(r_s\left(VB(T, E_S, u)\right) = R_s > \frac{1}{2}R_s \geq \delta R_s\right)$. Otherwise, $n(S) > 1$, suppose that:

$$r_r\left(VB(T, E_S, u)\right) \leq \delta R_r \qquad \left(r_s\left(VB(T, E_S, u)\right) \leq \delta R_s\right),$$

evidently $S - u$ is still a δ-ρ-separator (δ-σ-separator) that is because $B = VB(T, E_S, u)$ is the only component of $T - (S - u)$ that is not in $T - S$ and B satisfies the required conditions of δ-ρ-separator (δ-σ-separator). □

Now we prove a result needed in Lemma 4.2:

Proposition 4.4. *Given $0 < \delta \leq \frac{1}{2}$ and a minimal δ-ρ-separator S (δ-σ-separator) of a spanning tree T of G, there exists a set of internally-disjoint δ-$\sigma\rho$-paths Y of T satisfying $|ext(Y)| \leq \left\lceil \frac{6}{\delta} \right\rceil^2 - 11\left\lceil \frac{6}{\delta} \right\rceil + 1$ and $\cup_{P \in Y} P = S$.*

Proof. Consider the sets of nodes U_1, U_2 and $U_{>2}$, where U_1 contains the leaves of S, U_2 the nodes with degree two in S and $U_{>2}$ the nodes with degree greater than two in S. Since S is a tree $|U_1| \geq |U_{>2}| + 2$, then the set $U = U_1 \cup U_{>2}$ satisfies $|U| \leq 2\,|U_1| + 2$.

We say that $u, v \in U$ are neighbors in U if for any $w \in U - \{u, v\}$, w do not belong to the path between u and v in T. Then we define Y_1 as:

$$Y_1 = \{P | P \text{ is a path in } T \text{ between two neighbors of } U\}.$$

We classify a path $P \in Y_1$ as *ρ-heavy* (*σ-heavy*) if $\rho_P^m > \delta \frac{R_r}{6}$ $\left(\sigma_P^m > \delta \frac{R_s}{6}\right)$, then we can divide a ρ-heavy (σ-heavy) path P into non-ρ-heavy (non-σ-heavy) sub-paths using $\left\lceil \frac{6\rho_P^m}{\delta R_r} \right\rceil - 1$ $\left(\left\lceil \frac{6\sigma_P^m}{\delta R_s} \right\rceil - 1\right)$ nodes for the division (that is, internal nodes of P which will be endpoints of the sub-paths generated by the division).

Since S is a minimal δ-ρ-separator (δ-σ-separator) of T each leaf u of S satisfies $r_r\left(VB(T, E_S, u)\right) > \delta R_r$ $\left(r_s\left(VB(T, E_S, u)\right) > \delta R_s\right)$ (proved by Fact 4.2). Then:

$$\sum_{P \in Y_1} \rho_P^m < R_r\left(1 - \delta\,|U_1|\right) \qquad \left(\sum_{P \in Y_1} \sigma_P^m < R_s\left(1 - \delta\,|U_1|\right)\right).$$

Denote by U_3 the set of all nodes used to divide all ρ-heavy (σ-heavy) paths of Y_1 into non-ρ-heavy (non-σ-heavy) paths, then:

$$|U_3| \leq \sum_{P \in Y_1}\left(\left\lceil \frac{6\rho_P^m}{\delta R_r} \right\rceil - 1\right) < \frac{6}{\delta R_r} R_r\left(1 - \delta\,|U_1|\right) \leq \left\lceil \frac{6}{\delta} \right\rceil - 6\,|U_1|.$$

Denote by Y_2 the set of paths obtained by dividing all ρ-heavy (σ-heavy) paths of Y_1 into non-ρ-heavy (non-σ-heavy) paths, then:

$$|ext(Y_2)| = U + U_3 \leq 2\,|U_1| - 2 + \left\lceil \frac{6}{\delta} \right\rceil - 6\,|U_1| \leq \left\lceil \frac{6}{\delta} \right\rceil - 10.$$

Now we can divide every σ-heavy (ρ-heavy) path in Y_2 into non-σ-heavy (non-ρ-heavy) paths, in order to obtain a set Y of δ-$\sigma\rho$-paths. For that, let U_4 be the set of all nodes used in the division of the σ-heavy (ρ-heavy) paths in Y_2:

$$|U_4| \leq \sum_{P \in Y_2}\left\lceil \frac{6\sigma_P^m}{\delta R_s} \right\rceil - 1 \leq \sum_{P \in Y_2}\left\lceil \frac{6}{\delta} \right\rceil - 1 \leq |Y_2|\left(\left\lceil \frac{6}{\delta} \right\rceil - 1\right)$$

where $|Y_2| = |ext(Y_2)| - 1$, then:

$$|ext(Y)| = |ext(Y_2)| + |U_4| \leq \left\lceil \frac{6}{\delta} \right\rceil^2 - 11\left\lceil \frac{6}{\delta} \right\rceil + 1.$$

Finally, observe that by construction the paths in Y are internally-disjoint. □

4.4. PTAS for m-SROCT

Using lemmata 4.1 and 4.2 we can state the following proposition:

Proposition 4.5. *Given* $0 < \delta \leq \frac{1}{2}$ *and a spanning tree* T *of* G*, there exists a* $\left(3\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right)\right)$*-star* X *of* G*, such that* $C(X) \leq \frac{1}{1-\delta}C(T)$.

Let T^* be an optimal spanning tree for m-WSDOCT over G, by Proposition 4.5 for any $0 < \delta \leq \frac{1}{2}$, there exists a $\left(3\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right)\right)$-star X of G such that $C(X) \leq \frac{1}{1-\delta}C(T^*)$. Since an optimal $\left(3\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right)\right)$-star X^* of G guarantees $C(X^*) \leq C(X)$, then $C(X^*) \leq \frac{1}{1-\delta}C(T^*)$.

Lemma 4.3. *Given* $0 < \delta \leq \frac{1}{2}$ *an optimal* $\left(3\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right)\right)$*- star of* G *is a* $\frac{1}{1-\delta}$*-approximation for* m-WSDOCT.

The results of lemmata 3.1 and 4.3 complete the necessary tools for providing the PTAS:

Theorem 4.1. *There exists a* PTAS *for* m-SROCT*, such that a* $(1 + \frac{\delta}{1-\delta})$*-approximation can be found in* $O\left(n^{6\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right)+1} \log^2(n)\right)$ *time complexity where* $0 < \delta \leq \frac{1}{2}$.

5. Conclusions

In this work we present a PTAS for m-SROCT an NP-hard particular case of OCT. The best previously known result for m-SROCT was a 2-approximation algorithm due to [10]. Many questions remain open regarding OCT and related problems. One could improve the approximation ratio for m-WSDOCT, SROCT or other particular cases of OCT. In future works we will attempt to answer this question for some of these problems.

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