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NP-completeness of the game *Kingdomino*™

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ABSTRACT

Kingdomino™ is a board game designed by Bruno Cathala and edited by *Blue Orange* since 2016. The goal is to place 2×1 dominoes on a grid layout, and get a better score than other players. Each 1×1 domino cell has a color that must match at least one adjacent cell, and an integer number of crowns (possibly none) used to compute the score. We prove that even with full knowledge of the future of the game, it is NP-complete to decide whether a given score is achievable at *Kingdomino*™.

1. Introduction

Kingdomino™ is a 2-4 players game where players, turn by turn, place 2×1 dominoes on a grid layout (each player has its own board, independent of others). Each domino has a color on each of its two 1×1 cells, and when a player is given a domino to place on its board, he or she must do so with a color match along at least one of its edges. Also, if a domino *can* be placed (with at least one possible color match) then it *must* be placed, and if it cannot be placed then it is discarded. Finally, each player starts from a 1×1 tower matching any color. The winner of the game is the player that has the maximum score among the competitors. The computation of score will be precised in Section 3, it is basically a weighted sum of the number of cells in each monochromatic connected components on the player's board.

The purpose of this article is to prove that, even with full knowledge of the future of the game (*i.e.* the sequence of dominoes he or she will have to place), a player willing to know whether a given score is achievable is faced with an NP-complete decision problem.

Section 2 reviews some results around games complexity and domino problems, Section 3 presents our theoretical modeling of *Kingdomino*™, Section 4 illustrates the combinatorial explosion of possibilities, and Section 5 proves the NP-hardness result.

2. Computational complexity of games with dominoes

Kingdomino™ has been studied in [1], where the authors compare different strategies to play the game, via numerical simulations.

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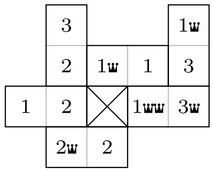


Fig. 1. Example of K-tiling by ($\boxed{2♛ \mid 2}$, $\boxed{1 \mid 2}$, $\boxed{1♛ \mid 1}$, $\boxed{1♛♛ \mid 3♛}$, $\boxed{3 \mid 1♛}$, $\boxed{2 \mid 3}$), with score $0+1+4+9+2+0=16$ (regions are ordered from top to bottom, left to right).

Understanding the computational complexity of games has raised some interest in the computer science community, and numerous games have been proven to be complete for NP or co-NP. Examples include *Minesweeper* [2,3], *SET* [4], *Hanabi* [5], some *Nintendo* games [6] and *Candy Crush* [7].

Domino tiling problems are a cornerstone for computer science, from undecidable ones [8–10] to simple puzzles [11–13]. Tiling some board with dominoes under constraints has already been seen to be NP-complete, and constructions vary according to the model definition [14–17]. The model of [18] is close to *Kingdomino*^{TM}, its construction can be adapted to prove that starting from a board with some dominoes already placed, deciding whether it can be completed to achieve at least a given score is NP-complete. The challenging part of the present work is to start from nothing else but the tower of *Kingdomino*^{TM}.

3. Model and problem statement

In order to apply the theory of computational complexity to *Kingdomino*^{TM}, we will consider a one player model which concentrates on an essential aspect of the game: how to maximize one's score, which is the source of domino choices. Also, in the game *Kingdomino*^{TM} there is a fixed set of colors (6) and a fixed multiset of dominoes (48, some dominoes have more than one occurrence), but we will abstract these quantities to be any finite (multi)set.

A *domino* is a 2×1 rectangle, with one *color* among a finite set on each of its two 1×1 *cells*. A domino also has a number of *crowns* on each of its cells, used to compute the score. For convenience we consider colors to be integers, and represent a domino as follows: $\boxed{1♛ \mid 2}$ for the domino with one cell of color 1 with one crown, and one cell of color 2 with no crown. A *tiling* is an overlap free placement of dominoes on the $\mathbb{Z}^2$ board, with a special cell at position $(0,0)$ called the *tower*. Given a sequence of n dominoes $\tau = (\boxed{c_1 \mid c_2}, \ldots, \boxed{c_{2n-1} \mid c_{2n}})$ possibly with crowns, a *K-tiling by* τ is a tiling respecting the following constraints defined inductively.

- The tiling with only the tower at position $(0,0)$ is a K-tiling by $\varnothing$ (case $n=0$).
- Given a K-tiling by the $n-1$ first dominoes of τ, the last domino of colors c_{2n-1} and c_{2n} can be placed on a pair of adjacent positions, if and only if at least one of its two cells is adjacent to a cell of the same color that has already been placed, or is adjacent to the tower. It is lost (not placed) if and only if it can be placed nowhere. This gives a K-tiling by the n dominoes of τ.

Hence dominoes must be placed in the order given by the sequence τ. Note that the definition of K-tiling does not take into consideration the crowns. They are only used to compute the score, as we will explain now.

The *score* of a K-tiling is the sum, for each monochromatic connected component (called *region*), of its number of cells times the number of crowns it contains. Note that a color may give rise to more than one region, and that a *region* scores no point if it contains no crown. We will say that some cell (resp. domino) must be *connected* to some region, to mean that it (resp. one of its two cells) must belong to this monochromatic connected component.

Definitions are illustrated on Fig. 1. We are ready to state the problem.

K-tiling problem
input: a sequence of dominoes τ and an integer s.
question: is there a K-tiling by τ with score at least s?

Given a tiling where each domino of the sequence τ is identified (a potential solution, *aka* certificate), one verifies domino after domino that it is indeed a K-tiling by τ, and computes the score to verify that it is indeed at least s, in polynomial time. Hence **K-tiling problem** belongs to NP.

Remark that our modeling of *Kingdomino*^{TM} discards the official game rule regarding a bounding box for player's boards, where dominoes must be placed inside a square of size 5×5 or 7×7 containing the tower.

Table 1
Number of K-tilings reaching the maximum possible score, for some small sequences of dominoes. Rotations and axial symmetries are counted only once, and the positions of crowns are not taken into account (in parenthesis the full counts are given).

Dominoes	$\left(\boxed{\begin{smallmatrix}1\\1\text{♛}\end{smallmatrix}}, \boxed{\begin{smallmatrix}1\\1\end{smallmatrix}}, \boxed{\begin{smallmatrix}1\\1\end{smallmatrix}}, \boxed{\begin{smallmatrix}1\\1\end{smallmatrix}}\right)$	$\left(\boxed{\begin{smallmatrix}1\\1\text{♛}\end{smallmatrix}}, \boxed{\begin{smallmatrix}2\\2\text{♛}\end{smallmatrix}}, \boxed{\begin{smallmatrix}3\\3\text{♛}\end{smallmatrix}}, \boxed{\begin{smallmatrix}4\\4\text{♛}\end{smallmatrix}}\right)$	$\left(\boxed{\begin{smallmatrix}2\text{♛}\\1\text{♛}\end{smallmatrix}}, \boxed{\begin{smallmatrix}4\text{♛}\\3\text{♛}\end{smallmatrix}}, \boxed{\begin{smallmatrix}6\text{♛}\\5\text{♛}\end{smallmatrix}}, \boxed{\begin{smallmatrix}8\text{♛}\\7\text{♛}\end{smallmatrix}}\right)$
1st (score 2)	2 (24)	2 (24)	4 (24)
2nd (score 4)	19 (752)	13 (400)	52 (400)
3rd (score 6)	253 (35448)	63 (4032)	504 (4032)
4th (score 8)	3529 (2176064)	141 (18048)	2256 (18048)

4. Counting K-tilings

To give an idea of the combinatorial explosion one faces when playing*Kingdomino*^{TM} or when deciding some **K-tiling problem** instance, we propose in Table 1 to count the number of possible K-tilings for some small sequences of dominoes. These exact results were obtained by brute force numerical simulations.

Table 1 may be compared to the number of domino tilings of a $2n \times 2n$ square, appearing in the *Online Encyclopedia of Integer Sequences* under reference `A004003` [19]: 1, 2, 36, 6728, 12988816, 258584046368, 53060477521960000, ...

5. NP-hardness of K-tiling problem

In this section we prove the main result of the article.

Theorem 1. ***K-tiling problem*** *is* NP*-hard.*

Proof. We make a polynomial time many-one reduction from **4-Partition problem**, which is known to be strongly NP-complete [20]. This is important, since we encode the instance of **4-Partition problem** in unary into an instance of **K-tiling problem** (basically, with $28x$ domino cells for each item of size x).

> **4-Partition problem**
> *input:* n items of integer sizes $x_1, \ldots, x_n$, and $m = \frac{n}{4}$ bins of size k,
> with n a multiple of four, $x_i > 0$ for all i, and $\sum_{i=1}^{n} x_i = km$.
> *question:* is it possible to pack[1] the n items into the m bins,
> with exactly four items (whose sizes thus sum to k) per bin?

Given such an instance of **4-Partition problem**, we first multiply by 28 all item and bin sizes (for technical reasons to be explained later) and consider the equivalent instance with n items of strictly positive integer sizes $x_1 \leftarrow 28x_1, \ldots, x_n \leftarrow 28x_n$ and m bins of size $k \leftarrow 28k$ (for convenience we keep the initial notations with x and k). We then construct (in polynomial time from a unary encoding) the following sequence of dominoes τ:

1. *guardians* $\boxed{1\text{♛} \mid 1}, \boxed{2 \mid 2}, \boxed{3 \mid 3}, \boxed{4 \mid 4},$
2. *square* $\left(\boxed{1 \mid 1}\right)^{18m^2-6},$
3. *contour* $\boxed{1 \mid 5\text{♛}}, \boxed{5 \mid 6\text{♛}}, \boxed{6 \mid 7\text{♛}}, \boxed{7 \mid 8\text{♛}}, \boxed{8 \mid 1}, \boxed{8 \mid 1}, \boxed{8 \mid 2}, \boxed{8 \mid 9\text{♛}}, \left(\boxed{9 \mid 9}\right)^{9m-8}, \boxed{9 \mid 10\text{♛}}, \boxed{10 \mid 11\text{♛}},$
$\boxed{11 \mid 12\text{♛}}, \boxed{12 \mid 13\text{♛}}, \boxed{13 \mid 14\text{♛}}, \ldots, \boxed{3m+10 \mid 3m+11\text{♛}}, \boxed{3m+11 \mid 3m+12\text{♛}}, \boxed{3m+12 \mid 5},$
4. *guide* $\left(\boxed{6 \mid 7}\right)^{18m^2+12m},$
5. *arms* $\left(\boxed{10 \mid 11}\right)^{\frac{k}{4}+2}, \left(\boxed{13 \mid 14}\right)^{\frac{k}{4}+2}, \left(\boxed{16 \mid 17}\right)^{\frac{k}{4}+2}, \ldots, \left(\boxed{3m+10 \mid 3m+11}\right)^{\frac{k}{4}+2}$
6. *anchors* $\left(\boxed{12 \mid 3m+13}\right)^2, \left(\boxed{15 \mid 3m+13}\right)^2, \ldots, \left(\boxed{3m+9 \mid 3m+13}\right)^2,$
7. *items* for each x_i we have $\boxed{3m+13 \mid 3m+13+i\text{♛}}, \left(\boxed{3m+13+i \mid 3m+13+i}\right)^{\frac{x_i}{2}-1},$
8. *zippers* $\boxed{11 \mid 3m+13+n+1\text{♛}}, \boxed{3m+13+n+1 \mid 13}, \boxed{14 \mid 3m+13+n+2\text{♛}}, \boxed{3m+13+n+2 \mid 16}, \ldots, \boxed{3m+8 \mid 4m+13+n\text{♛}},$
$\boxed{4m+13+n \mid 3m+10},$

[1] Of course an item cannot be split.

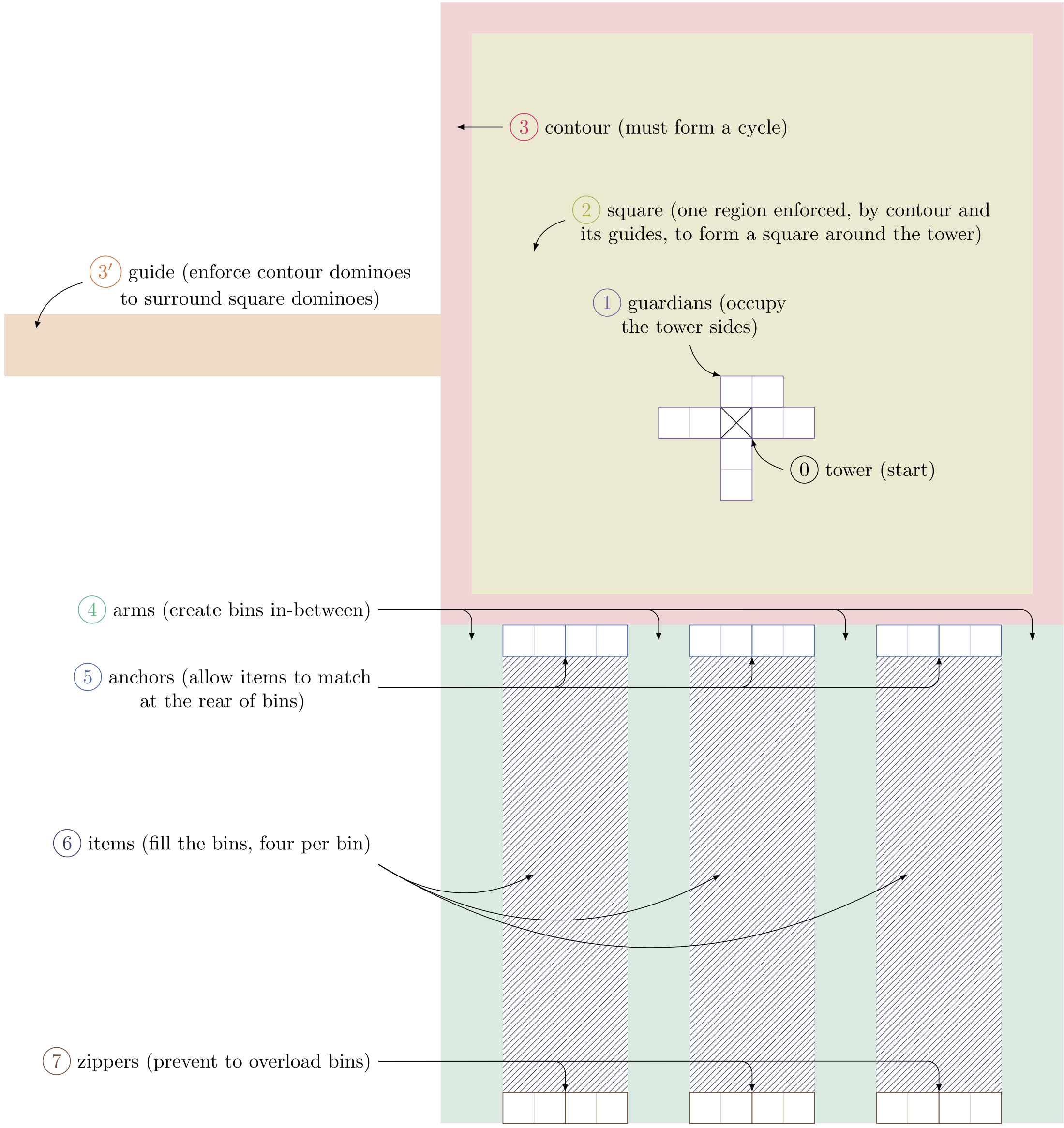
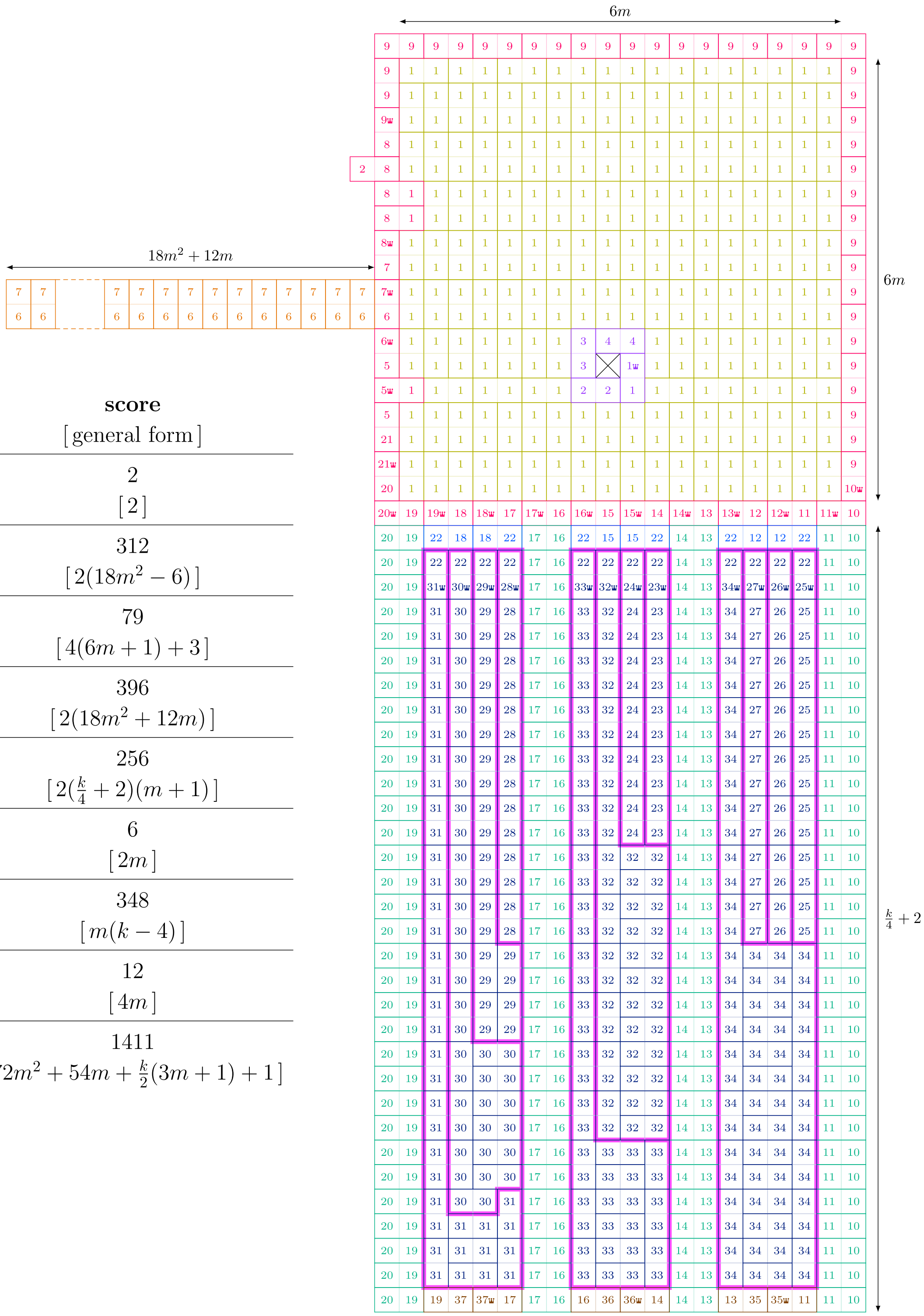


Fig. 2. Sketch of a K-tiling of score s by τ in the reduction from **4-Partition problem** to **K-tiling problem**. Circled numbers from 1 to 7 allow to follow the process of dominoes placement, chronologically. The main idea is to create $m = \frac{n}{4}$ bins of a given area (hatched on the figure), that can be filled with groups of *items* dominoes (whose quantities/areas correspond to item sizes) if and only if all items can be packed into the m bins (with exactly four items, of sum k, per bin).

and the target score $s = 72m^2 + 54m + \frac{k}{2}(3m+1) + 1$. This is our instance of **K-tiling problem**, with the idea of the reduction presented on Fig. 2. Let us now prove that there exists a packing of the n items into the m bins of size k with four items per bin if and only if there exists a K-tiling by τ with score at least s.

$\boxed{\Rightarrow}$ Suppose there exists a packing of the n items into the m bins of size k with exactly four items per bin, and let X_j be the set of items in bin j. We construct the following K-tiling by τ (see Fig. 3).

1. Place the four *guardians* dominoes around the tower.
2. Around this create a square of size $6m \times 6m$ with the *square* dominoes $\boxed{1 \mid 1}$, leaving three dents empty on the left border of the square at the fifth, twelfth and thirteenth positions from the bottom left corner (the square has area $36m^2$, minus 9 cells already taken by the *guardians* dominoes and the tower, minus 3 dents, hence exactly the $18m^2 - 6$ *square* dominoes are required).
3. Make a path clockwise around the square with the *contour* dominoes in this order, filling the three dents with cells of color 1, leaving the cell of color 2 outside, and starting with the first dent at the fifth position above the bottom left



dominoes	score [general form]
soldiers	2 [2]
square	312 [$2(18m^2-6)$]
contour	79 [$4(6m+1)+3$]
guide	396 [$2(18m^2+12m)$]
arms	256 [$2(\frac{k}{4}+2)(m+1)$]
anchors	6 [$2m$]
items	348 [$m(k-4)$]
zippers	12 [$4m$]
total	1411 [$72m^2+54m+\frac{k}{2}(3m+1)+1$]

Fig. 3. To reduce the height of this figure, original sizes have only been multiplied by 4 instead of 28. A K-tiling by τ with score s (hence solving the **K-tiling problem** instance), from a solution to the **4-Partition problem** instance with $n = 12$, $m = 3$, $k = 120$ (originally $k = 30$), and item sizes 12, 12, 16, 16, 16, 16, 24, 40, 40, 48, 48, 72 (originally 3, 3, 4, 4, 4, 4, 6, 10, 10, 12, 12, 18): first bag $72 + 16 + 16 + 16$, second bag $48 + 48 + 12 + 12$, third bag $40 + 40 + 24 + 16$. Domino colors are *guardians*, *square*, *contour*, *guide*, *arms*, *anchors*, *items* (groups are highlighted), and *zippers*. The anchor color $3m + 13$ (on which groups of item dominoes can match) equals 22 on this example. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

corner of the square (the contour has length $4(6m+1)$, corresponding exactly to the $12m+4$ *contour* dominoes with four dents).

4. Stack all *guide* dominoes on the left of the $\boxed{7\text{♛} \mid 6}$ domino of the border.
5. Stack all *arms* dominoes, color by color, below the corresponding dominoes of the border. Observe that they match exactly one domino over three of the bottom border, creating $m+1$ stacks of length $\frac{k}{4}+2$, and therefore m bins of size $k+8$ in between.
6. Place a pair of *anchors* dominoes per bin, matching the existing colors, as on Fig. 3.
7. Place *items* dominoes corresponding to items of X_j in bin j, filling k cells of each bin and leaving the last row of four cells empty.
8. Close each bin with the corresponding *zippers* dominoes (anchors *dominoes* fill four cells, consequently the *items* dominoes leave four cells in each bin, exactly the number of cells required for the *zipper* dominoes to match colors onto the arms on both ends[2]).

The score of this K-tiling is s, as detailed on Fig. 3.

$\boxed{\Leftarrow}$ This is the challenging part of the proof, where we will argue that the construction of a K-tiling by τ with score at least s is compelled to have the structure described above and illustrated on Fig. 3, which corresponds to solving the **4-Partition problem** instance. The proofs of some claims are postponed.

Suppose there exists a K-tiling by τ with score s. First notice that s is an upper bound on the score one can obtain with a K-tiling by τ, as it is the sum for each color of the number of cells of this color times the number of crowns on cells of this color. It must therefore correspond to a K-tiling with one region per color, except for colors $2, 3, 4$ and $3m+13$ which have no crown. This will be the main assumption that will guide us as we study the dominoes chronologically. Also remark that all dominoes must be placed: at the beginning colors $2, 3, 4$ can always match the *tower*, and afterwards colors $2, 3m+13$ appear only on dominoes with another color bringing some necessary points to the sum s.

1. There is no choice but to place the four *guardians* dominoes on the four sides of the tower. As a consequence, we don't have to treat the particular case of the tower anymore.
2. All *square* dominoes are monochromatic, hence they form one large region of color 1 (remark that they can, and therefore must, all be placed). We have now $9+2(18m^2-6)=36m^2-3$ cells occupied by some dominoes or the tower.
3. For the *contour* dominoes the three cells of color 1 must be connected to the unique region of color 1 since this color will not appear anymore. *Contour* dominoes must be arranged in a cycle with some possible defects, that we define now. The idea is that colors 5 to 8 and 10 to $3m+12$ will have a simple path arrangement, whereas color 9 may form a potato like connected component in which a path has to be identified. Let a *pseudo-cycle* be a K-tiling by *contour* dominoes such that, when cells of colors 1 and 2 are discarded:

   - colors 5 to 8 and 10 to $3m+12$ form a simple path (starting from the 5♛ cell of the first *contour* domino, the successor of each cell being either its partner domino cell, or a cell of the next domino, with the successor of the color 5 from the last $\boxed{3m+12 \mid 5}$ being the first 5♛ cell), and
   - a connected component of color 9 connects two distinguished cells of color 9: the 9♛ cell of domino $\boxed{8 \mid 9\text{♛}}$, to the 9 cell of domino $\boxed{9 \mid 10\text{♛}}$.

   The length of a pseudo-cycle is the length of its simple path plus the length of a shortest path (inside the connected component of color 9) between the two distinguished cells of color 9.

   Claim 1. *All* contour *dominoes must be placed (to reach score s), and they must form a pseudo-cycle, of length at most* $4(6m+1)$.

   The pseudo-cycle is connected to the region of color 1 via three dominoes $\boxed{1 \mid 5\text{♛}}$, $\boxed{8 \mid 1}$, $\boxed{8 \mid 1}$:

   - two of them are intended to frame the $\boxed{6 \mid 7\text{♛}}$ domino,
   - and the last one is for parity of cells number, since we have an odd number of occupied cells so far that is intended to form a square of even side length.

   The pseudo-cycle of *contour* dominoes may have the region of color 1 either inside its outer face, or inside its inner face (in this case the pseudo-cycle surrounds the region of color 1).
4. The *guide* dominoes enforce that the cycle of *contour* dominoes surrounds the region of color 1.

[2] Note that the pattern of placement sketched on Fig. 3 can be extended to pack each bin with *items* dominoes corresponding to any four items of sizes summing to k, and leaving four cells on the bottom end for the *zippers* dominoes.

Claim 2. *The* guide *dominoes must be stacked one after the other in a straight segment, rooted at the analogous* contour *domino*

Claim 3. *The pseudo-cycle of* contour *dominoes must have the region of color* 1 *inside its inner face.*

After this step we have $36m^2 - 3$ occupied cells surrounded by a cycle of $4(6m+1)$ cells with three additional cells (with color 1) inside the cycle, hence $(6m)^2$ cells inside the cycle which is just long enough to make a square of side $6m+2$ around it. However, any other shape would either require a too long path, or leave a too small area inside, as stated in the following Claim.

Claim 4. *The maximum area inside a cycle of* $4(6m+1)$ *cells contains* $(6m)^2$ *cells, and is achieve by a square shape of sides* $6m+1$.

<u>Intermediate conclusion:</u> at this point we have a square of *contour* cells with the tower, *guardians* and *square* dominoes inside, three dents of color 1 inside, one dent of color 2 outside, and a stack of *guide* dominoes starting from the corresponding $\boxed{6 \mid 7\text{♛}}$ *contour* domino (the reader can refer to Fig. 3 for an illustration).

We will argue thereafter why the contour is well aligned, with *contour* dominoes $\boxed{10 \mid 11\text{♛}}$ to $\boxed{3m+10 \mid 3m+11\text{♛}}$ all on the same side of the square.

5. The *arm* dominoes must create m bins.

Claim 5. *The* arm *dominoes must be placed into* $m+1$ *bicolor stacks of length* $\frac{k}{4}+2$ *starting from the corresponding* contour *dominoes (separated by four positions), and joined with a pair of* zipper *dominoes.*

This also explains why *contour* dominoes are well aligned, with the third line of *contour* dominoes all on the same side of the square.

<u>Intermediate conclusion:</u> after these dominoes the K-tiling by τ with score s must have created m bins (with $m+1$ arms) of size $4 \times (\frac{k}{4}+2)$. This size explains why the bin size (and consequently all item sizes) of the original **4-Partition problem** instance is converted to a multiple of 4.

6. Each pair of *anchors* dominoes must be placed so that each color already present in the contour (from 12 to $3m+9$) form one region because these are the last dominoes with these colors. So there is a pair of *anchor dominoes* at the rear of each bin. Remark that color $3m+13$ has no crown hence it can be split into multiple regions. The purpose of this color is to be an anchor inside each bin, intended for groups of *items* dominoes to match.
7. For each item x_i we have a group of *items* dominoes, where the first domino of color $3m+13$ allows to match a bin anchor, and then all other dominoes of the group will form one region from this anchored domino (with the unique color $3m+13+i$ for each i), for a total of x_i cells.

Claim 6. *For each bin, at most four groups of* items *dominoes can match its anchors.*

<u>Intermediate conclusion:</u> As all n groups of *items* dominoes must be placed on the board to reach score s, we must have exactly four groups of *items* dominoes in each of the $m = \frac{n}{4}$ bins (corresponding to the values of four items from the **4-Partition problem** instance).

8. The *zippers* dominoes have the purpose of closing bins, with one pair of zippers matching the colors of each bin's arms. They must join the two arms of each bin with a path of length four (because of the unique pair of cell colors $3m+13+n+1$ to $4m+13+n$). However this is possible if and only if no *items* dominoes exceed a volume of $4 \times (\frac{k}{4}+1)$ inside the bin (leaving the last row of four positions of each bin for the zipper), *i.e.* each bin contains four groups of *items* dominoes for a sum of at most k cells (anchors already occupy four cells). Observe that when they contain a total of at most $\frac{k}{2}$ dominoes, it is always possible to place four groups of *items* dominoes in a bin and leave the last row for a pair of *zippers* dominoes (as on the example of Fig. 3).

<u>Conclusion:</u> to reach score s with a K-tiling by τ, a player must close the zipper on top of each of the m bins and therefore trap inside each of them four items of sum at most k, for a total of $4m = n$ items, therefore solving the **4-Partition problem** instance. □

We now present the proofs of the different Claims, and recall their statements.

Claim 1. *All* contour *dominoes must be placed (to reach score s), and they must form a pseudo-cycle, of length at most* $4(6m+1)$.

Proof of Claim 1. For the simple path part, by induction on *contour* dominoes with some cell of color 5 to 8 or 10 to $3m+12$, we have two simple paths concatenated by the last $\boxed{3m+12 \mid 5}$ domino:

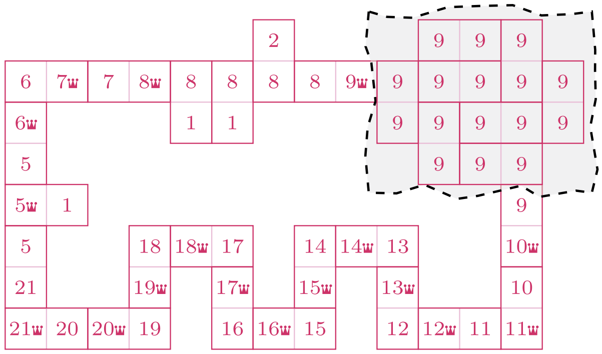


Fig. 4. *Contour* dominoes must form a pseudo-cycle, with a connected component of color 9 (dashed, containing the $9m-8$ $\boxed{9 \mid 9}$ dominoes) including a (shortest) path of length at most $2(9m-8)$, connecting the two extremities of a simple path containing all other cells (excluding cells of color 1 and 2). Domino by domino, the length of the simple path is therefore $1+2+2+2+1+1+1+2$ plus $2+2(3m+1)+4$, giving a total length of the pseudo-cycle upper bounded by $4(6m+1)$.

- either there is no occurrence of a color after the *contour* dominoes hence it must directly form one region (case of dominoes with color[3] 8, and color 5 in the last *contour* domino which concatenates the two simple paths; *e.g.* when placing domino $\boxed{3m+12 \mid 5}$ its cell of color 5 must match the existing region),
- or there is no other placed occurrence of one of its colors apart from the previous *contour* domino (case of all other *contour* dominoes; *e.g.* when placing domino $\boxed{6 \mid 7𝕨}$ its cell of color 6 must match domino $\boxed{5 \mid 6𝕨}$),
- or, for the $\boxed{10 \mid 11𝕨}$ domino, we create the second simple path.

For the connected component of color 9 the argument is straightforward, since the group of $\boxed{9 \mid 9}$ dominoes must form one region, and be connected to the two remaining cells of color 9, in order to have a unique connected component of color 9. For the length calculation, see Fig. 4. □

Claim 2. *The* guide *dominoes must be stacked one after the other in a straight segment, rooted at the analogous* contour *domino*

Proof of Claim 2. *Guide* dominoes are the last occurrences of colors 6 and 7, hence each of these colors must end up forming one region. The basic idea behind this proof is that when a $\boxed{6 \mid 7}$ domino is not correctly stacked it creates two regions of a color, which has to be reconnected via $\boxed{6 \mid 7}$ dominoes, which necessarily creates two regions of the other color, which has to be reconnected via $\boxed{6 \mid 7}$ dominoes, which necessarily creates two regions of the other color, *etc*, hence the two colors cannot simultaneously end up forming one region.

To formalize this intuition, we introduce some definitions. Let the *free outside region* denote the infinite set of grid cells with no domino on them (*free cells*), which are connected together. Given a K-tiling by some prefix of τ until some *guide* dominoes (between 1 and $18m^2+12m$), let the *home-region* of color $c \in \{6,7\}$ be the region of the cell of color c from the $\boxed{6 \mid 7𝕨}$ *contour* domino.

We say that a cell of color $c \in \{6,7\}$ is *zigzag-separated* from its home-region when:

- it does not belong to its home-region, and
- there does not exist a straight segment[4] of free cells and cells of color c, connecting it to its home-region.

We say that a cell of color $c \in \{6,7\}$ is *alternating-zigzag-separated* from its home-region when, for $\bar{c} = 13-c$:

- it does not belong to its home-region, and
- any continuation of the K-tiling (placement of more $\boxed{6 \mid 7}$ *guide* dominoes) connecting it to its home-region would leave a cell of color $\bar{c}$ zigzag-separated from the home-region of color $\bar{c}$.

Let us now argue that, given a K-tiling by some prefix of τ until some *guide* dominoes (between 1 and $18m^2+12m$):

[3] The use of colors 1 and 2, though already present, changes nothing to this argument.

[4] A *straight segment* being a set of cells of the form $\{(x,y),(x,y)+e,(x,y)+2e,\dots,(x,y)+ze\}$ for some $x,y \in \mathbb{Z}$, $z \in \mathbb{N}$ and $e \in \{(0,1),(1,0),(0,-1),(-1,0)\}$.

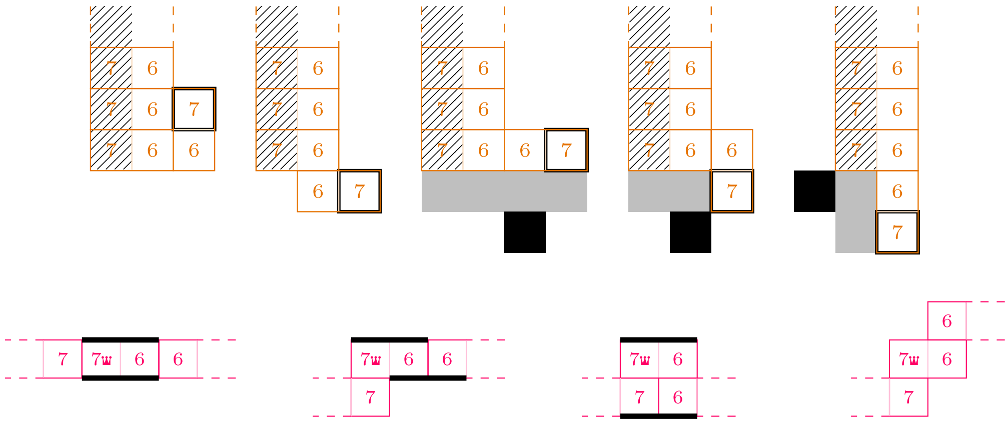


Fig. 5. Top: up to rotation, symmetry and swap of colors, five possible ways to not correctly stack a *guide* domino; in the two first cases a cell of color 7 (bold) is zigzag-separated from its home-region (hatched); and in the three other cases, if $\boxed{6 \mid 7}$ *guide* dominoes are placed so that a straight segment (gray) connects the cell of color 7 (bold) to its home-region (hatched), then there would necessarily exists a cell of color 6 (black) zigzag-separated from its home-region. Hence in any case there exists a zigzag-separated cell. Bottom: up to rotation and symmetry, possible stack positions are highlighted, depending on the placement of *contour* dominoes. For any stack position and number of correctly stacked dominoes, the argument on the top holds.

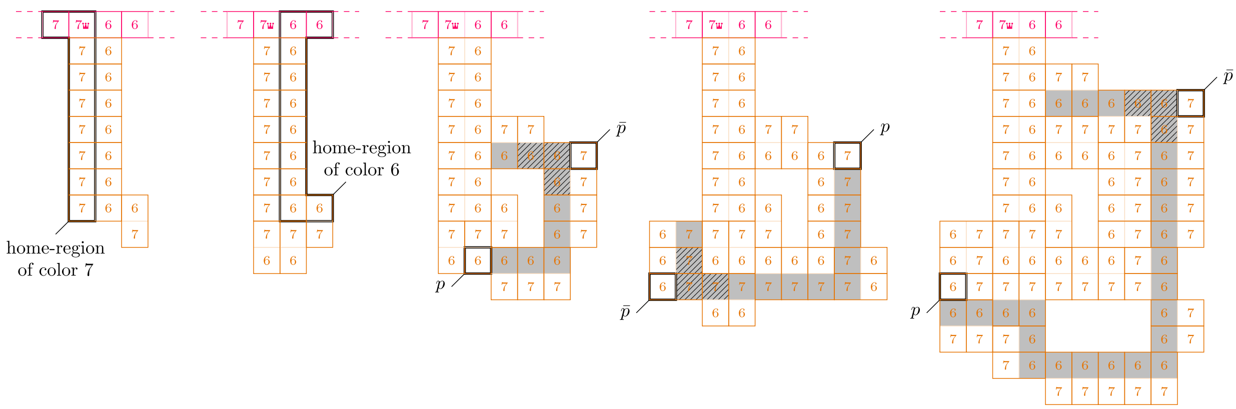


Fig. 6. In order to reconnect a zigzag-separated cell p of color $c \in \{6, 7\}$ to its home-region, a zigzag-separated cell $\bar{p}$ of color $\bar{c} = 13 - c$ is created. Indeed, any path (gray) of cells of color $c \in \{6, 7\}$ connecting cell p to its home-region acts as a barrier, and it contains some turn (dashed) which, due to the use of $\boxed{6 \mid 7}$ *guide* dominoes, necessarily leaves at least one cell $\bar{p}$ of color $\bar{c}$ zigzag-separated from the home-region of color $\bar{c}$. As a consequence, zigzag-separated cell p is alternating-zigzag-separated.

a. if a cell of color $c \in \{6, 7\}$ is zigzag-separated, then it is alternating-zigzag-separated,
b. if a *guide* domino is not correctly stacked as stated in the Claim, then a cell of color $c \in \{6, 7\}$ is zigzag-separated.

The combination of Items *b.* (base case) and *a.* (induction) allows to conclude that it would be impossible to end up having one region of each color, since there would always exist a cell disconnected from its home-region. As a consequence, in order to reach score s, the Claim must hold.

Item *b.* follows a simple case disjunction presented on Fig. 5, and Item *a.* holds since in order to have one region of each color and reach score s, it is necessary to connect the zigzag-separated cell p to its home-region. However, since this path is not a straight segment, it contains some turn which leaves a cell $\bar{p}$ of color $\bar{c}$ zigzag-separated from the home-region of color $\bar{c}$ (the path connecting p to the home-region of color c separates $\bar{p}$ from the home-region of color $\bar{c}$, see Fig. 6). □

Claim 3. *The pseudo-cycle of* contour *dominoes must have the region of color* 1 *inside its inner face.*

Proof of Claim 3. Observe that if the pseudo-cycle of *contour* dominoes (Claim 1) has the region of color 1 on its outer face, then the $18m^2 + 12m$ stacked $\boxed{6 \mid 7}$ *guide* dominoes (Claim 2) would have been placed inside the pseudo-cycle (at most $\frac{4(6m+1)}{2}$ of them) or inside the region of color 1 (at most $18m^2 - 6$ of them), but they are too numerous so it would be impossible to have simultaneously a unique region for colors 6 and 7 (see Fig. 7). □

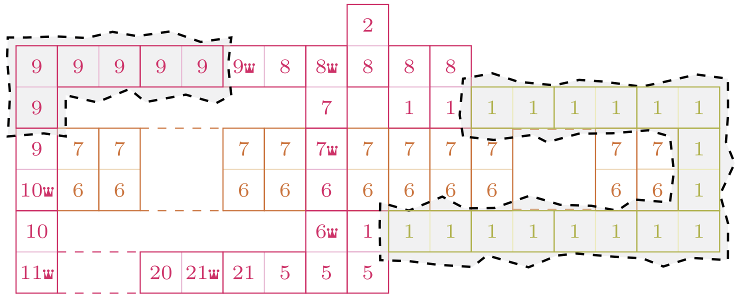


Fig. 7. If the pseudo-cycle of *contour* dominoes has the region of color 1 on its outer face, then one cannot stack all the $18m^2 + 12m$ $\boxed{6 \mid 7}$ dominoes (the region of 1 contains $2(18m^2 - 6)$ cells, and the contour contains $4(m + 1)$ cells).

Claim 4. *The maximum area inside a cycle of* $4(6m + 1)$ *cells contains* $(6m)^2$ *cells, and is achieve by a square shape of sides* $6m + 1$.

Proof of Claim 4. First remark that it is enough to consider rectangular shapes, because any *L-shape* cropping some part of the inside area can be reversed to include this area instead of excluding it (hence increasing strictly the area inside the cycle). Then we have a rectangle of sides a and b (from 2 to $2(6m + 1)$), such that $2(a + b) = 4(6m + 1)$ is fixed, and we want to maximize its area, *i.e.* the product ab. It follows that $b = 2(6m + 1) - a$ and we want to maximize $a(2(6m + 1) - a)$, a quadradic equation whose derivative reaches zero at $a = 6m + 1$, hence the area is maximum for a square shape of sides $6m + 1$, containing $(6m)^2$ cells. □

Claim 5. *The* arm *dominoes must be placed into* $m+1$ *bicolor stacks of length* $\frac{k}{4} + 2$ *starting from the corresponding* contour *dominoes (separated by four positions), and joined with a pair of* zipper *dominoes.*

Proof of Claim 5. For each group of $\frac{k}{4} + 2$ *arm* dominoes the argument is analogous to the proof of Claim 2, with two differences.

- For the base case, possible stack positions (Fig. 5 for *guide* dominoes) are more restricted, due to the established square shape of *contour* dominoes (Claims 1 and 4) surrounding *square* dominoes (Claim 3).
- There is one extra cell of colors $11, 13, 14, 16, \ldots, 3m + 8, 3m + 10$ after *arm* dominoes, in *zippers* dominoes.

We now present how to adapt the argumentation structure from the proof of Claim 2, in order to take into account these two differences.

First, we can already notice that each pair of *zippers* dominoes contain a unique color (between $3m + 13 + n + 1$ and $4m + 13 + n$), hence in order to have one region of each color and reach score s they must form paths of length four, whose ends are connected to the regions of the respective colors (between 11 and $3m + 10$).

Contrary to the four base cases presented on Fig. 5 (bottom), only the two first cases are now possible, as presented on Fig. 8, which also considers all ways to not correctly stack a first *arm* domino. It shows that there is only one case where an extra *zippers* cell may be useful, and that it leads to the impossibility to reach score s. As a consequence, the first domino of each arm must be correctly stacked.

The induction is identical to the proof of Claim 2: one can observe on Figs. 5 (top) and 6 that an extra *zippers* cell of color 6 or 7 is not enough to get one region of each color. As a consequence, for any couple of colors $3p + 10, 3p + 11$ with $0 \le p \le m$, the sequence of $\frac{k}{4} + 2$ $\boxed{3p + 10 \mid 3p + 11}$ *arm* dominoes must also be arranged as a stack starting from the corresponding *contour* dominoes.

The remark first made about *zippers* corresponds to the second part of the Claim. □

Claim 6. *For each bin, at most four groups of* items *dominoes can match its anchors.*

Proof of Claim 6. Each item has size at least 28 and therefore corresponds to at least 14 dominoes, however after placing a pair of *anchors* dominoes and four of these minimum size groups of 14 *items* dominoes in any possible way, no anchor cell of color $3m + 13$ is available for a fifth group of *items* dominoes (see details on Fig. 9). This argument explains why all item and bin sizes of the original **4-Partition problem** instance have been multiplied by 28 (and not simply by 4): so that each group of *items* dominoes is large enough to enforce that at most four per bin can match the anchor. □

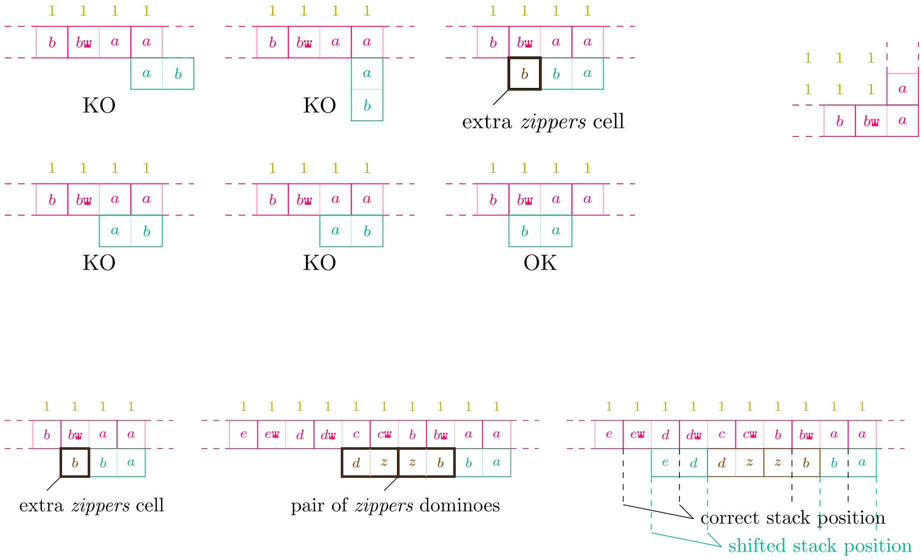


Fig. 8. Presentation of the base case for Claim 5. Top and middle: given that *contour* dominoes form a square surrounding *square* dominoes (Claims 1, 3 and 4), there are only two cases to consider for the positioning of *arm* dominoes. The first six configurations (left) present, up to rotation, symmetry and swap of colors, all placements of the first dominoe in the first case for some arm domino of colors a, b with $a \in \{10, 13, 16, \dots, 3m + 10\}$ and $b = a + 1$. One can see that an extra *zippers* cell of color b may be useful to get one region of each color a and b only in one configuration (shown in the bottom part). For the others, four marked "KO" require more *arm* dominoes, and one marked "OK" corresponds to a correctly stacked *arm* domino. The second case is presented on the last configuration (right), any placement of a first *arm* domino other than the intended one leads to an impossibility by a reasoning analogous to the first cases (note that in this second case there is no color symmetry). Bottom: when a first *arm* domino is not correctly stacked but we can get one region of each color a and b using the extra *zippers* cell of color b (left), the couple of *zippers* dominoes (via color $z = 3m + 13 + n + \frac{a-7}{3}$ appearing only in this couple) must be connected to color $d = c + 1$, with $c = b + 1$, on the other end (middle). This is possible only with an *arm* domino of the next group ($e = d + 1$) also not correctly stacked (right, with the same "shift" of the stack of *arm* dominoes, sketched). However, this prevents the two *anchors* dominoes containing cells of color c to match *contour* dominoes of color c, hence these two *anchors* dominoes are lost and score s cannot be reached.

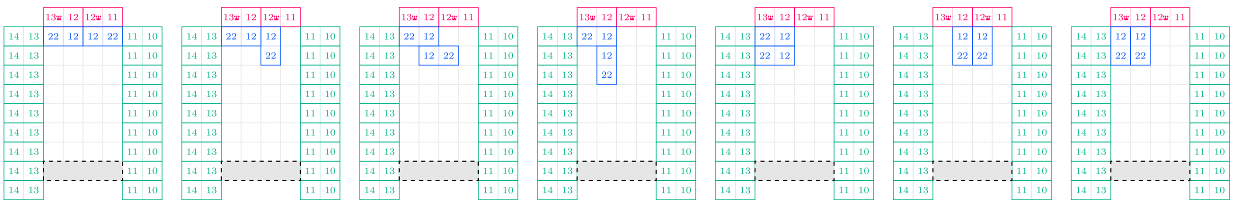


Fig. 9. Up to axial symmetry, seven different ways to place a pair of anchors in a bin (colors of the first bin are taken as an example). After placing the first group of at least 14 *items* dominoes, at least one position on the eighth row of the bin (dashed) is occupied (one can simply count available positions); after placing the second group, at least a second position of the eighth row is occupied; after the third group a third one; and after the fourth group the eighth row of the bin is full of *items* dominoes. However, after these four groups of *items* dominoes, cells of anchor color $3m + 13$ cannot exceed the seventh row (the third row after the pair of *anchors* dominoes, plus one for each group of *items* dominoes), consequently no more group of *items* dominoes can match an anchor color and take place inside the bin.

6. Conclusion

Theorem 1 establishes that *Kingdomino™* shares the feature of many fun games: it requires to solve instances of an NP-complete problem. Finding efficient moves is therefore[5] a computationally hard task, and players may feel glad to encounter good solutions.

As we have seen in Section 4, the number of possible K-tilings may grow rapidly. The main difficulty in designing of the NP-hardness reduction to the **K-tiling problem**, has been to find an initial sequence of dominoes which imposes a rigid structure (with very few possible K-tiling reaching a maximum score), and still allows to be continued in order to implement some strong NP-complete problem (given by the instance from the reduction).

Our modeling of the game *Kingdomino™* abstracts various aspects of the game (as board games are finite, this is necessary), and our construction in Theorem 1 is frugal in terms of crowns, but it is opulent in terms of colors (we have not tried to diminish the usage of colors). An open question is whether the **K-tiling problem** is still NP-hard if the number of colors is bounded?

[5] Unless P = NP.

It would also be interesting to integrate the multi player notion of strategy to the abstract modeling of the game. This somewhat involved process in the official *Kingdomino*^{TM} rules may be simplified as follows. Given,

- k K-tilings $K_1, \ldots, K_k$ by some sequences of dominoes $\tau_1, \ldots, \tau_k$, and
- an unordered set of dominoes τ,

the k players

1. construct k sequences of dominoes $\tau'_1, \ldots, \tau'_k$ by picking one domino at a time from τ (turn by turn, until $\tau'_1, \ldots, \tau'_k$ form a partition of τ), and then
2. each player P_i plays its sequence τ'_i from K_i.

We ask whether player $i \in \{1, \ldots, k\}$ has a winning strategy.

Remark that this modeling also discard two rules of the official *Kingdomino*^{TM} game. First, at the domino picking stage, the order in which players pick dominoes is fixed, and does not depend on the previously picked dominoes as in the official game. Second, the game is separated in two stages: a domino picking stage where players construct their τ'_i until τ is empty, and then a domino placement stage where players place their τ'_i, whereas in the official game these alternate. Also note that the strategy part of this multi player game is on the first stage only, the second stage only consists for each player P_i to maximize its score given K_i and τ'_i.

For any k this problem is solvable in polynomial-space (PSPACE), because one can enumerate all possible game plays (all sequences of dominoes choices $\tau'_1, \ldots, \tau'_k$, and for each of them the best achievable score of each player) and discover whether player P_i has a winning strategy or not (the existence of a winning strategy corresponds to the satisfiability of a quantified propositional formula).

Note that, similarly to other multi players games, starting from empty boards (only the tower for each player, *i.e.* $\tau_1 = \cdots = \tau_k = \varnothing$), a strategy stealing argument would lead to the conclusion that the first player always has a winning strategy.[6] As a consequence, adding non-empty starting boards $K_1, \ldots, K_k$ is necessary, and hopefully makes a hardness proof easier to construct. However players' boards are independent of each other, in the sense that each player plays its sequence of dominoes only on *its own board*, which makes the setting a bit different from PSPACE-hardness results encountered in the literature about multi players games, such as *Hex* [21,22], *Checkers* [23], *Go* [24] and other two players games with perfect information [25].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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[6] For two players: by contradiction suppose P_2 has a winning strategy; P_1 first takes any domino d and then follows the winning strategy of P_2 on τ (*i.e.* P_1 picks dominoes according to the choices of P_2 as if she had never taken domino d and P_2 had started the game); if at some point P_1 needs to take domino d (according to the strategy being stolen), then she takes any available domino d' and the reasoning goes on with d substituted by d'. Player P_1 can always steal the moves of P_2, and therefore (by hypothesis) construct a sequence τ'_1 leading to a score higher than τ'_2.

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