WEA 2008

Experimental Evaluation of an Exact Algorithm for the Orthogonal Art Gallery Problem

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History

- Originally posed by Victor Klee in 1973
  How many guards are sufficient to see every point in the interior of an n-wall Art Gallery room?
- Example: Basilique St. Sernin, Toulouse

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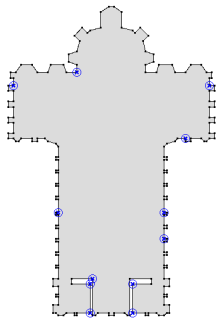
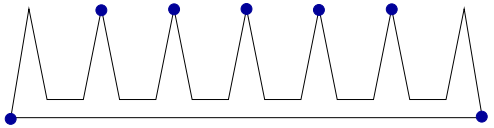


Art Gallery Theorem

- Chvátal proved:
  $\lfloor \frac{n}{3} \rfloor$ *guards are occasionally necessary and always sufficient to cover a polygon of n vertices*



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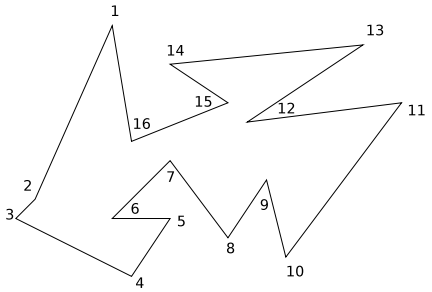

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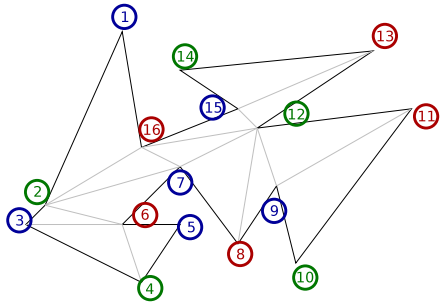

Art Gallery Theorem

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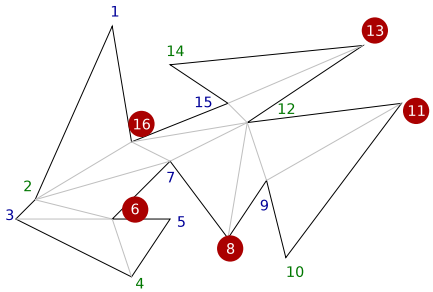
- Fisk had a simpler proof. It leads to placement of $\lfloor \frac{n}{3} \rfloor$ guards



Art Gallery Theorem

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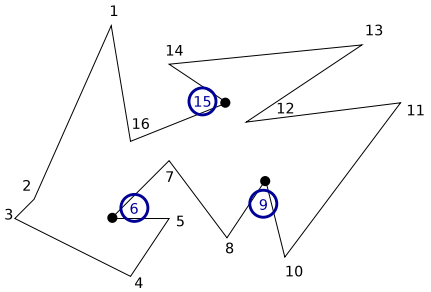
- But it may not the minimum!



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Art Gallery Theorem

- Chvátal proved:
  $\lfloor \frac{n}{3} \rfloor$ *guards are occasionally necessary and always sufficient to cover a polygon of n vertices*
- The Minimization Problem is NP-Hard [Lee & Lin (1985)]

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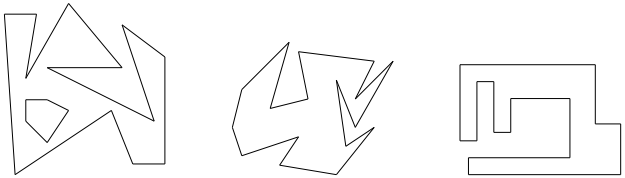
Variations on the Art Gallery Problem

- Types of polygons



- Types of guards
  - Stationary guards
    - Vertex guards
    - Point guards
  - Mobile guards
    - Edge guards
    - Diagonal guards
- External visibility
- See surveys by O'Rourke, Shermer, Urrutia, Ghosh.

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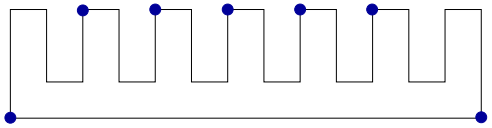
Orthogonal Art Gallery Problem (OAGP)

- Kahn *et al.* (1983) proved that $\lfloor \frac{n}{4} \rfloor$ *guards are occasionally necessary and always sufficient to cover an ortho-gon of n vertices*
  - Necessity



  - Sufficiency: four-color a quadrilateralization of P (mimics Fisk's proof).

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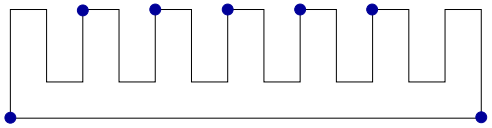


  - Sufficiency: four-color a quadrilateralization of P (mimics Fisk's proof).
- The minimization problem is NP-Hard, too [Schuchardt *et al.* (1995)].

Some known Results

- log n-approximation algorithm [Ghosh (1987)]
- log c_{opt}-approximation algorithm [Efrat and Har-Peled (1998)]
- Eidenbenz (2002) showed AGP is APX-hard (for minimum vertex, point and edge-guards)
- Erdem and Sclaroff (2006) approximation algorithm through fixed grid using an Integer Programming heuristic
- Tomás *et al.* (2006): an exact constraint programming approach through successive approximations

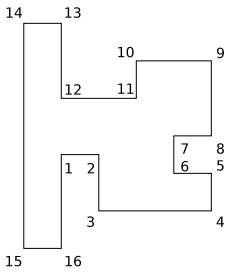
What we set out to do

To study the minimization version of OAGP for polygons without holes and stationary vertex-guards:

- To design an exact algorithm to optimally solve the problem through discretization
- To show the practical viability of the algorithm
- To study various discretization strategies
- To conduct and analyze experimental results

Vertex Guard Set G

- A Vertex Guard Set G is any subset of vertices such that $\bigcup_{g \in G} V(g) = P$



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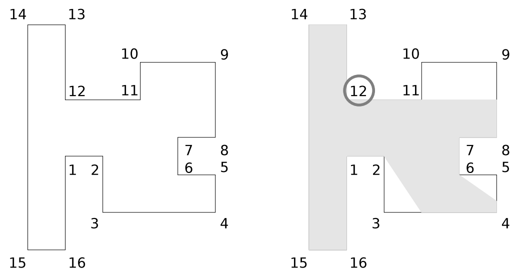

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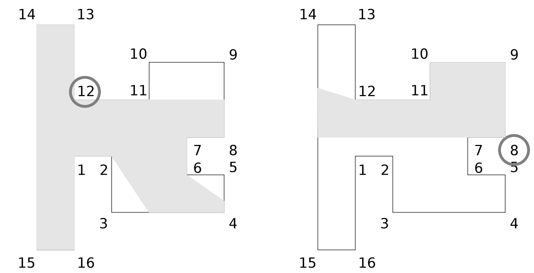

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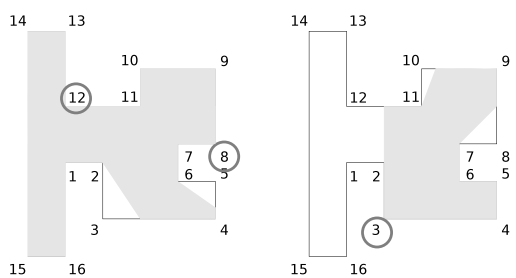

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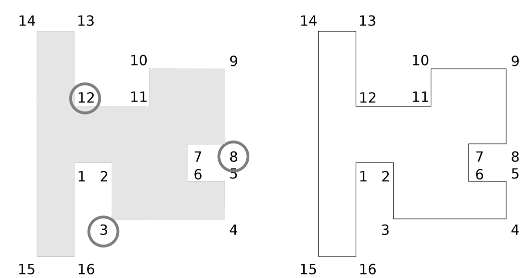

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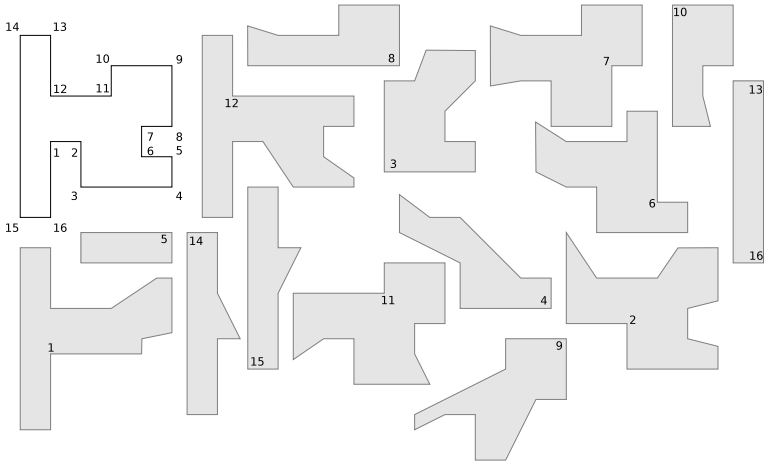
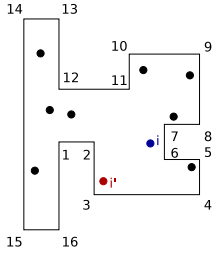
Set Cover Problem



The Discretized Problem

- In order to solve OAGP we:
  - Model the problem as a discrete combinatorial problem
  - Discretize P into a set of points $D(P)$
  - Encode the visibility of $D(P)$ by the vertices, on a binary matrix:
    For each point $i, i' \in D(P)$,
    $a_{ij} = 1$ if i is visible from $j \in V_P$ and
    $a_{i'j} = 0$ if i' is not visible from $j \in V_P$



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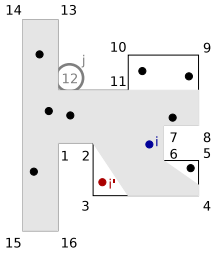


Formulation

- The set $Z = \{j \in V_P \mid x_j = 1\}$ is called a solution set
- (1) states that each point $i \in D(P)$ is visible from at least one selected guard
- The objective function minimizes the cardinality z of the solution set Z
- Solution might not form a guard set for P

$$z = \min \sum_{j \in V_P} x_j$$

$$\text{s.t.} \sum_{j \in V_P} a_{ij} x_j \geq 1, \forall i \in D(P) \quad (1)$$

$$x_j = \begin{cases} 1, & \text{if } j \text{ belongs to the solution} \\ 0, & \text{otherwise.} \end{cases}$$

$$a_{ij} = \begin{cases} 1, & \text{if } i \in V(j) \\ 0, & \text{otherwise.} \end{cases}$$

Initial Discretization of P

- The number of uncovered regions depends on the choice of $D(P)$
  - Tradeoff between speed and accuracy
  - We should generate a small number of constraints

Strategy (betting on)

Discretization strategies

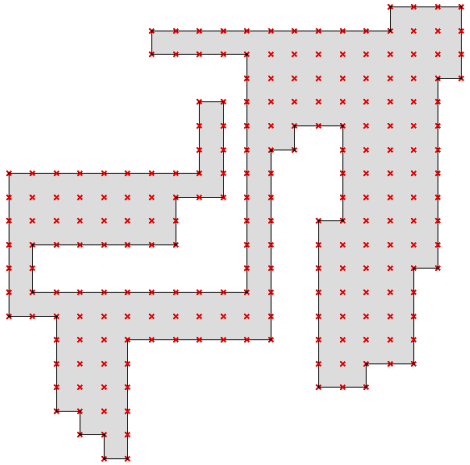
- The number of uncovered regions depends on the choice of $D(P)$
  - Tradeoff between speed and accuracy
  - We should generate a small number of constraints

Strategy (betting on)

- **Regular grid** (density)
  [1]: up to 200 vertices with few iterations in < 100 secs
- **Induced grid** (reflex angles cause hidden areas)
- **Just vertices** (convex vertices identify hard to guard areas)
- (Based on) **AVP**s (ingenious but has costly preprocessing)

Discretization strategies

- **Regular grid** (density)



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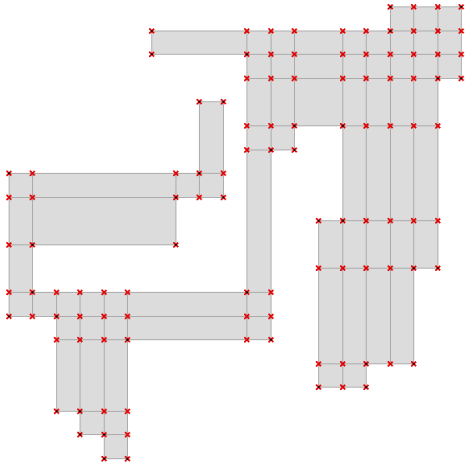

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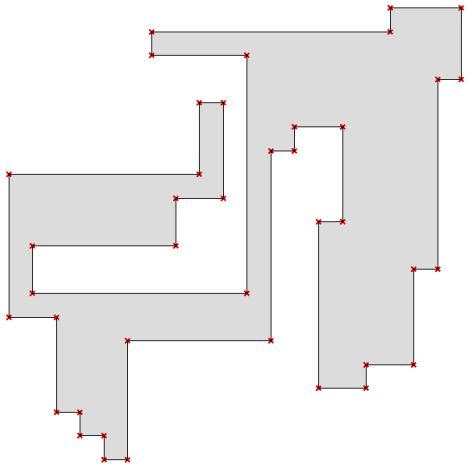

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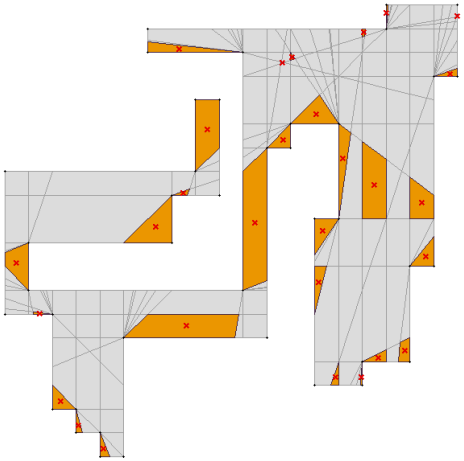

Viable Solution

- A solution Z of the discretized problem is *viable* if Z is a vertex guard set for P

Theorem

Let Z be an optimal solution of an instance $I(P, D(P))$ of the discretized Orthogonal Art Gallery problem. If Z is viable then Z is optimal for the original problem.

Proof. Let Z^* be an optimal vertex guard set for P. Since Z^* is also a vertex guard cover for $D(P)$, we must have $|Z^*| \geq |Z|$. On the other hand, since Z^* is viable, it follows that $|Z| \geq |Z^*|$. □

Preprocessing Phase

- Build $D(P)$: an initial discretization of P
- Compute the visibility region $V(u)$ for each vertex u
- Locate each point $p \in D(P)$ w.r.t. each visibility region
- Return the initial formulation for the Set Cover Problem

Solution Phase

1: **repeat**
2: $Z \leftarrow$ Solution(SetCover($P, D(P)$));
3: **for** each uncovered region R **do**
4: $c \leftarrow$ centroid of R;
5: $D(P) \leftarrow D(P) \cup \{c\}$;
6: Add a new row, r, to the set of constraints corresponding to the new uncovered point c:
$\sum a_{rj} x_j \geq 1$ where, $\forall j \in V_P$,
$a_{rj} \leftarrow$ **Boolean**$(c \in V(j))$;
7: **end for**
8: **until** Z is viable

Iterations

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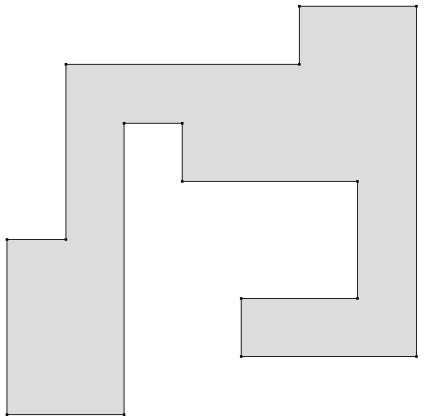

Polygon

Iterations

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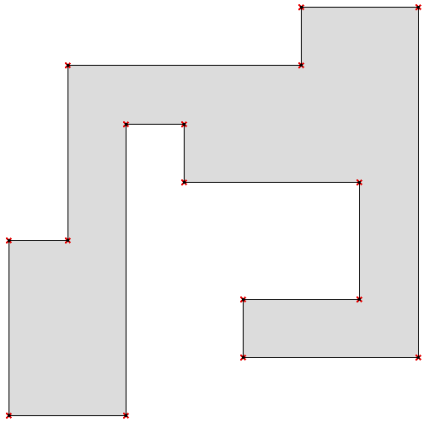

Initial Discretization

Iterations

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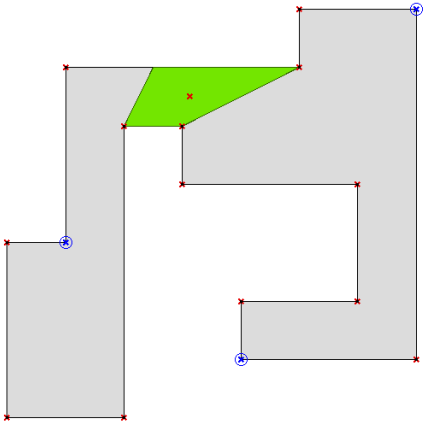

First Iteration

Iterations

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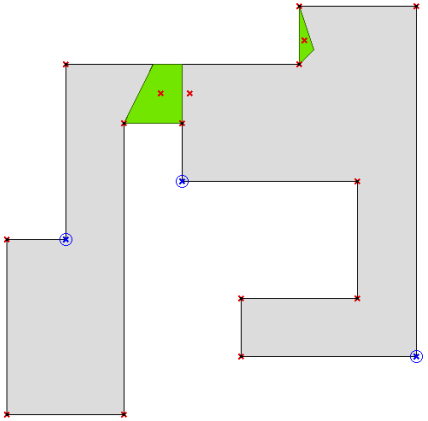

Second Iteration

Iterations

1: **repeat**
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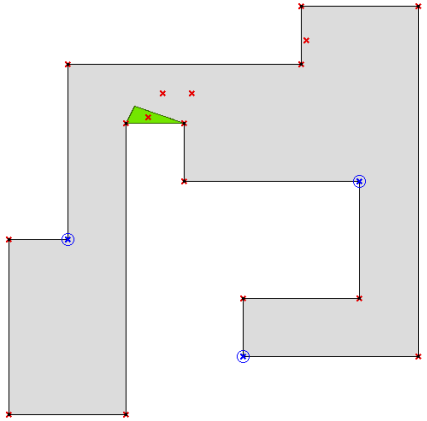

Third Iteration

Iterations

1: **repeat**
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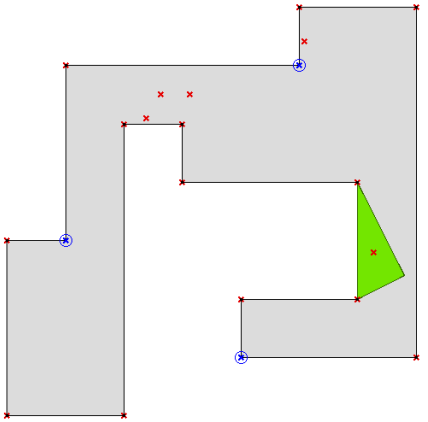

Fourth Iteration

Iterations

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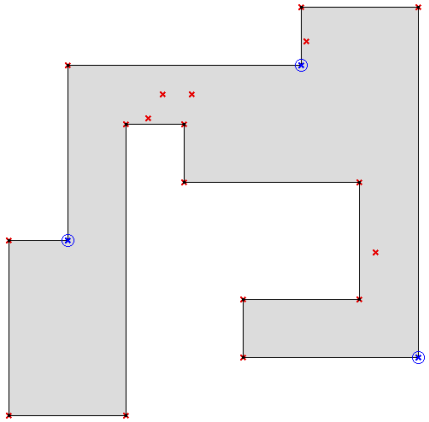


Fifth Iteration: viable solution

What Remains To Be Seen

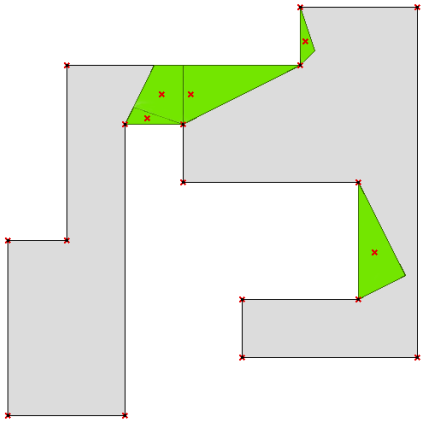
- That the algorithm converges
- How to improve the initial discretization strategy to reduce the number of iterations
- Is it possible to reduce the iterations to one? At what cost?
- What types of polygons we considered
- An analysis of the experimental results

How many uncovered regions can there be?

- Convergence will follow from an upper bound on the number of uncovered regions
- As a byproduct, we might see how to improve the initial discretization strategy

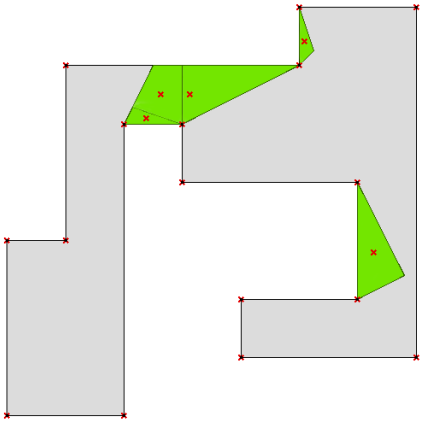
How many uncovered regions can there be?

- Could we predict these regions *before* running the algorithm?



How many uncovered regions can there be?

- Better yet: could we pre-identify *all* relevant regions regardless of discretization?



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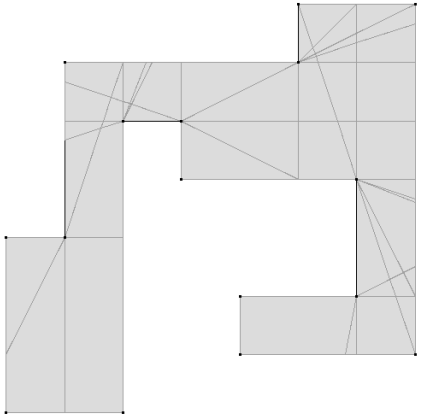

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Improving the Initial Discretization

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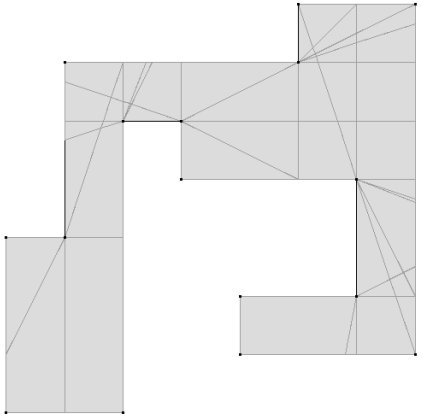
- Better yet: could we pre-identify *all* relevant regions regardless of discretization?
- They are (unions of) faces of the visibility arrangement $\propto \{P, \text{guard-candidates}\}$
- We call these faces *Atomic Visibility Regions* (AVPs). There are $O(n^3)$ of them [Bose *et al.* (2002)].



M.C. Couto, C.C. de Souza & P.J. de Rezende — An Exact Algorithm for the Art Gallery Problem

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Modeling
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Experimental Setup
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How many uncovered regions can there be?

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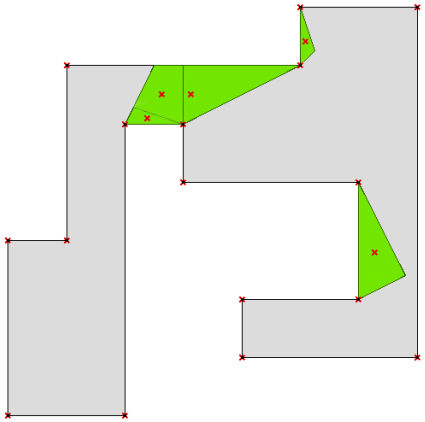

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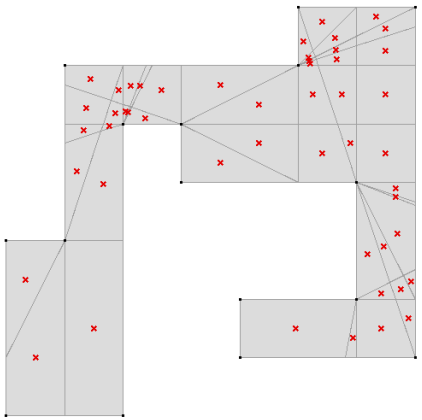
- This proves that the algorithm converges!



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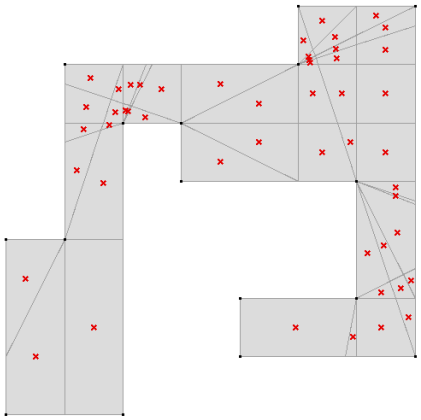
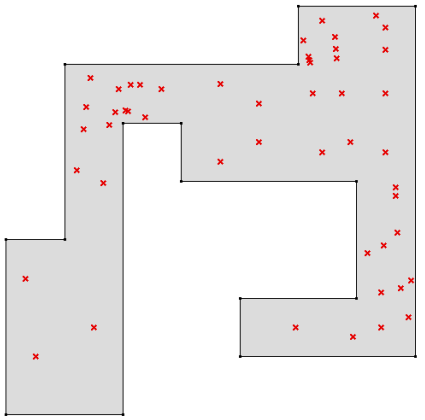
Improving the Initial Discretization

- Coverage of a centroid × implies its AVP is covered.



Improving the Initial Discretization

- Coverage of a centroid × implies its AVP is covered.
- This means that we can solve the Art Gallery problem through a single instance of SCP.
- But this may lead to a very large instance of SCP.



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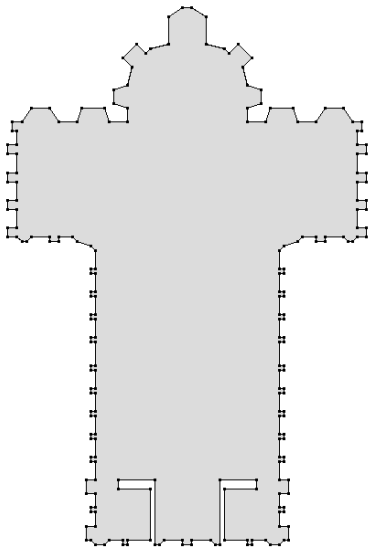

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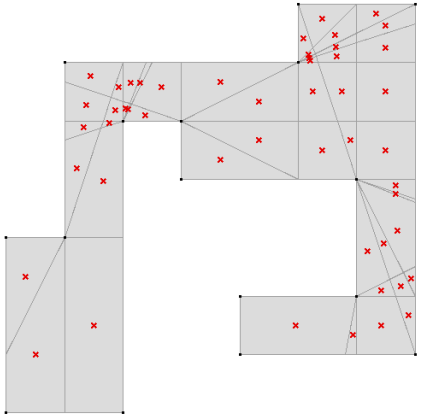
Improving the Initial Discretization

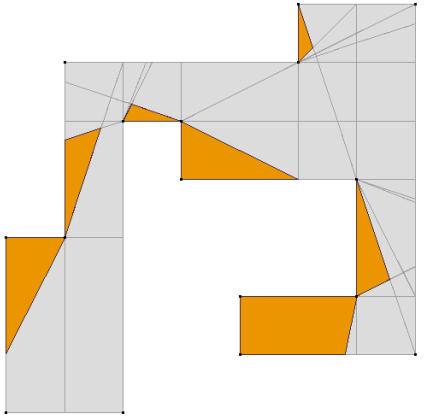
- Can we do better?



Improving the Initial Discretization

- Can we do better?
- Consider a reduced set of AVPs: "Shadow AVPs"

  An AVP $\mathcal{V}$ is a *shadow AVP* if it is not visible from any of the vertices whose (proper) visibility edges generated $\mathcal{V}$.



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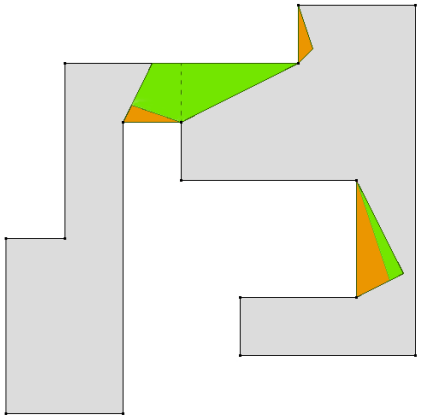

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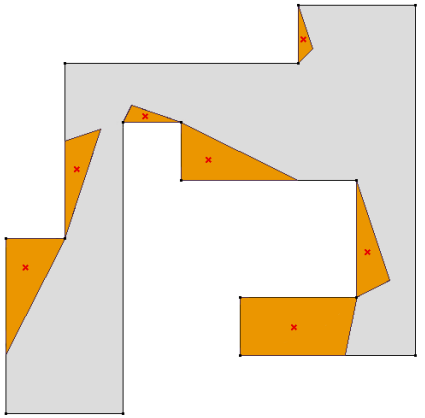

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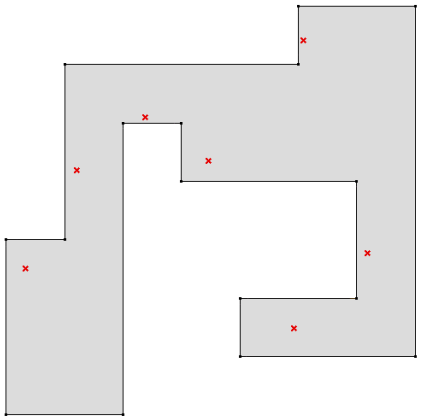

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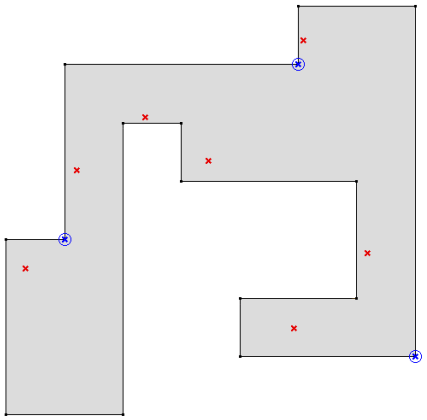

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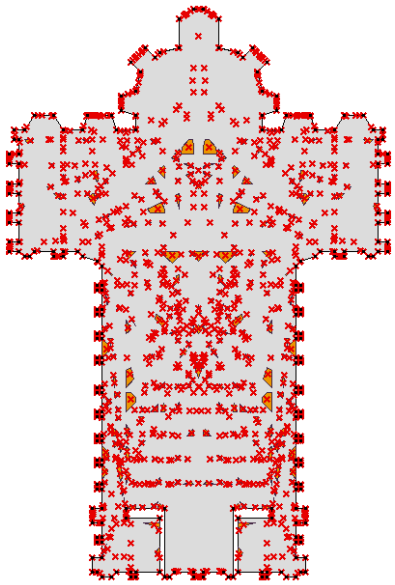
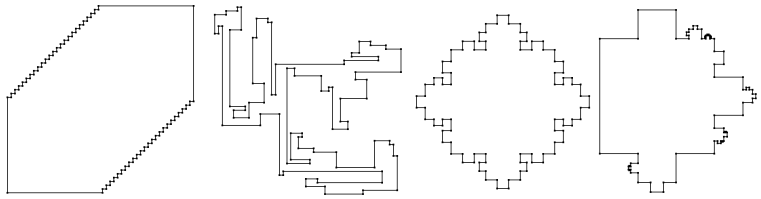


Instances

- 1833 instances of orthogonal polygons without holes (20 to 1000 vertices)
- Polygon Classes:

Large Area || Random || Complete von Koch || Random von Koch



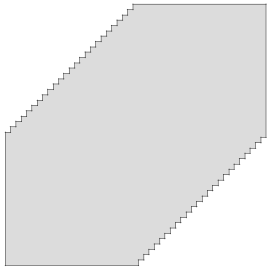
- Available at:
  `www.ic.unicamp.br/~cid/Problem-instances/Art-Gallery/`
  (with over 10000 other instances)

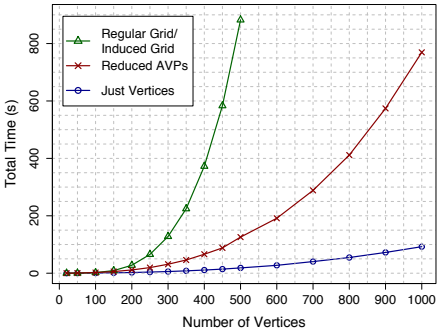
Computational Environment

- The implementation was done in C++ on top of CGAL 3.2.1
- The Integer Linear Programming solver was Xpress-Optimizer v17.01.02 – to solve instances of the Set Cover Problem
- The algorithm was tested on a PC featuring a Pentium IV at 3.4 GHz and 1 GB of memory.

Total Time

- Large area polygons

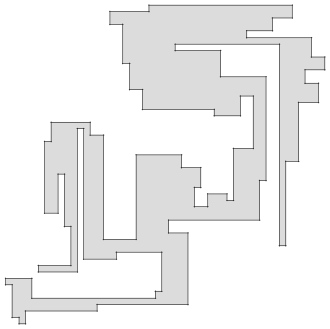




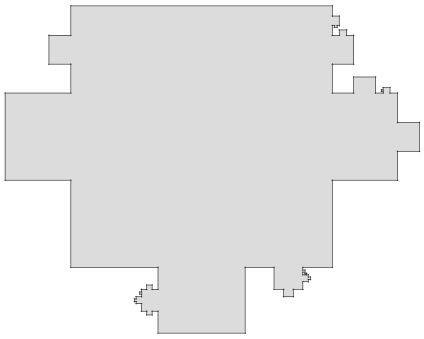
- Five-fold increase in polygon size over Tomás *et al.* [13] within 96 secs

Final Discretization Sizes

- Random polygons



- Random von Koch polygons



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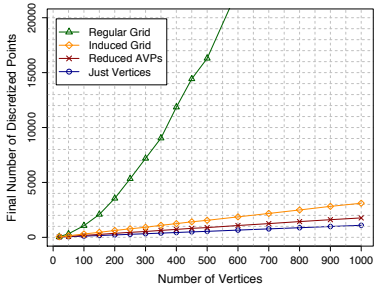

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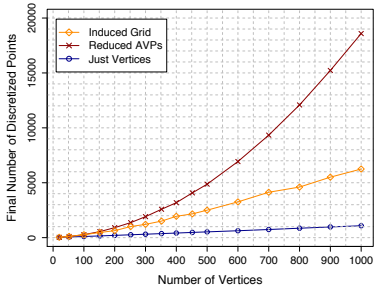
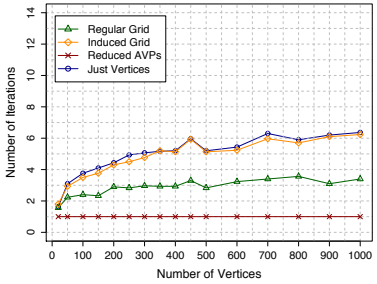


- *Regular grid* is impractical
- *Reduced AVPs* is well-suited for random polygons
- *Just Vertices* is the most economical

Number of iterations

- Random polygons



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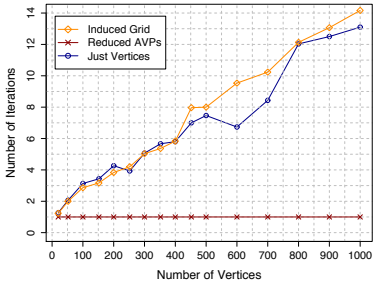
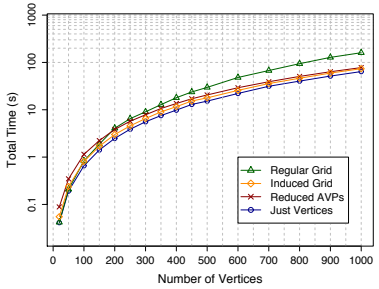


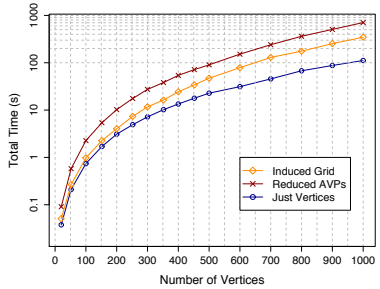
- Obviously one single iteration for Reduced AVPs
- Number of iterations negligible compared to $O(n^3)$ theoretical bound
- For Random von Koch polygons, there is a linear increase, but it is still small

Total Time

- Random polygons



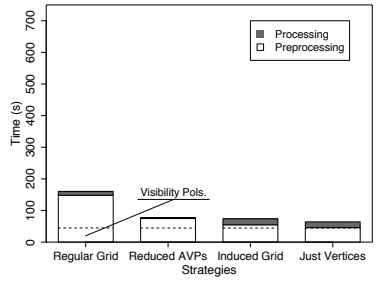
- Random von Koch polygons



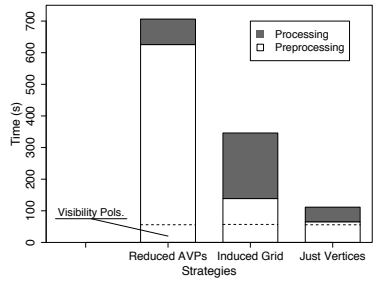
- Note: log×linear graphs
- Random von Koch polygons: three strategies have similarly shaped time curves
- RvKs are harder problems: ∴ *Just Vertices* is a robust strategy

Preprocessing × Processing Time

- Random polygons



- Random von Koch polygons

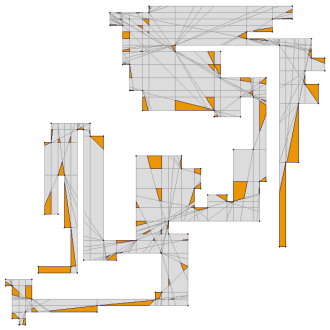


- *Preprocessing* $\in \mathbb{P}$ while *Processing* $\in \mathbb{NP}$-hard and yet, for up to 1000 vertices, *Processing* remains faster

Reduced AVPs is "ingenious but has costly preprocessing": lots of AVPs to sort out

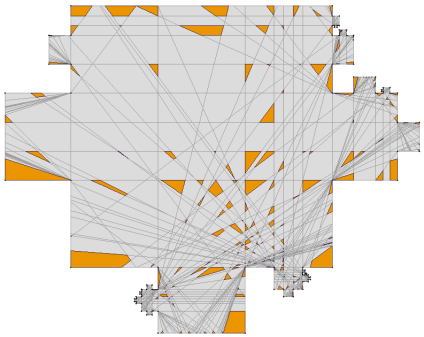
Preprocessing × Processing Time

- Random polygons



- 100 vertices, 2216 edges, 1085 AVPs, 99 shadow AVPs

- Random von Koch polygons



- 100 vertices, 8794 edges, 4420 AVPs, 264 shadow AVPs

Summary

- We obtained an exact algorithm for the *Orthogonal Art Gallery problem*.
- We proved the convergence and correctness of the algorithm.
- Computational tests showed that the algorithm is very efficient in practice.
- Instance sizes of up to 1000 vertices were solved in:

	Polygons Classes			
n	Random	Large Area	RvK	CvK
100	0.65	0.58	0.73	0.97
500	15.21	18.47	22.73	29.35
1000	64.13	92.41	111.55	♯

Total Time (seconds): *Just Vertices* strategy.

What next?

- Simple non-orthogonal polygons without holes (done)
- Multiple coverage (done)
- Solve for point-guards (in progress)
- Solve for edge-guards (next challenge)
- Simple orthogonal polygons with holes
- Extend to general polygons with holes