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Some families of 0-rotatable graceful caterpillars*

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Abstract

A graceful labelling of a tree $T$ is an injective function $f: V(T) \to \{0, 1, \ldots, |E(T)|\}$ such that $|\{f(u) - f(v) : uv \in E(T)\}| = 1, 2, \ldots, |E(T)|$. A tree $T$ is said to be 0-rotatable if, for any $v \in V(T)$, there exists a graceful labelling $f$ of $T$ such that $f(v) = 0$. In this work, it is proved that the following families of caterpillars are 0-rotatable: caterpillars with a perfect matching; caterpillars obtained by identifying a central vertex of a path $P_n$ with a vertex of $K_2$; caterpillars obtained by linking one leaf of the star $K_{1,s-1}$ to a leaf of a path $P_n$ with $n \geq 3$ and $s \geq \lceil \frac{n}{2} \rceil$; and caterpillars with diameter five or six. These results reinforce the conjecture that all caterpillars with diameter at least five are 0-rotatable.

1 Introdução

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A labelling of $G$ is an injective function $f: V(G) \to \mathbb{Z}_{\geq 0}$. Under labelling $f$, the label of a vertex $v \in V(G)$ is $f(v)$, and the (induced) label of an edge $uv \in E(G)$ is the absolute difference of the labels of its ends, $|f(u) - f(v)|$. Given a labelling $f$ of $G$, denote by $L^f_V$ the set of vertex labels under $f$ and denote by $L^f_E$ the set of induced edge labels under $f$. Labelling $f$ is a graceful labelling if $L^f_V \subseteq \{0, 1, \ldots, |E(G)|\}$ and $L^f_E = \{1, \ldots, |E(G)|\}$. We say that $G$ is graceful if it has a graceful labelling. A labelling $f$ of $G$ is an $\alpha$-labelling if $f$ is graceful and there exists an integer $k \in \{0, 1, \ldots, |E(G)|\}$ such that, for each edge $uv \in E(G)$, either $f(u) \leq k < f(v)$, or $f(v) \leq k < f(u)$.

In 1967, Rosa [10] introduced four types of labellings of graphs, among them graceful labelings and $\alpha$-labelings, and posed the Graceful Tree Conjecture, which states that all trees are graceful. Rosa proved that the Graceful Tree Conjecture is a strengthened version of the well-known Ringel-Kotzig Conjecture which states that $K_{2m+1}$ has a cyclic decomposition into subgraphs isomorphic to a given tree $T$ with $m$ edges. The Graceful Tree Conjecture is a very important open problem in Graph Theory, with hundreds of papers about it [6].

As soon as one starts investigating graceful labellings of trees, it becomes clear the importance of knowing how to construct graceful labellings with the label 0 appearing in a given vertex. There are at least two results in the literature that stress the importance of label 0 in a graceful labelling of a tree $T$: first, it is easy to grow a gracefully labelled tree $T$ by adding $k$ new leaves to the 0-labelled vertex and expand the graceful labelling by assigning labels $|E(T)| + 1, \ldots, |E(T)| + k$ to these new leaves. Second, Huang et al. [8] showed that it is possible to combine any tree with an

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α-labelling and any tree with a graceful labelling, by identifying the vertices labelled 0, such that the resultant tree is graceful. A tree $T$ is 0-rotatable if, for any $v \in V(T)$, there exists a graceful labelling $f$ of $T$ such that $f(v) = 0$.

The importance of the 0-rotatability of trees was first noted by Rosa in his seminal paper [10], in which the author stated, without proof, that all paths are 0-rotatable. Ten years later, the author published a proof of this result [11]. Meanwhile, in 1969, some examples of non-0-rotatable trees were discovered [5]. As an example, the smallest non-0-rotatable tree is shown in Figure 1. Posteriorly, Chung and Hwang [4] investigated the 0-rotatability of a product of trees called $\Delta$-construction and proved that if two trees $T$ and $T'$ are 0-rotatable, then their product $T \Delta T'$ is also 0-rotatable. Using this result, the authors showed that every caterpillar whose non-leaf vertices have the same degree is 0-rotatable.

Figure 1: This tree does not have a graceful labelling that assigns label 0 to the black vertex.

In 2004, Bussel [2] showed that all trees with diameter at most three are 0-rotatable. Additionally, the author showed that there exist non-0-rotatable trees with diameter four. In fact, he completely determined the non-0-rotatable trees of diameter four. In order to do this, the author used the following result:

**Theorem 1** (Bussel [2]). Let $T$ be a tree of diameter four such that its center $v$ has degree two. Let $v_1, v_2$ be the vertices adjacent to $v$ and $m_1, m_2$ be the number of leaves adjacent to $v_1, v_2$, respectively. Assume $m_1 \geq m_2$. The tree $T$ has a graceful labelling $f$ with $f(v) = 0$ if and only if there exist integers $x$ and $r$ such that $m_1 = (m_2 + 2 - x)(r - 1) - x$, with: (i) $x, r$ not both odd; (ii) $2 \leq r \leq |E(T)|/2$; and (iii) $0 \leq x \leq \min\{r - 1, m_2\}$. \hfill $\Box$

Let $D$ denote the class of diameter-four trees whose center has degree two and that do not satisfy the conditions of Theorem 1. Let $D'$ be the class of trees built by identifying a leaf of an arbitrary path $P_n$, $n \geq 1$, with the center of a tree in $D$. Bussel [2] proved that, given a tree $T$ with diameter four, $T$ is 0-rotatable if and only if $T \notin D'$. Additionally, he showed that all trees with at most 14 vertices and that are not 0-rotatable belong to the class $D'$. Thus, based on these results, the author posed the following conjecture:

**Conjecture 2** (Bussel [2]). The class $D'$ contains all non-0-rotatable trees.

From the time it was first studied, 0-rotatability of trees has been considered a possible way to approach the Graceful Tree Conjecture, and also a challenging problem by itself. Even for arbitrary caterpillars the result is not known. In fact, note that, if Conjecture 2 is true, then it implies that every caterpillar with diameter at least five is 0-rotatable.

In this work we investigate the Conjecture 2 restricted to caterpillars and prove that the following families of caterpillars are 0-rotatable: (i) caterpillars with a perfect matching; (ii) caterpillars obtained by identifying a central vertex of a path $P_n$ with a vertex of $K_2$; (iii) caterpillars obtained by linking one leaf of the star $K_{1,s-1}$ to a leaf of a path $P_n$, $n \geq 3$ and $s \geq \lceil \frac{n}{2} \rceil$; and (iv) caterpillars with diameter five or six. These results reinforce the conjecture that every caterpillar with diameter at least five is 0-rotatable. In particular, the last two families show that, for each integer $d \geq 5$, there exist 0-rotatable caterpillars with diameter $d$ and arbitrary number of vertices.

In the next section, we present additional definitions as well as classic results and techniques that are used in our proofs. The main results are presented in Section 3.
2 Preliminaries

A matching $M$ of a graph $G$ is a set of pairwise nonadjacent edges of $G$. A vertex $v \in V(G)$ is saturated by $M$ if $v$ is incident with an edge of $M$. If $M$ saturates all the vertices of $G$, then $M$ is a perfect matching. Given a tree $T$ with perfect matching $M$, the contree of $T$ is the tree $T'$ obtained from $T$ by contracting all the edges of $M$.

Broersma and Hoede [1] introduced the concept of strongly graceful labellings of trees, defined as follows. Let $T$ be a tree with a perfect matching $M$. A labelling $f$ of $T$ is strongly graceful if $f$ is a graceful labelling and if $f(u) + f(v) = |E(T)|$ for every edge $uv \in M$. The authors proved that the Graceful Tree Conjecture is true if and only if every tree with a perfect matching has a strongly graceful labelling. They also proved the following result.

**Lemma 3** (Broersma and Hoede [1]). Let $T$ be a tree with a perfect matching $M$ and $uv \in M$, $u, v \in V(T)$. Let $T'$ be the contree of $T$ and let $x \in V(T')$ be the vertex corresponding to edge $uv$. If $T'$ has a graceful labelling $f'$, with $f'(x) = 0$, then $T$ has two strongly graceful labellings $f_1$ and $f_2$, such that: (i) $f_1(u) = 0$ and $f_1(v) = |E(T)|$; (ii) $f_2(u) = |E(T)|$ and $f_2(v) = 0$.

Given a graceful labelling $f$ of a tree $T$, the complementary labelling of $f$ is the labelling $\overline{f}$ defined by $\overline{f}(v) = |E(T)| - f(v)$ for each $v \in V(T)$. Note that the complementary labelling is also a graceful labelling since: (i) $f(v)$ is an injection from $V(T)$ to $\{0, \ldots, |E(T)|\}$; and (ii) for each $uv \in E(T)$, $|\overline{f}(u) - \overline{f}(v)| = |(|E(T)| - f(u)) - (|E(T)| - f(v))| = |f(v) - f(u)|$.

A tree $T$ is a path $P_n$ if its vertices can be arranged in a linear sequence such that two vertices are adjacent if and only if they are consecutive in the sequence. The next lemmas are related to $\alpha$-labellings and graceful labellings of paths and are used in Section 3.

**Lemma 4** (Rosa [11]). Let $P_n$ be a path, $n \geq 1$, and let $v \in V(P_n)$. Then,

(i) there exists an $\alpha$-labelling $f$ of $P_n$ such that $f(v) = 0$ if and only if $v$ is not the central vertex of $P_n$.

(ii) if $v$ is the central vertex of $P_n$, then $P_n$ has a graceful labelling $f$ such that $f(v) = 0$.

**Lemma 5** (Cattell [3]). Let $P_n$ be a path and $v \in P_n$. For any $i \in \{0, \ldots, n-1\}$, there exists a graceful labelling $f$ of $P_n$ with $f(v) = i$ whenever at least one of the following conditions is true:

(i) $n$ is even;

(ii) $n \equiv 5$ or $9 \pmod{12}$;

(iii) given a bipartition $\{X, Y\}$ of $P_n$ with $|X| \geq |Y|$, $v \in X$;

(iv) $i \neq \frac{n-1}{2}$.

Let $T$ be a tree and $v \in V(T)$. Denote by $N_k(v)$ the set of neighbours of $v$ with degree $k$. The distance $d(u, v)$ between two vertices $u, v \in V(T)$ is the number of edges in the unique path connecting $u$ and $v$ in $T$. The eccentricity of a vertex $u \in V(T)$ is defined as $\epsilon(u) = \max\{d(u, v) : v \in V(T)\}$, the diameter as $\text{diam}(T) = \max_{v \in V(T)} \epsilon(v)$, and the radius as $\text{radius}(T) = \min_{v \in V(T)} \epsilon(v)$. A vertex $v \in V(T)$ is a central vertex of $T$ if $\epsilon(v) = \text{radius}(T)$. A spine of $T$ is a path $P \subset T$ such that its ends have maximum eccentricity in $T$. Given a tree $T$ with spine $P$, we say that $T$ is a caterpillar if all vertices of $T$ are either contained in $P$, or are at distance exactly one from $P$. The next result states that every caterpillar has an $\alpha$-labelling.
Lemma 6 (Rosa [10]). Let \( T \) be a caterpillar and \( v \in V(T) \) be a vertex which either has maximum eccentricity or is adjacent to a vertex of maximum eccentricity. Then, \( T \) has an \( \alpha \)-labelling \( f \) such that \( f(v) = 0 \).

Let \( u, v, w \) be distinct vertices of a tree \( T \), such that \( w \) is adjacent to \( u \). We call transfer the operation of deleting edge \( uv \) from \( T \) and adding edge \( vw \). After the transfer operation, we say that vertex \( w \) has been transferred or moved from \( u \) to \( v \). For any two distinct vertices \( u \) and \( v \) of a gracefully labelled tree \( T \), the notation \( u \rightarrow v \) means that we moved some vertices incident with vertex \( u \) to vertex \( v \). We say that a transfer \( u \rightarrow v \) applied to a graceful tree is safe if the resulting tree is also graceful. The following lemma states when a transfer performed on a graceful tree generates another graceful tree.

Lemma 7 (Hrnčiar and Haviar [7]). Let \( f \) be a graceful labelling of a tree \( T \) and let \( u, v \in V(T) \) be two distinct vertices. If \( u \) is adjacent to (not necessarily distinct) leaves \( u_1, u_2 \in V(T) \), such that \( u_1 \neq v, u_2 \neq v \) and \( f(u_1) + f(u_2) = f(u) + f(v) \), then the tree \( T' \) obtained from \( T \) by moving \( u_1, u_2 \) from \( u \) to \( v \) is also graceful.

A \( u \rightarrow v \) transfer is said to be of the first type if the labels of the transferred vertices are the labels in set \( \{k, k+1, \ldots, k+p\} \), where \( f(u) + f(v) = k + (k+p) \). A transfer of the first type is also denoted by \( u \xrightarrow{[k,k+p]} v \). Note that, in a transfer of the first type, the labels of transferred vertices constitute a set of consecutive integers. On the other hand, a \( u \rightarrow v \) transfer is of the second type if the labels of the transferred vertices are the labels in set \( \{k, k+1, \ldots, k+p\} \cup \{l, l+1, \ldots, l+p\} \), where \( f(u) + f(v) = k + l + p \). A transfer of the second type is also denoted by \( u \xrightarrow{[k,k+p],[l,l+p]} v \).

In a transfer of the second type, the labels of the transferred vertices can be partitioned into two sets of same cardinality, where each set is composed by consecutive integers. Figure 2 illustrates these concepts.

![Graceful trees](image)

Figure 2: Tree \( T' \) is obtained from \( T \) by transfer \( 11 \xrightarrow{[5,7]} 1 \) of the first type; on the other hand, tree \( T'' \) is obtained from \( T \) by applying transfer \( 11 \xrightarrow{[3,5],[7,9]} 1 \) of the second type.

The next lemma establishes additional conditions under which it is possible to make safe transfers in a graceful tree.

Lemma 8 (Mishra and Panigrahi [9]). Let \( T \) be a tree with a graceful labelling \( f \) satisfying the following properties:

(i) there exist distinct vertices in \( T \) with labels \( a, \ldots, a+r_1, b-r_2, \ldots, b \) such that \( a < b \), \( a + r_1 < b - r_2 \), and \( r_1, r_2 \in \mathbb{Z}_{\geq 0} \);

(ii) the vertex with label \( a \) is adjacent to a set of vertices \( S \) with labels \( s, \ldots, s+p \), such that:

(a) \( p \geq 2 \);

(b) \( \{s, \ldots, s+p\} \cap \{a, \ldots, a+r_1, b-r_2, \ldots, b\} = \emptyset \); and
(c) for $0 \leq i < \lfloor \frac{n}{2} \rfloor$, either $(s+i+1)+(s+p-i) = a+b$ or $(s+i)+(s+p-1-i) = a+b$.

Then, the following statements are true:

(a) if $|S|$ is odd, then it is possible to make a safe transfer $a \rightarrow b$ of the first type followed by a safe transfer $b \rightarrow (a+1)$ of the second type, keeping an odd number of vertices at vertex $a$ and a positive even number of vertices at $b$, moving the rest of the vertices to $a+1$.

(b) if $|S|$ is even, then it is possible to make a sequence of safe transfers of the second type $a \rightarrow b \rightarrow a+1 \rightarrow b-1 \rightarrow a+2 \rightarrow b-2 \rightarrow \ldots \rightarrow z$, where $z = a+r_1$ or $z = b-r_2$, keeping a positive even number of vertices of $S$ at each vertex of the sequence.

3 Results

In this section, we prove our main results. We start by showing an interesting result which is useful for proving that certain families of trees with a perfect matching are 0-rotatable.

**Theorem 9.** Let $T$ be a tree with a perfect matching. If the contree of $T$ is 0-rotatable, then $T$ is 0-rotatable.

**Proof.** Let $T$ be a tree with perfect matching $M$ and $uv \in M$. Let $T'$ be the contree of $T$ and $x \in V(T')$ be the vertex corresponding to edge $uv$. Suppose $T'$ is 0-rotatable. Hence, $T'$ has a graceful labelling $f'$ such that $f'(x) = 0$. Thus, by Lemma 3, $T$ has two strongly graceful labellings $f_1$ and $f_2$ such that: $f_1(u) = 0$ and $f_1(v) = |E(T)|$; $f_2(u) = |E(T)|$ and $f_2(v) = 0$. Therefore, there exist strongly graceful labellings of $T$ which assign label 0 to vertices $u$ and $v$. Since $uv$ is an arbitrary edge of $M$, we conclude that $T$ is 0-rotatable.

**Corollary 10.** Every caterpillar with a perfect matching is 0-rotatable.

**Proof.** The result follows from Theorem 9 and the fact that the contree of a caterpillar with a perfect matching is a path, which is 0-rotatable by Lemma 4.

Theorem 14 and Theorem 16 prove that two families of caterpillars are 0-rotatable. Before presenting these results, it is necessary to establish some auxiliary lemmas.

**Lemma 11.** Let $X, Y, Z$ be nonempty sets such that:

(i) $|Y| \geq \max\{|X|, |Z|\}$;

(ii) $X = \{0, \ldots, |X| - 1\}$;

(iii) $Y = \{|X|, \ldots, |X| + |Y| - 1\}$; and

(iv) $Z = \{|X| + |Y|, \ldots, |X| + |Y| + |Z| - 1\}$.

Then, for every $l \in X \cup Z$, there exists $t \in Y$ for which $|l - t| = |Y|$.

**Proof.** The result follows by letting $t = l + |Y|$ when $l \in X$, and letting $t = l - |Y|$ when $l \in Y$.

**Lemma 12.** Let $T$ be either a path $P_n$, with $n \geq 1$, or a star $K_{1,n-1}$, with $n \geq 2$. Let $v \in V(T)$ be a leaf of $T$, $t$ be a positive integer and $S = \{t, t+1, \ldots, t+n-1\}$. Then, for each $i \in S$, there exists a labelling $f : V(T) \rightarrow S$ such that $f(v) = i$ and $L_{E(T)}^f = \{1, \ldots, n-1\}$.
Lemma 13. If a tree $T$ has an $\alpha$-labeling $f$, then there exists a bipartition $\{A, B\}$ of $T$ such that $L_A^f = \{0, \ldots, |A| - 1\}$ and $L_B^f = \{|A|, \ldots, |A| + |B| - 1\}$. 

Proof. Let $P_n = v_1 \cdots v_n$ be a path, with $n \geq 1$. Let $T$ be the caterpillar obtained by identifying a vertex of $K_2$ with the central vertex $v_{\lfloor \frac{n}{2} \rfloor}$ of $P_n$. Let $v_1, \ldots, v_n$ be the vertices of the spine of $T$ and let $v_{n+1}$ be the leaf adjacent to $v_{\lfloor \frac{n}{2} \rfloor}$.

If $\text{diam}(T) \in \{1, 2, 3, 4\}$, the result follows from Lemma 6 and Corollary 10. Now, consider $\text{diam}(T) \in \{5, 6, 7\}$. By Lemma 6, for $v \in \{v_1, v_2, v_{n-1}, v_n\}$, there exists a graceful labelling $f$ of $T$ such that $f(v) = 0$. Moreover, Figure 3 exhibits two distinct graceful labellings $f_5^1, f_6^2$ of $T$ with $\text{diam}(T) = 5$, such that $f_5^1(v_3) = 0$ and $f_6^2(v_4) = 0$. The complementary labelling of $f_5^1$ assigns label 0 to $v_7$. Figure 4 exhibits three distinct graceful labellings $f_6^1, f_6^2, f_6^3$ of $T$ with $\text{diam}(T) = 6$, such that $f_6^1(v_3) = 0$, $f_6^2(v_4) = 0$, and $f_6^3(v_5) = 0$. The complementary labelling of $f_6^2$ assigns label 0 to $v_8$. Finally, Figure 5 exhibits three distinct graceful labellings $f_7^1, f_7^2, f_7^3$ of $T$ with $\text{diam}(T) = 7$, such that $f_7^1(v_3) = 0$, $f_7^2(v_4) = 0$, and $f_7^3(v_5) = 0$. The complementary labelling of $f_7^2$ assigns label 0 to $v_9$, and the result follows.

Now, we consider the remaining case in which $\text{diam}(T) \geq 8$. Let $P \subset T$ be the subgraph induced by vertex set $\{v_1, v_2, \ldots, v_{\lfloor \frac{n}{2} \rfloor}, v_{n+1}\}$ and let $Q \subset T$ be the subgraph induced by vertex set $V(T) \setminus V(P)$. Let $n_P$ and $n_Q$ denote the order of $P$ and $Q$, respectively, and let $m_P$ and $m_Q$ denote the sizes of $P$ and $Q$, respectively. Note that both $P$ and $Q$ are paths. Moreover, since $\text{diam}(T) \geq 8$, $\text{diam}(P) \geq 5$.

First, we prove that, for $v \in V(P)$, there exists a graceful labelling $f$ of $T$ such that $f(v) = 0$. By Lemma 4, $P$ has an $\alpha$-labelling $g: V(P) \rightarrow \{0, 1, \ldots, m_P\}$ such that $g(v) = 0$. By Lemma 13, there exists a bipartition $\{A, B\}$ of $P$ such that $L_A^g = \{0, 1, \ldots, |A| - 1\}$ and $L_B^g = \{|A|, \ldots, |A| + |B| - 1\}$. 

Figure 3: Two graceful labellings of a caterpillar $T$ with $\text{diam}(T) = 5$.

Figure 4: Three graceful labellings of a caterpillar $T$ with $\text{diam}(T) = 6$.

Figure 5: Three graceful labellings of a caterpillar $T$ with $\text{diam}(T) = 7$. 

Theorem 14. Every caterpillar obtained by identifying a vertex of $K_2$ with a central vertex of $P_n$ is 0-rotatable.
Using this bipartition, we modify $g$ in order to obtain another labelling $f_P$ of $P$ as follows:

$$f_P(u) = \begin{cases} g(u), & \text{if } u \in A; \\ g(u) + n_Q, & \text{if } u \in B. \end{cases}$$

Therefore, we obtain $f_P : V(P) \to A \cup B'$ such that $A = \{0, 1, \ldots, |A| - 1\}$ and $B' = \{|A| + n_Q, |A| + 1 + n_Q, \ldots, |A| + |B| - 1 + n_Q\}$. Since each label in $B$ was increased by $n_Q$, $L_{E(P)}^{f_P} = \{1 + n_Q, 2 + n_Q, \ldots, m_P + n_Q\}$.

Note that the vertex labels $|A|, |A| + 1, \ldots, |A| + n_Q - 1$ are missing in $f_P$, as well as the induced edge labels $1, 2, \ldots, n_Q$. Let $C = \{|A|, |A| + 1, \ldots, |A| + n_Q - 1\}$ and let $l = f_P(v_{[n/2]}^{n/2})$. Next, we show that there exists an integer $t \in C$, such that $|l - t| = |C| = n_Q$.

By the definition of $P$, we have that $|A| + |B'| = n_P = \lceil \frac{n}{2} \rceil + 1$. Moreover, since $P$ is a path, one of the following holds: (i) $|A| = |B'| = \lceil \frac{n}{2} \rceil + 1 \div 2$; (ii) $|A| = \lceil \frac{n}{2} \rceil + 1 \div 2$ and $|B'| = |A| + 1$; or (iii) $|B'| = \lceil \frac{n}{2} \rceil + 1 \div 2$ and $|A| = |B'| + 1$. Since $|C| = n_Q = \lceil \frac{n}{2} \rceil$ and $\lceil \frac{n}{2} \rceil > \lceil \frac{n}{2} \rceil + 1 \div 2$ for $n \geq 9$, we obtain that $|C| > |A|$ and $|C| > |B'|$. Thus, considering $X = A, Y = C, Z = B'$, and $l$ as previously chosen, by Lemma 11, there exists $t \in Y$, such that $|l - t| = |Y| = |C|$, as required.

By Lemma 12, there exists a labelling $f_Q : V(Q) \to C$ such that: (i) $f_Q(v_{[n/2]}^{n/2}+1) = t$; and (ii) $L_{E(Q)}^{f_Q} = \{1, \ldots, n_Q - 1\}$. Define a labelling $f : V(T) \to \{0, 1, \ldots, |E(T)|\}$ such that:

$$f(u) = \begin{cases} f_P(u), & \text{if } u \in P; \\ f_Q(u), & \text{if } u \in Q. \end{cases}$$

Labelling $f$ is a graceful labelling of $T$ since: (i) $f$ is an injective function from $V(T)$ to $\{0, 1, \ldots, m_P + m_Q + 1 = |E(T)|\}$; (ii) the induced edge labels of $Q$ are $1, 2, \ldots, n_Q - 1$; (iii) the induced edge labels of $P$ are $n_Q + 1, n_Q + 2, \ldots, |E(T)|$; and (iv) $f(v_{[n/2]}^{n/2}+1) = n_Q$.

In order to conclude the proof, we have to show that there exists a graceful labelling $f$ such that $f(v) = 0$ for each vertex $v \in V(Q)$. It can be done by the previous reasoning, considering $V(P) = \{v_{[n/2]}^{n/2}, v_{n/2+1}\}$ and $V(Q) = V(T) \backslash V(P)$.

Theorem 16 proves that every caterpillar obtained by linking one leaf of star $K_{1,s-1}$ to a leaf of path $P_n$, with $n \geq 3$ and $s \geq \lceil \frac{n}{2} \rceil$, is 0-rotatable. In our proof we use a specific labelling of a caterpillar which is presented in the next lemma.

**Lemma 15.** Let $T$ be the caterpillar obtained by linking one leaf of star $K_{1,s-1}$, $s \geq 3$, to a leaf of path $P_5$. If $v$ is the central vertex of $P_5$, then there exists a graceful labelling $f$ of $T$ such that $f(v) = 0$.

**Theorem 16.** Let $T$ be the caterpillar obtained by linking one leaf of the star $K_{1,s-1}$ to a leaf of the path $P_n$. If $n \geq 3$ and $s \geq \lceil \frac{n}{2} \rceil$, then $T$ is 0-rotatable.

**Proof.** Let $P_n = v_1 \cdots v_n$ and $V(K_{1,s-1}) = \{x_1, \ldots, x_s\}$, with $x_s$ its central vertex. Let $T$ be the caterpillar obtained by linking $x_1$ to $v_1$. Thus, $T$ has vertex set $V(T) = V(K_{1,s-1}) \cup V(P_n)$ and edge set $E(T) = E(K_{1,s-1}) \cup E(P_n) \cup \{x_1v_1\}$.

Suppose $n \geq 3$ and $s \geq \lceil \frac{n}{2} \rceil$. In the following, we prove that there exists a graceful labelling $f$ of $T$ such that $f(v) = 0$ for every $v \in V(T)$. We consider two cases depending on which subgraph, $K_{1,s-1}$ or $P_n$, vertex $v$ belongs to.

**Case 1.** $v \in V(K_{1,s-1})$. 

By Lemma 6, for every \( v \in \{x_2, \ldots, x_s\} \), there exists a graceful labelling \( f \) of \( T \) such that \( f(v) = 0 \). Therefore, in order to conclude this case, it remains to show that there exists a graceful labelling \( f \) of \( T \) such that \( f(x_1) = 0 \).

Let \( H_1 \) and \( H_2 \) be subgraphs of \( T \) induced by the vertex set \( \{v_1, x_1, x_2, \ldots, x_s\} \) and \( \{v_2, v_3, \ldots, v_n\} = V(T) \setminus V(H_1) \), respectively. Define a graceful labelling \( h_1 : V(H_1) \rightarrow \{0, \ldots, s\} \) as follows: (i) \( h_1(x_i) = i - 1 \), for \( 1 \leq i \leq s \); and (ii) \( h_1(v_1) = s \). Since \( h_1(x_s) = s - 1 \) and its neighbours have labels \( 0, 1, \ldots, s - 2 \), the edges incident with \( x_s \) have induced labels \( 1, 2, \ldots, s - 1 \). Moreover, since \( h_1(x_1) = 0 \) and \( h_1(v_1) = s \), the edge \( x_1v_1 \) has label \( s \). Next, we modify \( h_1 \) in order to obtain another labelling \( h'_1 \):

\[
   h'_1(v) = \begin{cases} h_1(v), & \text{if } v \in \{x_1, x_2, \ldots, x_{s-1}\}; \\ h_1(v) + |V(H_2)|, & \text{if } v \in \{x_s, v_1\}. \end{cases}
\]

Since \( n \geq 3 \) and \( |V(H_2)| = n - 1 \), \( |V(H_2)| \geq 2 \). By the definition, \( h'_1(x_s) = n + s - 2 \). Moreover, the neighbours of \( x_s \) have labels \( 0, 1, \ldots, s - 2 \) under \( h'_1 \). Therefore, \( L_{E(H_1)}^{h'_1} = \{n, n + 1, \ldots, n + s - 2\} \). Also, since \( h'_1(x_1) = 0 \) and \( h'_1(v_1) = n + s - 1 \), we have that \( h'_1(x_1v_1) = n + s - 1 \). Thus, we conclude that the vertex labels \( s - 1, s, \ldots, s + n - 3 \) are missing, as well as the edge labels \( 1, 2, \ldots, n - 1 \).

Since \( H_2 \) is a path with \( |V(H_2)| \geq 2 \), by Lemma 12, \( H_2 \) has a labelling \( h_2 : V(H_2) \rightarrow \{s - 1, s, \ldots, s + n - 3\} \) such that \( h_2(v_2) = s \) and \( L_{E(H_2)}^{h_2} = \{1, 2, \ldots, n - 2\} \). We define labelling \( f : V(T) \rightarrow \{0, 1, \ldots, |E(T)|\} \) as follows:

\[
   f(v) = \begin{cases} h'_1(v), & \text{if } v \in V(H_1); \\ h_2(v), & \text{if } v \in V(H_2). \end{cases}
\]

Labelling \( f \) is graceful since: (i) \( f \) is an injective function from \( V(T) \) to \( \{0, \ldots, |E(T)|\} \); and (ii) \( L_{E(H_2)}^{f} = \{1, \ldots, n - 2\}, L_{E(H_1)}^{f} = \{n, \ldots, n + s - 1\} \), and \( |f(v_2) - f(v_1)| = |s - (n + s - 1)| = n - 1 \). Thus, \( L_{E(T)}^{f} = \{1, \ldots, n + s - 1\} \) and the result follows.

**Case 2.** \( v \in V(P_n) \).

If \( n = 5 \) and \( v \) is the central vertex of \( P_5 \), the result follows by Lemma 15. Thus, consider \( n \neq 5 \) or \( v \) different from the central vertex of \( P_5 \). By Lemma 4, since \( v \) is not the central vertex of \( P_5 \), path \( P_n \) has an \( \alpha \)-labelling \( g \) such that \( g(v) = 0 \). By Lemma 13, there exists a bipartition \( \{A, B\} \) of \( P_n \) such that \( L_A = \{0, 1, \ldots, |A| - 1\} \) and \( L_B = \{|A|, \ldots, |A| + |B| - 1\} \). Using this bipartition, we modify \( g \) in order to obtain another labelling \( f_P \) of \( P_n \). For each \( u \in V(P_n) \), define

\[
   f_P(u) = \begin{cases} g(u), & \text{if } u \in A; \\ g(u) + s, & \text{if } u \in B. \end{cases}
\]

Thus, we obtain the labelling \( f_P : V(P_n) \rightarrow A \cup B' \), such that \( A = \{0, 1, \ldots, |A| - 1\} \) and \( B' = \{|A| + s, |A| + s + 1, \ldots, |A| + s + |B| - 1\} \). Since each label in \( B \) was increased by \( s \), \( L_{E(P_n)}^{f_P} = \{1 + s, 2 + s, \ldots, n - 1 + s = |E(T)|\} \). Note that the vertex labels \( |A|, |A| + 1, \ldots, |A| + s - 1 \) are missing in \( f_P \), as well as the induced edge labels \( 1, 2, \ldots, s \). Let \( C = \{|A|, |A| + 1, \ldots, |A| + s - 1\} \) and let \( l = f_P(v_1) \). Next, we show that there exists an integer \( t \in C \), such that \( |l - t| = |C| = s \).

Consider \( X = A, Y = C, Z = B' \), and \( l \) as previously chosen. Since \( |C| \geq \lceil \frac{n}{2} \rceil \), by Lemma 11, there exists \( t \in Y \), such that \( |l - t| = |Y| = |C| \), as required. By Lemma 12, there exists a labelling
\( f_K : V(K_{1,s-1}) \to C \), such that: (i) \( f_K(x_1) = t \); and (ii) \( L^f_{K_{1,s-1}} = \{1,2,\ldots,s-1\} \). Thus, define labelling \( f : V(T) \to \{0,1,\ldots,|E(T)|\} \) as follows:

\[
\begin{align*}
f(u) &= \begin{cases} 
  f_P(u), & \text{if } u \in V(P_n); \\
  f_K(u), & \text{if } u \in V(K_{1,s-1}).
\end{cases}
\end{align*}
\]

Labelling \( f \) is graceful since: (i) \( f \) is an injective function from \( V(T) \) to \( \{0,1,\ldots,|E(T)|\} \); and (ii) \( L^f_{E(K_{1,s-1})} = \{1,\ldots,s-1\} \), \( L^f_{E(P_n)} = \{s+1,\ldots,s+n-1\} \) and \( f(x_1v_1) = s \). Therefore, \( L^f_{E(T)} = \{1,\ldots,|E(T)|\} \) and the result follows.

\[ \square \]

### 3.1 Caterpillars with diameter five

The main result of this section is Theorem 19, which states that every caterpillar \( T \) with diameter five is 0-rotatable. In order to prove this result, for each non-leaf vertex \( v \in V(T) \), we construct a graceful labelling \( f \) of \( T \) that assigns label 0 to \( v \) and assigns label \( |E(T)| \) to any leaf \( u \in V(T) \) adjacent to \( v \). Consequently, we use its complementary labelling \( \overline{f} \) in order to obtain \( \overline{f}(u) = 0 \) and \( \overline{f}(v) = |E(T)| \). Since \( \overline{f} \) is also a graceful labelling and \( f \) is constructed considering an arbitrary non-leaf vertex \( v \) of \( T \), we conclude that \( T \) is 0-rotatable.

The above mentioned labellings are obtained either directly from Lemma 6, or by modifying one of the trees presented in Figure 6. These trees are modified by transfer operations and need some properties presented in Lemma 17.

Given two finite sets of integers \( A \) and \( B \), we say that \( A < B \) if \( \max\{a : a \in A\} < \min\{b : b \in B\} \). An ordered pair \( \{(r,s,t),\mathcal{N}\} \) with \( r,s,t \in \mathbb{Z}_{\geq 0} \) and \( \mathcal{N} \subseteq \mathbb{Z}_{\geq 0} \) is called a special pair if it satisfies the following conditions:

\begin{enumerate}
  \item[(i)] \( r \leq s \) and \( t = r + s \);
  \item[(ii)] given the index set \( \mathcal{I} = \{0,t-1,t,t+1\} \) and \( \mathcal{N} = \{n_i : n_i \in \mathbb{Z}_{\geq 0}, i \in \mathcal{I}, \text{ and } n_0 \geq 1\} \), then \( \sum_{i \in \mathcal{I}} n_i = s - r + 1 \); and
  \item[(iii)] exactly one of the following conditions holds:
    \begin{enumerate}
      \item[(a)] \( n_i \) is even for \( i \in \mathcal{I}\setminus\{0\} \);
      \item[(b)] \( n_{t-1} \equiv n_{t+1} \equiv t \pmod{2} \) and \( n_t \not\equiv t \pmod{2} \).
    \end{enumerate}
\end{enumerate}

**Lemma 17.** Let \( \{(r,s,t),\mathcal{N}\} \) be a special pair. Let \( T \) be a tree and let \( f \) be a graceful labelling of \( T \) satisfying the following properties:

\begin{enumerate}
  \item[(i)] \( T \) has a vertex \( v \) that is adjacent to a set \( S \) of vertices with labels \( r, r+1, \ldots, s \);
  \item[(ii)] for each \( i \in \{t-1,t,t+1\} \), if \( T \) has a vertex \( v_i \) such that \( f(v) + f(v_i) = i \), then \( v_i \not\in S \).
\end{enumerate}

Then, for each \( i \in \{t-1,t,t+1\} \), it is possible to safely transfer \( n_i \) vertices of \( S \) from \( v \) to \( v_i \). \( \square \)

The main result of this section is Lemma 18. In order to present it, we need an additional definition: the model-tree \( T_d(c_1,c_2,\ldots,c_{d-1}) \) is the caterpillar with diameter \( d \) and spine \( P = u_0 \cdots u_d \) such that, for \( i \in \{1,\ldots,d-1\} \), vertex \( u_i \) is adjacent to exactly \( c_i \) leaves. Figure 6 shows three model-trees with special graceful labellings.

**Lemma 18.** Let \( T \) be a caterpillar with diameter five. Let \( v \in V(T) \) be a central vertex of \( T \) and \( w \in \{v\} \cup N_1(v) \). Then, \( T \) has a graceful labelling \( f \) such that \( f(w) = 0 \).
Case 1. \(|U_1| \equiv |U_4| \pmod{2}\) or \(|U_1| \equiv 1 \pmod{2}\) and \(|U_4| \equiv 0 \pmod{2}\).

Initially, we modify \(T\) in order to obtain a model-tree \(T' = T_5(a+1, 0, b, 1)\), with \(a = |U_1| + |U_4|\), \(b = |U_3|\), and \(V(T') = V(T)\), changing the edge set of \(T\) as follows: \(E(T') = (E(T)\setminus\{u_4w: w \in U_3\}) \cup \{u_4w: w \in U_4\}\). Tree \(T'\) has a graceful labelling \(f\) such that \(f(u_2) = 0\), as illustrated in Figure 6(a). Next, we show that it is possible to safely transfer \(|U_4|\) leaves from \(u_1\) to \(u_4\), obtaining a graceful labelling for \(T\).

Let \(m = |E(T')| = |U_1| + |U_3| + |U_4| + 5 = a + b + 5\). Let \(r, s, t\) be three positive integers such that \(r = b + 3\), \(s = m - 2\), and \(t = r + s = m + b + 1\). Let \(I = \{0, t - 1, t, t + 1\}\) be an index set and let \(N = \{n_i: i \in I\}\) such that \(n_0 = |U_1| + 1\), \(n_t = |U_4|\), \(n_{t-1} = n_{t+1} = 0\). Note that the ordered pair \((\{r, s, t\}, N)\) is a special pair since: (i) \(r \leq s\) and \(t = r + s\); (ii) \(n_0 \geq 1\) and \(\sum_{i \in I} n_i = a + 1 = s - r + 1\); (iii) when \(|U_4| \equiv 0 \pmod{2}\), \(n_i\) is even, for \(i \in I\setminus\{0\}\); and (iv) when \(|U_1| \equiv |U_4| \equiv 1 \pmod{2}\), we have that \(n_{t-1} \equiv n_{t+1} \equiv t \pmod{2}\) and \(n_t \not\equiv t \pmod{2}\). Moreover, note that: (i) the vertex \(u_1 \in V(T')\) is adjacent to a set of leaves \(S\) with labels \(b + 3, \ldots, m - 2\); (ii) vertex \(u_4 \not\in S\) and \(f(u_1) + f(u_4) = m + b + 1 = t\). Therefore, by Lemma 17, considering \(u_1 = v\) and \(u_4 = v_t\), we can safely transfer \(n_t = |U_4|\) leaves of set \(S\) from vertex \(u_1\) to vertex \(u_4\).

Case 2. \(|U_1| \equiv |U_3| \equiv 0 \pmod{2}\) and \(|U_4| \equiv 1 \pmod{2}\).

If \(|E(T)| = 6\), the result follows from the graceful labelling of \(T\) depicted in Figure 6(b). Thus, suppose \(|E(T)| \geq 7\). By the same reasoning of the previous case, we construct \(T' = T_5(1, 0, 0, a+1)\), with \(a = |U_1| + |U_3| + |U_4|\) and \(V(T') = V(T)\), changing the edge set of \(T\) as follows: \(E(T') = \{(u_i, u_j): i, j \in I, i \neq j, n_i + n_j = a + 1\}\) and adding a new vertex \(v\) with label 0 adjacent to \(u_1\) and \(u_4\). Next, we show that it is possible to safely transfer \(|U_4|\) leaves from \(u_1\) to \(u_4\), obtaining a graceful labelling for \(T'\).
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$(E(T) \setminus \{u_4 w: w \in U_1 \} \cup \{u_3 w: w \in U_3\}) \cup \{u_4 w: w \in U_1 \cup U_3\}$. Tree $T'$ has a graceful labelling $f$ such that $f(u_2) = 0$, illustrated in Figure 6(b). Next, we show that it is possible to safely transfer $|U_1|$ leaves from $u_4$ to $u_i$, for $i \in \{1, 3\}$. Let $m = |E(T')| = |U_1| + |U_3| + |U_4| + 5 = a + 5$. We consider two subcases.

**Subcase 1.** $|U_1| > 0$.

Since $f(u_1) + f(u_4) = m + 2$ and $u_4$ has two leaves with labels $m - 1$ and $3$, by Lemma 7, we can safely transfer this pair of leaves from vertex $u_4$ to vertex $u_1$. Additionally, let $r, s, t$ be three positive integers such that $r = 4$, $s = m - 3$, and $t = r + s = m + 1$. Let $I = \{0, t - 1, t, t + 1\}$ be an index set and let $N = \{n_i: i \in I\}$ such that $n_0 = |U_4| + 1$, $n_{t-1} = |U_3|$, $n_t = 0$ and $n_{t+1} = |U_1| - 2$. Note that the ordered pair $([r, s, t], N)$ is a special pair since: (i) $r \leq s$ and $t = r + s$; (ii) $n_0 \geq 1$, $\sum_{i \in I} n_i = a - 1 = s - r + 1$; (iii) $n_i \equiv 0 \pmod{2}$ for $i \in I \setminus \{0\}$. Moreover, note that: (i) vertex $u_4$ is adjacent to a set of leaves $S$ with labels $4, \ldots, m - 3$; (ii) vertices $u_1, u_3 \not\in S$, $f(u_3) + f(u_4) = t - 1$, and $f(u_1) + f(u_4) = t + 1$. Therefore, by Lemma 17, we can safely transfer $n_{t-1} = |U_3|$ leaves from $u_4$ to $u_3$ and $n_{t+1} = |U_1| - 2$ leaves from $u_4$ to $u_1$.

**Subcase 2.** $|U_1| = 0$.

In this subcase, it is sufficient to safely transfer $|U_3|$ leaves from $u_4$ to $u_3$. Since $f(u_3) + f(u_4) = (m - 2) + 2 = m$, we move the pairs of leaves with labels in the set $\{3 + i, m - 3 - i: 0 \leq i < |U_3|/2\}$ from $u_4$ to $u_3$. Since $(3 + i) + (m - 3 - i) = m$, by Lemma 7, the tree obtained after these transfers is graceful.

**Case 3.** $|U_1| \equiv 0 \pmod{2}$ and $|U_3| \equiv |U_4| \equiv 1 \pmod{2}$.

Let $u \in U_3$. As in previous cases, modify $T$ so as to obtain a model-tree $T' = T_0(1, 0, 1, a + 1)$, with $a = |U_1| + |U_3| + |U_4| - 1$ and $V(T') = V(T)$, changing the edge set of $T$ as follows: $E(T') = (E(T) \setminus \{u_4 w: w \in U_1\} \cup \{u_3 w: w \in U_3\}) \cup \{u_4 w: w \in U_1 \cup U_3\}$. Tree $T'$ has a graceful labelling $f$ such that $f(u_2) = 0$, as illustrated in Figure 6(c).

Next, we show how to safely transfer $|U_1|$ leaves from $u_4$ to $u_1$. Moreover, by construction, exactly one leaf of $U_3$ is adjacent to vertex $u_3$ in $T'$. Hence, we also show how to safely transfer $|U_3| - 1$ leaves from $u_4$ to $u_3$. Let $m = |E(T')| = |U_1| + |U_3| + |U_4| + 5 = a + 6$. Let $r, s, t$ be three positive integers such that $r = 5$, $s = m - 2$, and $t = r + s = m + 3$. Let $I = \{0, t - 1, t, t + 1\}$ be an index set and let $N = \{n_i: i \in I\}$ such that $n_0 = |U_4|$, $n_{t-1} = |U_1|$, $n_t = |U_3| - 1$ and $n_{t+1} = 0$. The ordered pair $([r, s, t], N)$ is a special pair since: (i) $r \leq s$ and $t = r + s$; (ii) $n_0 \geq 1$, $\sum_{i \in I} n_i = a - s - r + 1$; and (iii) $n_i$ is even for $i \in I \setminus \{0\}$. Moreover, note that: (i) the vertex $u_4$ is adjacent to a set of leaves $S$ with labels $5, \ldots, m - 2$; (ii) vertices $u_1, u_3 \not\in S$, $f(u_1) + f(u_4) = t - 1$, and $f(u_3) + f(u_4) = t$. Therefore, by Lemma 17, we can safely transfer $n_{t-1} = |U_1|$ leaves from $u_4$ to $u_1$ and $n_t = |U_3| - 1$ leaves of $S$ from $u_4$ to $u_3$. This concludes the proof.

**Theorem 19.** If $T$ is a caterpillar with diameter five, then $T$ is 0-rotatable.

**Proof.** The result follows from Lemma 6 and Lemma 18.

### 3.2 Caterpillars with diameter six

The main result of this section is Theorem 22, which states that every caterpillar with diameter six is 0-rotatable. The technique used to prove this result is the same used to prove Theorem 19. Accordingly, Lemma 20 and Lemma 21 present auxiliary results needed in the proof of Theorem 22. Furthermore, Figure 7 shows four model-trees of diameter six with graceful labellings $f$ such that $f(u_3) = 0$, that are used in Lemma 20.
Lemma 20. Let $T$ be a caterpillar with diameter six, let $v \in V(T)$ be the central vertex of $T$, and let $w \in \{v\} \cup N_1(v)$. Then, $T$ has a graceful labelling $f$ such that $f(w) = 0$.

Proof. Let $T$ be a caterpillar with diameter six and spine $P = u_0u_1v_2u_3u_4v_5u_6$. For each $i \in \{1, 2, 3, 4, 5\}$, define $U_i$ as the set of leaves from $V(T) \setminus \{u_0, u_6\}$ that are adjacent to $u_i$. Note that $u_3$ is the unique central vertex of $T$. As shown in the proof of Lemma 18, we can assume $|U_3| = 0$.

In our proof, we consider five cases depending on the parities of the $|U_i|$s. In order to do this, we introduce the following definition: given tree $T$, we assign $T$ a 5-tuple $(p_1, p_2, \ldots, p_4, p_5)$ such that, for each $i \in \{1, 2, 4, 5\}$, $p_i$ is the parity of $|U_i|$. Since $p_i \in \{0, 1\}$, there exist 16 distinct 5-tuples.

Case 1. Tree $T$ is assigned one of the following 5-tuples: $(0, 0, -0, 0), (1, 0, -0, 1), (1, 0, -0, 0), (1, 1, -1, 0)$.

Let $T' = T_0(a + 1, 0, 0, 1)$, with $a = |U_1| + |U_2| + |U_3| + |U_5|$, $V(T') = V(T)$, and $E(T') = (E(T))^2 \cup \{u_2w: w \in U_2 \cup \{w: v \in U_4 \cup \{u_1w: w \in U_5\}\} \}$. Thus, $T'$ has a graceful labelling $f$ such that $f(u_3) = 0$, as illustrated in Figure 7(a).

Let $m = |E(T')| = a + 6$, $r = 3$, $m' = m - 3$, and $r + s = m$. Also, let $I = \{0, t - 1, t, t + 1\}$ and $N = \{n_i: i \in I\}$ with $n_0 = |U_1| + 1$, $n_t - 1 = |U_4|$, $n_t = |U_5|$, and $n_{t + 1} = |U_2|$. Note that ordered pair $(r, s, t, N)$ is a special pair since: (i) $r < s$ and $t = r + s$; (ii) $n_0 \geq 1$ and $\sum_{i \in I} n_i = a + 1 = s + r + 1$; (iii) if $T'$ is assigned $(0, 0, -0, 0)$ or $(1, 0, -0, 0)$, then $n_1$ is even for $i \in I \setminus \{0\}$; and (iv) if $T'$ is assigned $(1, 0, -0, 1)$ or $(1, 1, -1, 0)$, then $n_{t - 1} \equiv n_{t + 1} \equiv t \pmod{2}$ and $n_t \neq t \pmod{2}$. Moreover: (i) vertex $u_1 \in V(T')$ is adjacent to a set $S$ of leaves with labels 3, . . . , $m - 3$; and (ii) $u_2, u_4, u_5 \not\in S$, $f(u_1) + f(u_2) = t + 1$, $f(u_1) + f(u_4) = t + 1$, and $f(u_1) + f(u_5) = t$.

Therefore, by Lemma 17, for $i \in \{2, 4, 5\}$, we can safely transfer $|U_i|$ leaves from $u_1$ to $u_i$.

Case 2. Tree $T$ is assigned one of the following 5-tuples: $(1, 0, -1, 1), (0, 0, -0, 1)$.

Let $v \in U_4$; we modify $T$ so as to obtain another tree $T' = T_0(a + 1, 0, 0, 1)$, with $a = |U_1| + |U_2| + |U_3| + |U_5| - 1$, $V(T') = V(T)$, and $E(T') = (E(T))^2 \cup \{u_2w: w \in U_2 \cup \{w: v \in U_4 \cup \{u_1w: w \in U_5\}\} \}$. Thus, $T'$ has a graceful labelling $f$ such that $f(u_3) = 0$, as illustrated in Figure 7(b).

Let $m = |E(T')| = a + 7$ and $r, s, t$ be three positive integers such that $r = 4$, $s = m - 3$, and $t = r + s = m + 1$. Let $I = \{0, t - 1, t, t + 1\}$ be an index set and $N = \{n_i: i \in I\}$ be such that $n_0 = |U_1| + 1$, $n_t - 1 = |U_4|$, $n_t = |U_5| - 1$, and $n_{t + 1} = |U_2|$. Note that ordered pair $(r, s, t, N)$ is a special pair. Moreover: (i) vertex $u_1 \in V(T')$ is adjacent to a set $S$ of leaves with labels 4, . . . , $m - 3$; (ii) $u_2, u_4, u_5 \not\in S$, $f(u_1) + f(u_2) = t + 1$, $f(u_1) + f(u_4) = t$, and $f(u_1) + f(u_5) = t - 1$.

Figure 7: Scheme of four model-trees of diameter six with graceful labellings.
Therefore, by Lemma 17, for \( i \in \{2, 5\} \), we can safely transfer \(|U_i|\) leaves from \( u_1 \) to \( u_i \) and we can also safely transfer \(|U_4| - 1\) leaves from \( u_1 \) to \( u_4 \).

**Case 3.** Tree \( T \) is assigned one of the following 5-tuples: \((0, 0, -1, 1), (1, 1, -1, 1), (1, 0, -1, 1)\).

Let \( v_4 \in U_1 \) and \( v_5 \in U_2 \). Let \( T' = T_0(a + 1, 0, 0, 1, 2) \), with \( a = |U_1| + |U_2| + |U_4| + |U_5| - 2 \), \( V(T') = V(T) \), and \( E(T') = (E(T) \setminus \{u_2w : w \in U_2\} \cup \{u_4w : w \in U_4\} \cup \{u_5w : w \in U_5\}) \cup \{u_1w : w \in U_2 \cup (U_4 \setminus v_4) \cup (U_5 \setminus v_5)\} \). Thus, \( T' \) has a graceful labelling \( f \) such that \( f(u_3) = 0 \), as illustrated in Figure 7(c).

Let \( m = |E(T')| = a + 8, r = 4, s = m - 4, \) and \( t = r + s = m \). Let \( \mathcal{I} = \{0, t, t - 1\} \) and \( \mathcal{N} = \{n_i : i \in \mathcal{I}\} \) with \( n_0 = |U_1| + 1, n_{t-1} = |U_5| - 1, n_t = |U_2|, \) and \( n_{t+1} = |U_4| - 1 \). Then, using the same reasoning of the previous case, one can see that \((\{r, s, t\}, \mathcal{N})\) is a special pair. Moreover, vertex \( u_1 \in V(T') \) is adjacent to a set \( \mathcal{S} \) of leaves with labels \( 4, \ldots, m - 4; u_2, u_4, u_5 \notin \mathcal{S}, f(u_1) + f(u_2) = t, f(u_1) + f(u_4) = t + 1, \) and \( f(u_1) + f(u_5) = t - 1 \). Therefore, by Lemma 17, we can safely transfer \(|U_2|\) leaves from \( u_1 \) to \( u_2 \) and, for \( i \in \{4, 5\} \), we can safely transfer \(|U_i| - 1\) leaves from \( u_1 \) to \( u_i \).

**Case 4.** Tree \( T \) is assigned the 5-tuple \((0, 1, -1, 0, 0)\).

Let \( v_2 \in U_2 \) and \( v_4 \in U_4 \). Let \( T' = T_0(a + 1, 0, 1, 1, 0) \), with \( a = |U_1| + |U_2| + |U_4| + |U_5| - 2, V(T') = V(T) \), and \( E(T') = (E(T) \setminus \{u_2w : w \in U_2\} \cup \{u_4w : w \in U_4\} \cup \{u_5w : w \in U_5\}) \cup \{u_1w : w \in U_5 \cup (U_2 \setminus v_2) \cup (U_4 \setminus v_4)\} \). Thus, \( T' \) has a graceful labelling \( f \) such that \( f(u_3) = 0 \), as illustrated in Figure 7(d).

Let \( m = |E(T')| = a + 8, r = 4, s = m - 4, \) and \( t = r + s = m \). Let \( \mathcal{I} = \{0, t, t - 1\} \) and \( \mathcal{N} = \{n_i : i \in \mathcal{I}\} \) with \( n_0 = |U_1| + 1, n_{t-1} = |U_4| - 1, n_t = |U_5|, \) and \( n_{t+1} = |U_2| - 1 \). Note that \((\{r, s, t\}, \mathcal{N})\) is a special pair. Moreover, vertex \( u_1 \in V(T') \) is adjacent to a set \( \mathcal{S} \) of leaves with labels \( 4, \ldots, m - 4; u_2, u_4, u_5 \notin \mathcal{S}, f(u_1) + f(u_2) = t + 1, f(u_1) + f(u_4) = t - 1, \) and \( f(u_1) + f(u_5) = t \). Therefore, by Lemma 17, we can safely transfer \(|U_5|\) leaves from \( u_1 \) to \( u_5 \) and, for \( i \in \{2, 4\} \), we can safely transfer \(|U_i| - 1\) leaves from \( u_1 \) to \( u_i \).

**Case 5.** Tree \( T \) is assigned one of the following 5-tuples: \((0, 0, -0, 0, 1), (0, 1, -1, 1), (0, 1, -0, 0), (0, 1, -0, 0, 1), (1, 1, -0, 0, 1)\).

For \( a, b, c, d \) non-negative integers, \( T_6(a, b, 0, c, d) \) is isomorphic to \( T_6(d, c, 0, b, a) \). Thus, the trees in this case are isomorphic to trees treated in Case 1, Case 2, and Case 3, and the result follows, concluding the proof.

For the next lemma, consider the eight model-trees exhibited in Figure 8, each of which with a graceful labelling \( f \) with \( f(u_2) = 0 \).

**Lemma 21.** Let \( T \) be a caterpillar with diameter six and spine \( P = u_0u_1u_2u_3u_4u_5u_6 \). Also, let \( w \in \{u_2, u_4\} \cup N_1(u_2) \cup N_1(u_4) \). Then, \( T \) has a graceful labelling \( f \) such that \( f(w) = 0 \).

**Proof.** Let \( T \) be a caterpillar with diameter six and let \( P = u_0u_1u_2u_3u_4u_5u_6 \) be its spine. For each \( i \in \{1, 2, 3, 4, 5\} \), define \( U_i \) as the set of leaves from \( V(T) \setminus \{u_0, u_6\} \) that are adjacent to \( u_i \). We prove the result for \( u_2 \) and the proof for \( u_4 \) is analogous. As shown in the proof of Lemma 18, we can assume \( U_2 = \emptyset \).

In our proof, we consider eight cases depending on the parities of the \(|U_i|\)s. In order to do this, we assign \( T \) a 5-tuple \((p_1, -p_3, p_4, p_5)\) such that, for each \( i \in \{1, 3, 4, 5\} \), \( p_i \) is the parity of \(|U_i|\).

**Case 1.** Tree \( T \) is assigned one of the following 5-tuples: \((0, -0, 0, 0, 0), (0, -0, 0, 0, 1), (0, -1, 0, 1), (1, -0, 1, 1)\).
Thus, \( T \) and \( E \) can safely transfer \( N \) and \( f \), assigned. Let \( E = \{ T \} \). In this case, modify \( T \) so as to obtain another tree \( T' = T_6(1, 0, 0, 0, a + 1) \), with \( a = |U_1| + |U_3| + |U_4| + |U_5| \). Thus, as illustrated in Figure 8(a), \( T' \) has a graceful labelling \( f \) such that \( f(u_2) = 0 \). Next, we show how to safely transfer \( |U_i| \) leaves from \( u_5 \) to \( u_i \) for \( i \in \{1, 3, 4\} \).

Let \( m = |E(T')| = a + 6 \), \( r = 3 \), \( s = m - 3 \), and \( t = r + s = m \). Let \( I = \{0, t - 1, t, t + 1\} \) and \( N = \{n_i: i \in I\} \) with \( n_0 = |U_5| + 1 \), \( n_{t-1} = |U_4| \), \( n_t = |U_3| \), and \( n_{t+1} = |U_1| \). Note that \( \{r, s, t\}, \mathcal{N} \) is a special pair since: (i) \( r \leq s \) and \( t = r + s \); (ii) \( n_0 \geq 1 \) and \( \sum_{i \in I} n_i = a + 1 = s - r + 1 \); (iii) if \( T' \) is assigned \((0, -s, 0, 0)\) or \((0, -0, 0, 1)\), then \( n_i \) is even for \( i \in I \setminus \{0\}\); and (iv) if \( T' \) is assigned \((0, -1, 0, 1)\) or \((1, -0, 1, 1)\), then \( n_{t-1} \equiv n_{t+1} \equiv t \pmod{2} \) and \( n_t \neq t \pmod{2} \). Moreover, vertex \( u_5 \in V(T') \) is adjacent to a set of leaves \( \mathcal{S} \) with labels \( 3, \ldots, m-3 \), \( u_1, u_3, u_4 \not\in \mathcal{S} \), \( f(u_1) + f(u_3) = t + 1 \), \( f(u_3) + f(u_5) = t \), and \( f(u_4) + f(u_5) = t - 1 \). Therefore, by Lemma 17, we can safely transfer \( |U_i| \) leaves from \( u_5 \) to \( u_i \), for \( i \in \{1, 3, 4\} \).

Case 2. Tree \( T \) is assigned one of the following 5-tuples: \((0, -0, 1, 0)\), \((0, -0, 1, 1)\), \((1, -1, 1, 1)\).

Let \( v_4 \in U_4 \) and \( T' = T_6(1, 0, 0, 1, a + 1) \), with \( a = |U_1| + |U_3| + |U_4| + |U_5| - 1 \). Thus, \( T' \) has a graceful labelling \( f \) such that \( f(u_2) = 0 \), as illustrated in Figure 8(b).

Let \( m = |E(T')| = a + 7 \), \( r = 4 \), \( s = m - 3 \), and \( t = r + s = m + 1 \). Let \( I = \{0, t - 1, t, t + 1\} \) and \( N = \{n_i: i \in I\} \) with \( n_0 = |U_5| + 1 \), \( n_{t-1} = |U_3| \), \( n_t = |U_4| - 1 \), and \( n_{t+1} = |U_1| \). Note that \( \{r, s, t\}, \mathcal{N} \) is a special pair since: (i) \( r \leq s \) and \( t = r + s \); (ii) \( n_0 \geq 1 \) and \( \sum_{i \in I} n_i = a + 1 = s - r + 1 \); (iii) if \( T' \) is assigned \((0, -0, 1, 0)\) or \((0, -0, 1, 1)\), then \( n_i \) is even for \( i \in I \setminus \{0\}\); and (iv) if \( T' \) is assigned \((1, -1, 1, 1)\), then \( n_{t-1} \equiv n_{t+1} \equiv t \pmod{2} \) and \( n_t \neq t \pmod{2} \). Moreover, vertex \( u_5 \in V(T') \) is adjacent to a set of leaves \( \mathcal{S} \) with labels \( 4, \ldots, m-3 \), \( u_1, u_3, u_4 \not\in \mathcal{S} \), \( f(u_1) + f(u_5) = t + 1 \), \( f(u_3) + f(u_5) = t \), and \( f(u_4) + f(u_5) = t - 1 \). Therefore, by Lemma 17, we can safely transfer \( |U_i| \) leaves from \( u_5 \) to \( u_i \), for \( i \in \{1, 3, 4\} \).
$f(u_3) + f(u_5) = t - 1,$ and $f(u_4) + f(u_5) = t$. Therefore, by Lemma 17, for $i \in \{1, 3\}$, we can safely transfer $|U_i|$ leaves from $u_5$ to $u_i$ and we can also safely transfer $|U_4| - 1$ leaves from $u_5$ to $u_4$.

**Case 3.** Tree $T$ is assigned the 5-tuple $(1, -1, 1, 0)$.

Let $v_1 \in U_1$, $v_3 \in U_3$, and $v_4 \in U_4$. Modify $T$ so as to obtain $T' = T_6(2, 0, 1, 1, a + 1)$, with $a = |U_1| + |U_3| + |U_4| + |U_5| - 3$, $V(T') = V(T)$, and $E(T') = (E(T) \setminus \{u_4w: w \in U_1 \setminus \{v_1\}\} \cup \{u_3w: w \in U_3 \setminus \{v_3\}\} \cup \{u_4w: w \in U_4 \setminus \{v_4\}\})$ $\cup \{u_5w: w \in (U_1 \setminus \{v_1\}) \cup (U_3 \setminus \{v_3\}) \cup (U_4 \setminus \{v_4\})\}$. Thus, $T'$ has a graceful labelling $f$ such that $f(u_2) = 0$, as illustrated in Figure 8(c).

Let $m = |E(T')| = a + 9$, $r = 5$, $s = m - 4$, and $t = r + s = m + 1$. Let $I = \{0, t - 1, t, t + 1\}$ and $N = \{v_i: i \in I\}$ with $n_0 = |U_5| + 1$, $n_{t-1} = |U_4| - 1$, $n_t = |U_1| - 1$, and $n_{t+1} = |U_3| - 1$. Note that $\{(r, s, t), N\}$ is a special pair. Moreover, vertex $u_5 \in V(T')$ is adjacent to a set of leaves $S$ with labels $5, \ldots, m - 4, u_1, u_3, u_4 \notin S$, $f(u_1) + f(u_5) = t$, $f(u_3) + f(u_5) = t + 1$, and $f(u_4) + f(u_5) = t - 1$. Therefore, by Lemma 17, we can safely transfer $|U_i| - 1$ leaves from $u_5$ to $u_i$, $i \in \{1, 3, 4\}$.

**Case 4.** Tree $T$ is assigned the 5-tuple $(0, -1, 0, 0)$.

Let $v_3 \in U_3$. Consider $T' = T_6(1, 0, 1, 0, a + 1)$, with $a = |U_1| + |U_3| + |U_4| + |U_5| - 1$, $V(T') = V(T)$, and $E(T') = (E(T) \setminus \{u_4w: w \in U_1\} \cup \{u_3w: w \in U_3 \setminus \{v_3\}\} \cup \{u_4w: w \in U_4\}) \cup \{u_5w: w \in U_1 \cup (U_3 \setminus \{v_3\}) \cup (U_4 \setminus \{v_4\})\}$. Thus, $T'$ has a graceful labelling $f$ such that $f(u_2) = 0$, as illustrated in Figure 8(d).

Let $m = |E(T')| = a + 7$, $r = 4$, $s = m - 3$, and $t = r + s = m + 1$. Let $I = \{0, t - 1, t, t + 1\}$ and $N = \{v_i: i \in I\}$ with $n_0 = |U_3| + 1$, $n_{t-1} = |U_4|$, $n_t = |U_3| - 1$, and $n_{t+1} = |U_1|$. Note that $\{(r, s, t), N\}$ is a special pair. Moreover, vertex $u_5 \in V(T')$ is adjacent to a set of leaves $S$ with labels $4, \ldots, m - 3, u_1, u_3, u_4 \notin S$, $f(u_1) + f(u_5) = t + 1$, $f(u_3) + f(u_5) = t$, and $f(u_4) + f(u_5) = t - 1$. Therefore, by Lemma 17, for $i \in \{1, 4\}$, we can safely transfer $|U_i|$ leaves from $u_5$ to $u_i$ and we can also safely transfer $|U_3| - 1$ leaves from $u_3$ to $u_3$.

**Case 5.** Tree $T$ is assigned one of the following 5-tuples: $(0, -1, 1, 0)$, $(0, -1, 1, 1)$.

Let $v_3 \in U_3$ and $v_4 \in U_4$. Consider $T' = T_6(1, 0, 1, 1, a + 1)$, with $a = |U_1| + |U_3| + |U_4| + |U_5| - 2$, $V(T') = V(T)$, and $E(T') = (E(T) \setminus \{u_4w: w \in U_1\} \cup \{u_3w: w \in U_3 \setminus \{v_3\}\} \cup \{u_4w: w \in U_4\}) \cup \{u_5w: w \in U_1 \cup (U_3 \setminus \{v_3\}) \cup (U_4 \setminus \{v_4\})\}$. Thus, $T'$ has a graceful labelling $f$ such that $f(u_2) = 0$, as illustrated in Figure 8(e).

Let $m = |E(T')| = a + 8$, $r = 4$, $s = m - 4$, and $t = r + s = m$. Let $I = \{0, t - 1, t, t + 1\}$ and $N = \{v_i: i \in I\}$ with $n_0 = |U_3| + 1$, $n_{t-1} = |U_4| - 1$, $n_t = |U_1|$, and $n_{t+1} = |U_3| - 1$. Note that $\{(r, s, t), N\}$ is a special pair. Moreover, vertex $u_5 \in V(T')$ is adjacent to a set of leaves $S$ with labels $4, \ldots, m - 4, u_1, u_3, u_4 \notin S$, $f(u_1) + f(u_5) = t$, $f(u_3) + f(u_5) = t + 1$, and $f(u_4) + f(u_5) = t - 1$. Therefore, by Lemma 17, for $i \in \{3, 4\}$, we can safely transfer $|U_i| - 1$ leaves from $u_5$ to $u_i$, and we can also safely transfer $|U_3| - 1$ leaves from $u_3$ to $u_1$.

**Case 6.** Tree $T$ is assigned the 5-tuple $(1, -0, 1, 0)$.

Let $v_1 \in U_1$ and $v_4 \in U_4$. Let $T' = T_6(2, 0, 0, 1, a + 1)$, with $a = |U_1| + |U_3| + |U_4| + |U_5| - 2$, $V(T') = V(T)$, and $E(T') = (E(T) \setminus \{u_4w: w \in U_1 \setminus \{v_1\}\} \cup \{u_3w: w \in U_3\} \cup \{u_4w: w \in U_4 \setminus \{v_4\}\}) \cup \{u_5w: w \in (U_1 \setminus \{v_1\}) \cup U_3 \cup (U_4 \setminus \{v_4\})\}$. Thus, $T'$ has a graceful labelling $f$ such that $f(u_2) = 0$, as illustrated in Figure 8(f).

Let $m = |E(T')| = a + 8$, $r = 5$, $s = m - 3$, and $t = r + s = m + 2$. Let $I = \{0, t - 1, t, t + 1\}$ and $N = \{v_i: i \in I\}$ with $n_0 = |U_5| + 1$, $n_{t-1} = |U_4| - 1$, $n_t = |U_3|$, and $n_{t+1} = |U_1| - 1$. Note that $\{(r, s, t), N\}$ is a special pair. Moreover, vertex $u_5 \in V(T')$ is adjacent to a set of leaves $S$ with labels $5, \ldots, m - 3, u_1, u_3, u_4 \notin S$, $f(u_1) + f(u_5) = t + 1$, $f(u_3) + f(u_5) = t$, and $f(u_4) + f(u_5) = t - 1$. Therefore, by Lemma 17, for $i \in \{1, 4\}$, we can safely transfer $|U_i| - 1$ leaves from $u_5$ to $u_i$ and we can also safely transfer $|U_3| - 1$ leaves from $u_3$ to $u_3$. 

Some families of 0-rotatable graceful caterpillars
Case 7. Tree $T$ is assigned one of the following 5-tuples: $(1, -1, 0, 0)$, $(1, -1, 0, 1)$.

Subcase 1. $|U_4| = 0$.

Let $v_1 \in U_1$ and $v_3 \in U_3$. Modify $T$ so as to obtain $T' = T_0(2,0,1,a+1)$, with $a = |U_1| + |U_3| + |U_4| + |U_5| - 2$, $V(T') = V(T)$, and $E(T') = (E(T) \setminus \{(u_1w: w \in U_1 \setminus v_1) \cup \{u_3w: w \in U_3 \setminus v_3\}\}) \cup \{u_5w: w \in (U_1 \cup v_1) \cup (U_3 \setminus v_3)\}$. Thus, $T'$ has a graceful labelling $f$ such that $f(u_2) = 0$, as illustrated in Figure 8(g).

Let $m = |E(T')| = a + 8$, $r = 5$, $s = m - 3$, and $t = r + s = m + 2$. Let $I = \{0, t - 1, t, t + 1\}$ and $N = \{n_i: i \in I\}$ with $n_0 = |U_5| + 1$, $n_{t-1} = |U_1| - 1$, $n_t = |U_3| - 1$, and $n_{t+1} = 0$. Note that $(\{r,s,t\}, N)$ is a special pair. Moreover, vertex $u_5 \in V(T')$ is adjacent to a set of leaves $S$ with labels 5, $\ldots$, $m - 3$, $u_1, u_3 \notin S$, $f(u_1) + f(u_5) = t - 1$ and $f(u_3) + f(u_5) = t$. Therefore, by Lemma 17, we can safely transfer $|U_i| - 1$ leaves from $u_5$ to $u_i$, $i \in \{1, 3\}$.

Subcase 2. $|U_4| \geq 2$.

Let $v_1 \in U_1$, $v_3 \in U_3$, and $v_1^1, v_3^2 \in U_4$. Consider $T' = T_0(2,0,1,a+1)$, with $a = |U_1| + |U_3| + |U_4| + |U_5| - 4$, $V(T') = V(T)$, and $E(T') = (E(T) \setminus \{(u_1w: w \in U_1 \setminus v_1) \cup \{u_3w: w \in U_3 \setminus v_3\}\}) \cup \{u_4w: w \in U_4 \setminus \{v_1^1, v_3^2\}\}) \cup \{u_5w: w \in (U_1 \cup v_1) \cup (U_3 \setminus v_3) \cup (U_4 \setminus \{v_1^1, v_3^2\})\}$. Thus, $T'$ has a graceful labelling $f$ such that $f(u_2) = 0$, as illustrated in Figure 8(h).

Let $m = |E(T')|$, $r = 7$, $s = m - 3$, and $t = r + s = m + 4$. Let $I = \{0, t - 1, t, t + 1\}$ and $N = \{n_i: i \in I\}$ with $n_0 = |U_5| + 1$, $n_{t-1} = |U_3| - 1$, $n_t = |U_4| - 2$, and $n_{t+1} = |U_1| - 1$. Note that $(\{r,s,t\}, N)$ is a special pair. Moreover, vertex $u_5 \in V(T')$ is adjacent to a set of leaves $S$ with labels 7, $\ldots$, $m - 3$, $u_1, u_3, u_4 \notin S$, $f(u_1) + f(u_5) = t + 1$, $f(u_3) + f(u_5) = t - 1$, and $f(u_4) + f(u_5) = t$. Therefore, by Lemma 17, for $i \in \{1, 3\}$, we can safely transfer $|U_i| - 1$ leaves from $u_5$ to $u_i$ and we can also safely transfer $|U_4| - 2$ leaves from $u_5$ to $u_4$.

Case 8. Tree $T$ is assigned one of the following 5-tuples: $(1, -1, 0, 0)$, $(1, -1, 0, 1)$.

Let $T$ be as in the hypothesis. We modify $T$ in order to obtain another tree $T'$, with $V(T') = V(T)$ and $E(T') = (E(T) \setminus \{(u_1w: i \in \{1, 4, 5\}, w \in U_1 \cup \{u_0, u_6\}\}) \cup (u_3w: w \in U_1 \cup U_4 \cup U_5 \cup \{u_0, u_6\})$. Figure 9 shows a scheme of $T'$ with a graceful labelling $f$ such that $f(u_2) = 0$. Note that $u_3$ is adjacent to exactly $|U_1| + |U_3| + |U_4| + |U_5| + 2 = m - 4$ leaves. Next, we show how to perform a sequence of transfers in $T'$ so as to obtain a graceful labelling for $T$.

![Figure 9: Scheme of tree $T'$ with a graceful labelling.](image)

Since $f(u_3) + f(u_5) = m + 1$ and vertex $u_3$ is adjacent to vertices with labels 3, $\ldots$, $m - 2$, by Lemma 7, we can safely transfer all the $m - 4 - |U_3|$ leaves with labels in the interval $[3 + \frac{|U_3|}{2}, m - 2 - \frac{|U_3|}{2}]$ from vertex $u_3$ to vertex $u_5$. After this transfer, $u_3$ is adjacent to $|U_3|$ leaves and $u_5$ is adjacent to exactly $|U_1| + |U_4| + |U_5| + 2$ leaves.

Now, consider $a = f(u_5) = 1$, $b = f(u_1) = m - 1$, $r_1 = 1$, $r_2 = 0$, and the set $S$ of leaves adjacent to $u_5$ with labels $3 + \frac{|U_3|}{2}, \ldots, m - 2 - \frac{|U_4|}{2}$. By Lemma 8, it is possible to perform a sequence of safe transfers $u_5 \rightarrow u_1 \rightarrow u_4$, such that the resulting tree has $|U_5| + 1$ leaves at vertex $u_5$, $|U_1| + 1$ leaves at vertex $u_1$, and $|U_4|$ leaves at vertex $u_4$. This concludes the proof. 

Theorem 22. If $T$ is a caterpillar with diameter six, then $T$ is 0-rotatable.
Proof. The result follows from Lemma 6, Lemma 20 and Lemma 21.

References


