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Abstract

In order to discover how Quantum Mechanics would improve the power of computational devices, Quantum Computation was proposed by Feynman [6] in 1982 and since then some important breakthroughs in this area were achieved. The two main models, Quantum Turing Machines [4] and Quantum Circuits [5], allowed computer scientists, mathematicians and physicists to develop algorithms that extract computational power from the quantum structure of matter and compare their efficiency with the classical ones. The most expressive results were Shor's Algorithm [19], that factorizes integers in polynomial time, and Grover's Algorithm [8], that presents a quadratic speedup for unordered database search problem.

Within this context, the topic of Quantum Automata was proposed to study the influence of Quantum Mechanics in simpler computational models. Several quantum counterparts of classical automata were proposed, whose differences usually involve characteristics of the tape head and interaction with external elements, such as quantum and classical states and pushdown stacks.

In this work, we consider two-way finite automata with quantum and classical states (2QCFA). The classical states control the input acceptance, and define the tape head movement and the operations that will be applied on the quantum states, which are used as a quantum memory. Intuitively, it can be seen as a classical two-way deterministic finite automata (2DFA) with a fixed number of quantum bits. Given that quantum measurements are intrinsically probabilistic, we can calculate the probabilities of acceptance and rejection of a string w by a 2QCFA M and the expected number of steps for halting. We denote by $\mathcal{L}_\epsilon$ as the set of languages for which there is a 2QCFA that accepts all strings in the language with probability 1 and reject the strings not in the language with probability $1-\epsilon$, where $\epsilon > 0$. With this definition, it is not surprising that all regular languages can be recognized by 2QCFAs. On the other hand, it has already been shown that the model can recognize some context-free languages and even some non-context-free languages.

Ambainis and Watrous [2], in their original paper that proposed the 2QCFA model, present a 2QCFA that recognizes the deterministic context-free language $\{a^n b^n | n \geq 0\}$ in polynomial time. As noted by Qiu [15], simple variations of the 2QCFA recognize some similar languages, like $\{a^{kn} b^n | n > 0 | k \geq 1\}$ and $\{\{a,b\}^* |$ number of a's is equal to the number of b's $\}$. The language of palindromes over the alphabet $\{a, b\}$, a non-ambiguous non-deterministic context-free language, has also been shown to be

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recognized by 2QCFAs by Ambainis and Watrous [2], but in exponential time. It was also shown by Zheng *et al* [18] that the 2QCFA model recognizes the non-context-free languages $\{wcw|w \in \{a,b\}^*\}$ in exponential time, and by Grilo and Moura [7] that $\{a^n b^n c^n | n \geq 0\}$ is recognized in polynomial time.

In addition to specific language recognition by 2QCFAs, the literature also presents some closure properties of the set $\mathcal{L}_\epsilon$. Qiu [15] proved concatenation closure under some constraints, and also union, intersection and reversion closures. Macko [12] proved that the languages recognized by a somewhat different quantum automata model are closed under inverse homomorphism, and this result is easily extended to $\mathcal{L}_\epsilon$. Moreover, Macko [12] also proved that $\mathcal{L}_\epsilon$ is closed under homomorphism if and only if the model recognizes all the recursively enumerable languages.

We are currently looking for decidable languages not recognizable by 2QCFAs. Ideally, we would like to propose a generic method for proving that no 2QCFA could recognize an arbitrary language, as the Pumping Lemma does for the Regular Languages. Also, we are also interested in partial achievements like the proof that a specific language is not recognizable by 2QCFAs.

We conjecture that the language of balanced parenthesis and the language $L_< = \{a^i b^j | i < j\}$ cannot be recognized by 2QCFA. We are studying the recognition for the latter because (i) it is more well-behaved and (ii) there is a 2QCFA that recognizes its similar language $\{a^n b^n | n \geq 0\}$. We still have not proved that $L_<$ cannot be recognized by 2QCFA model but we have some partial results. We have proved that the 1QCFA model[1] cannot accept all $w \in L_<$ with probability 1 and reject all $w \notin L_<$ with non-zero probability. This result excludes the technique used in the 2QCFA to recognize $\{a^n b^n | n \geq 0\}$, which consists in using a 1QCFA multiple times in order to amplify the rejection probability.

Now we detail the results stated in the extended abstract. This work is the continuation of a paper that appeared in the VI Workshop-School in Quantum Computation and Information [7]. Here, we have extended it, including the results involving closure properties of 2QCFAs and some partial results on the computability of the model. In this work we present proof sketches only, referencing the original papers for further details, where necessary.

We begin by presenting some related works, followed by a formal description of the two-way finite automata with classical and quantum states (2QCFA) model. In Section 3, we show results involving closure properties of languages recognized by 2QCFAs. Then, we present some languages recognized by the model in Section 4 and some partial results of non-recognizability in Section 5. Finally, we conclude with some open problems.

We assume a basic knowledge of quantum computation and formal languages. For a revision of quantum computation, we recommend Nielsen and Chuang [14] as a more complete and broad reference, or Yanofsky and Mannucci [21] for a reference closer to Computer Scientists. Formal languages and automata are well presented in Hopcroft and Ullman [9] and Sipser [20].

[1] 1QCFA is a 2QCFA whose tape head is only allowed to move right

1 Related works

Initially, there were two independent definitions of Quantum Finite Automata: the Measure-Once one-way quantum finite automata (MO-1QFA), proposed by Moore and Crutchfield [13], and the Many-Measure one-way quantum finite automata (MM-1QFA), proposed by Kondacs and Watrous [11]. In both approaches, the set of languages recognized with bounded error by 1QFA was proved to be strictly contained in the set of regular languages.

Kondacs and Watrous [11] also defined the two-way quantum finite automata (2QFA) model, in which the head can move left, right or stay in the same tape cell at each move. The 2QFA model was proven to be strictly stronger than the deterministic finite automata model [11]. This model supports only quantum states, permitting the head to be at different cells of the input tape at the same moment. This characteristic brings up the problem that it could only be implemented with $\log |w|$ bits of information, for any input w. In order to solve this problem, Ambainis and Watrous [2] have proposed a variant of the 2QFA, called two-way finite automata with quantum and classical states (2QCFA). The difference between these models is that, while a configuration of a 2QFA consists of the superposition of quantum states and head positions, a 2QCFA configuration is defined by a classical state, the position of the head and a superposition of quantum states.

Ambainis and Watrous [2] proposed, in their original paper, 2QCFAs for two non-regular context-free languages (CFLs). Macko [12] studied some closure properties for 1.5QFA[2], and some of his results can be easily extended to 2QCFA. Then, Qiu [15] proved some closure properties about the class of languages recognized by 2QCFAs, and used these results to derive 2QCFAs for recognizing other languages. Later Zheng *et al* [17][18] proposed 2QCFA for accepting some other non-regular languages, including a non-context-free language. Grilo and Moura [7] have compared the previous results with the classical hierarchy and extended their results showing new languages recognized by the 2QCFA model.

2 The 2QCFA model

In this section we will describe formally the 2QCFA model and define the types of recognition that will be used in this paper.

Formally, a 2QCFA is a 9-tuple $M = (Q, S, \Sigma, \Theta, \delta, q_0, s_0, S_{acc}, S_{rej})$, where Q is the set of quantum states, S is the set of classical states, Σ is the input alphabet, Θ and δ are the evolution functions for the quantum and classical states, respectively, $q_0 \in Q$ and $s_0 \in S$ are the initial states for the quantum and classical states, respectively, and $S_{acc}, S_{rej} \subseteq S$ are the sets of accept and reject classical states. We assume $S_{acc} \cap S_{rej} = \emptyset$.

Definition 2.1 *The set of halting states is $S_{halt} = S_{acc} \cup S_{rej}$ and the set of non-halting states is $S_{non} = S \backslash S_{halt}$.*

For an input w, the tape has $|w| + 2$ squares: the first one, indexed by 0, contains the left-end marker †, the last one contains the right-end marker ‡ and the i-th position,

[2] 1.5QFA is an intermediate model between 1QFA and 2QFA

$1 \leq i \leq |w|$, contains the i-th symbol of the input w. We assume $\dagger$, $\ddagger \notin \Sigma$ and we call $\Gamma = \Sigma \cup \{\dagger, \ddagger\}$ the tape alphabet.

As dictated by quantum mechanics, an operation over quantum systems can be either a linear unitary transformation or a measurement. In the 2QCFA model, the action performed over the quantum states is determined by the classical state and the tape symbol under the head. So, the quantum evolution function Θ is defined as $\Theta : S_{non} \times \Gamma \to \mathcal{U}(\mathcal{H}(Q)) \cup \mathcal{M}(\mathcal{H}(Q))$, where $\mathcal{H}(Q)$ is the complex Hilbert space in which Q is a basis, $\mathcal{U}(\mathcal{S})$ is the set of unitary linear operations over the Hilbert space $\mathcal{S}$ and $\mathcal{M}(\mathcal{S})$ is the set of projective measurements[3] over the Hilbert space $\mathcal{S}$.

The transition function δ depends on whether Θ applies a measurement or a linear unitary transformation. In the former case, δ will map $S_{non} \times \Gamma \times R$ to $S \times D$, where R is the set of possible results of the measurement and D is the set of possible movements of the head, $D = \{-1, 0, 1\}$[4]. When Θ applies a linear unitary transformation, δ will use only the classical information of the current configuration, that is $\delta : S_{non} \times \Gamma \to S \times D$.

The initial configuration of a 2QCFA M is the classical initial state s_0, the head position over cell 0 and the quantum initial state is $|q_0\rangle$. Then, the computation of the 2QFCA consists of applying, consecutively, the transformation Θ and then the transformation δ, until the automaton reaches one of the classical halting states.

Since δ may depend on the result of quantum measurements, which are intrinsically probabilistic, the classical part of the 2QCFA also has a probabilistic nature. We say that the 2QCFA accepts a string w with probability p_{acc} if it enters an accept state with probability p_{acc} while computing over w. We can define the analog rejection probability p_{rej} as the probability that the 2QCFA enters in a reject state while computing over w. If we assume that the 2QCFA halts for all inputs, then the acceptance and rejection probabilities sum up to 1.

Definition 2.2 *A 2QCFA M recognizes a language L with zero error if M accepts w with probability 1 when $w \in L$ and M rejects w with probability 1 when $w \notin L$.*

Definition 2.3 *A 2QCFA M recognizes a language L with one-sided error ϵ if M accepts w with probability 1 when $w \in L$ and M rejects w with probability at least $1-\epsilon$ when $w \notin L$.*

Remark Another model in use is the 1QCFA. It can be seen as a 2QCFA in which the tape head always move to the right and when it reaches the right-end marker, the 1QCFA must either accept or reject the input string.

3 Closure

In this section we will discuss some closure properties for the set of languages recognized by 2QCFAs with a one-sided error. We will call this set of languages $\mathcal{L}_\epsilon$. Since a 2QCFA is usually parametrized by the desired error, which influences the expected halting time, it will follow that the error can be selected as small as desired.

[3]The use of POVM-type measurements does not change the power of the computational device [2]

[4]We assume that whenever the head is in the first(last) cell, it never moves left(right).

We start this Subsection showing that $\mathcal{L}_\epsilon$ is closed under the union operation.

Theorem 3.1 *If L_1 and L_2 are recognized by 2QCFAs with one-sided error ϵ in time $O(t_1(|w|))$ and $O(t_2(|w|)$, respectively, then $L_1 \cup L_2$ is recognized by a 2QCFA with one-sided error $2\epsilon - \epsilon^2$ in time $O(t_1(|w|) + t_2(|w|) + |w|)$.*

Sketch of proof. Let M_1 and M_2 be, respectively, 2QCFAs that recognize the languages L_1 and L_2 with one-sided error ϵ. We will construct a 2QCFA M that accepts the language $L_1 \cup L_2$ with one-sided error $2\epsilon - \epsilon^2$.

M starts by simulating the 2QCFA M_1. If, in this simulation, M_1 would accept the input string, M accepts as well. On the other hand, if M_1 would reject the input string, M stops simulating M_1 and starts simulating M_2. During the simulation of M_2, M will accept or reject the input string according to what M_2 would do.

If the input w is in $L_1 \cup L_2$, it is easy to see that M will accept it. However, if w is not in $L_1 \cup L_2$, the probability of accepting it is $1 - (1-\epsilon)^2 = 2\epsilon - \epsilon^2$.

If M_1 and M_2 halt in expected time $O(t_1(|w|))$ and $O(t_2(|w|))$, respectively, it is easy to see that M runs in expected time $O(t_1(|w|) + t_2(|w|) + |w|)$.

For further details of this proof, we recommend Theorem 2 of Qiu [15]. □

Next, we show a similar result for closure under intersection.

Theorem 3.2 *If L_1 and L_2 are recognized by 2QCFAs with one-sided error ϵ in time $O(t_1(|w|))$ and $O(t_2(|w|)$, respectively, then $L_1 \cap L_2$ is recognized by a 2QCFA with one-sided error $2\epsilon - \epsilon^2$ in time $O(t_1(|w|) + t_2(|w|) + |w|)$.*

Sketch of proof. Let M_1 and M_2 be, respectively, 2QCFAs that recognize the languages L_1 and L_2 with one-sided error ϵ. We will construct a 2QCFA M that accepts the language $L_1 \cap L_2$ with one-sided error $2\epsilon - \epsilon^2$.

The operation of M is very similar to the 2QCFA proposed in the proof of Theorem 3.1, the difference is that, while simulating M_1, M will reject the input string whenever M_1 would do, and instead of accepting it when M_1 would do, M starts to simulate M_2. In the simulation of M_2, M will accept or reject the input string according to what M_2 would do.

So, if $w \in L_1 \cap L_2$, M will accept it with probability 1. If $w \in L_1$ but $w \notin L_2$ (and in the symmetric case $w \notin L_1$ and $w \in L_2$), M will accept it with probability ϵ. If $w \notin L_1$ and $w \notin L_2$, M will accept it with probability $1 - (1-\epsilon)^2 = 2\epsilon - \epsilon^2$.

If M_1 and M_2 halt in expected time $O(t_1(|w|))$ and $O(t_2(|w|))$, respectively, it is easy to see that M runs in expected time $O(t_1(|w|) + t_2(|w|) + |w|)$.

For further details of this proof, we recommend Theorem 1 of Qiu [15]. □

If the alphabet of two languages recognized by 2QCFA are disjoint, their concatenation is also recognized by a 2QCFA, as shown by the next theorem.

Theorem 3.3 *Let $L_1 \subseteq \Sigma_1^*$ and $L_2 \subseteq \Sigma_2^*$, $\Sigma_1 \cap \Sigma_2 = \emptyset$. If L_1 and L_2 are recognized by 2QCFAs with one-sided error ϵ in time $O(t_1(|w|))$ and $O(t_2(|w|)$, respectively, then L_1L_2 is recognized by a 2QCFA with one-sided error $2\epsilon - \epsilon^2$ in time $O(t_1(|w|) + t_2(|w|) + |w|)$.*

Sketch of proof. Let M_1 and M_2 be, respectively, 2QCFAs that recognize the languages L_1 and L_2 with one-sided error ϵ. We will construct a 2QCFA M that accepts the language L_1L_2 with one-sided error $2\epsilon - \epsilon^2$.

M starts checking if $w = xy$, $x \in \Sigma_1^*$ and $y \in \Sigma_2^*$, and reject if it is not the case. Then, it simulates M_1 over x and treats all symbols in Σ_2 as right-end markers. If M_1 would accept the input string, then M starts simulating M_2 over y, treating all symbols in Σ_1 as left-end markers. If M_1 would reject x, M rejects w as well. While simulating M_2, M accepts or rejects w following what M_2 does when computing over y.

So, when w is not in the form xy it is rejected with probability 1. If $x \in L_1$ and $y \in L_2$, M will accept w with probability 1. If $x \in L_1$ but $y \notin L_2$ (and in the symmetric case $x \notin L_1$ and $y \in L_2$), M will accept it with probability ϵ. If $x \notin L_1$ and $y \notin L_2$, M will accept it with probability $1 - (1-\epsilon)^2 = 2\epsilon - \epsilon^2$.

If M_1 and M_2 halt in expected time $O(t_1(|w|))$ and $O(t_2(|w|))$, respectively, it is easy to see that M runs in expected time $O(t_1(|w|) + t_2(|w|) + |w|)$.

For further details of this proof, we recommend Theorem 5 of Qiu [15]. □

We will now show the result that $\mathcal{L}_\epsilon$ is closed under reversion.

Theorem 3.4 *If L is recognized by 2QCFA with one-sided error ϵ in time $O(t(|w|))$, then L^R is recognized by a 2QCFA with one-sided error ϵ in time $O(t(|w|) + |w|)$.*

Sketch of proof. As long as the tape head can move both ways, given a 2QCFA M that recognizes L with one-sided error ϵ, the 2QCFA M^R that recognizes L^R starts by moving to the last cell and then it simulates M, but reversing tape head movements. Clearly M^R will accept w with the same probability that M accepts w^R.

If M halts in expected time $O(t(|w|))$, it is easy to see that M^R runs in expected time $O(t(|w|) + |w|)$.

For further details of this proof, we recommend Theorem 4 of Qiu [15]. □

Finally, we present an extension of the work of Macko [12], that shows that languages recognized by 1.5QFAs[5] are closed under inverse homomorphisms.

Theorem 3.5 *Let $h : \Sigma' \to \Sigma^*$ be an homomorphism. If L is recognized by 2QCFA with one-sided error ϵ in time $O(t(|w|))$, then $h^{-1}(L)$ is recognized by a 2QCFA with one-sided error ϵ in time $O(t(|w|))$.*

Sketch of proof. Let M be a 2QCFA that recognizes the language L over the alphabet Σ^* with one-sided error ϵ. The idea for a 2QCFA M' that recognizes the language $h^{-1}(L)$ is similar to the one appearing in classical proofs that regular languages are closed under inverse homomorphisms.

In short, M' will simulate M by applying the classical and quantum transitions for $h(\sigma)$ whenever the tape head is over the symbol σ. We notice that there might be some auxiliary states that will be used in the simulation when there are measurements during the computation of $h(\sigma)$ by M.

[5] 1.5QFA is an intermediate model between 1QFA and 2QFA

If M halts in expected time $O(t(|w|))$, it is easy to see that M' runs in expected time $O(t(|w|))$ too.

Finally, it is easy to see that M' accepts w with the same probability that M would accept $h(w)$.

For further details of the proof for the closure of languages recognized by 1.5QFA under inverse homomorphism, we recommend Section 4 of Macko [12]. $\square$

4 Computability

In this section we discuss the computational power of 2QCFA, showing languages in different classes of the classical hierarchy that are recognized by 2QCFA. In Section 4.1, we prove that all regular languages can be recognized by 2QCFA. Then, in Section 4.2, we present some 2QCFAs that recognize different categories of CFLs. Finally, in Section section 4.3 we present two 2QCFA that recognize two non-CFLs.

4.1 Regular languages

The class of regular languages comprise those languages that can be recognized by deterministic finite automata (DFAs), one of the simplest computational models[6]. In this section we prove a straightforward theorem showing that all regular languages can be recognized by 2QCFA with zero error, and in linear time. The idea of the proof is to simply simulate a DFA that recognizes the language, ignoring the quantum states.

Theorem 4.1 *If L is a regular language, there is a 2QCFA M that recognizes L with zero error in linear time.*

Sketch of proof. Given a DFA D that recognizes L, construct a 2QCFA M whose classical transition function is equal to the transition function of D and whose quantum transition function consists in applying the identity transformation. When the tape head reaches the right-end marker, M accepts the input string if the current classical state is an accepting state of D, and reject it otherwise. $\square$

4.2 Context-free languages

Context-free languages (CFLs) are those languages recognized by push-down automata (PDA) which, informally, are non-deterministic machines with a finite control and an infinite push-down tape. Equivalently, the set of CFLs is the set of languages generated by context-free grammars (CFGs)[7].

In this section, we start with a deterministic CFL recognizable with one-sided error by 2QCFA. Then, in Subsection 4.2.2 we show two 2QCFAs that accept different non-deterministic and unambiguous CFLs. In the last subsection we show a 2QCFA that recognizes an inherently ambiguous CFL.

[6]A general introduction to regular languages and DFA is in chapter 1 of Sipser [20].

[7]A general introduction to CFLs, PDAs and CFGs can be found in chapter 2 of Sipser [20].

Remark When we say that a 2QCFA recognizes a language, we mean it recognizes the language with one-sided error probability ϵ, $0 < \epsilon < \frac{1}{2}$, unless we explicitly state otherwise.

4.2.1 2QCFAs and deterministic context-free languages

A deterministic context-free language (DCFL) is a language that can be recognized by a deterministic push-down automata (DPDA). The class of DCFLs are a proper subset of the context-free languages. Their importance in Computer Science stems from the fact that it is possible to construct efficient parsers for some subclasses of them[8].

Some DCFLs have been proven to be recognized by 2QCFA in polynomial time; such as the language in which the number of a's is equal to the number of b's [15], and the language of strings in the form xcy, wher $x, y \in \{a, b\}^*$ and $|x| = |y|$ [17].

In this subsection, we present a 2QCFA recognizing the deterministic CFL $\{a^n b^n | n \geq 0\}$ [2].

Theorem 4.2 *There is a 2QCFA M that recognizes the language $\{a^n b^n | n \geq 0\}$ with one-sided error ϵ, $0 < \epsilon < \frac{1}{2}$, and in time $O(|w|^4)$, where w is the input string.*

Sketch of proof. The quantum states of M are formed by one *qubit*, with initial state $|0\rangle$. First, the automata will reject the input string if it is not in the form $a^i b^j$, for some $i, j \geq 0$. After that, it will start checking if $i = j$, scanning the input string once and applying rotation operations on the plane spanned by the basis $\{|0\rangle, |1\rangle\}$. If the tape head is over an a, the state is rotated by an angle of $\sqrt{2}\pi$ and if the current input symbol is a b, the rotation angle is taken to be $-\sqrt{2}\pi$.

When the tape head reaches the last cell, a measurement is made. If $i = j$ after scanning the whole input string, the successive application of rotations will be equivalent to the identity transformation and the probability of measuring $|0\rangle$ will be 1. However, if $i \neq j$, there will be a non-zero probability of measuring $|1\rangle$. So, if the result of the measurement is $|1\rangle$, the input string is reject. Otherwise, a method based on random walks and coin flips, parametrized by the desired error probability ϵ, is performed. The end result will be the acceptance of the input string with a non-zero probability. If this method does not accept the input string, the whole process is repeated.

It can be proved, based on the rejection and the acceptance probabilities of each iteration, that the automata stops in $O(|w|^4)$ expected steps.

For more details, see Section 4 of Ambainis and Watrous [2]. $\square$

Remark It is easy to see that the language $L^k_= = \{a^{kn} b^n | n \geq 0\}, k \in \mathbb{N}$, for a fixed k, is recognizable by a 2QCFA.

The difference between the 2QCFA for L_k and the one presented in Theorem 4.2 is that the rotation angle for the symbol a is now taken to be $\frac{\sqrt{2}\pi}{k}$.

Remark Using Theorem 3.3 and Theorem 4.2 it can be shown that the languages $\{a^n b^n c^* | n \geq 0\}$ and $\{a^* b^n c^n | n \geq 0\}$ are recognizable by 2QCFA.

[8]More details about DCFLs and DPDAs can be found in chapter 10 of Hopcroft and Ullman [9].

4.2.2 2QCFAs and non-deterministic unambiguous context-free languages

Non-deterministic CFLs are the ones that cannot be recognized by deterministic PDAs. An unambiguous CFL is a language that can be generated by an unambiguous CFG[9]. We note that the class of DCFLs is a proper subset of the class of unambiguous CFLs, and the latter is a proper subclass of all non-deterministic CFLs.

Some results about recognition of non-deterministic unambiguous CFLs by 2QCFAs have been obtained. We show in this subsection a 2QCFA that recognizes the language $\{a^n b^n | n \geq 0\} \cup \{a^{2n} b^n | n \geq 0\}$ in polynomial time and another 2QCFA that recognizes palindromes over the alphabet $\{a, b\}$ in exponential time.

4.2.3 The language $\{a^n b^n | n \geq 0\} \cup \{a^{2n} b^n | n \geq 0\}$

Using results from the previous sections, we show that the language $L = \{a^n b^n | n \geq 0\} \cup \{a^{2n} b^n | n \geq 0\}$[10] is recognized by a 2QCFA with an arbitrarily small one-sided error.

Theorem 4.3 *There is a 2QCFA that accepts the language* $\{a^n b^n | n \geq 1\} \cup \{a^{2n} b^n | n \geq 1\}$ *with one-sided error probability* δ *in polynomial time, where* $0 < \delta < \frac{1}{2}$*.*

Sketch of proof. Let $\epsilon = 1 - \sqrt{1 - \delta}$. From Theorem 4.2 and Remark 4.2.1, we know that there are 2QCFAs M_1 and M_2 that recognize the languages $\{a^n b^n | n \geq 0\}$ and $\{a^{2n} b^n | n \geq 0\}$, respectively, in polynomial time and with one-sided error ϵ.

Applying Lemma 3.1, we get a 2QCFA that recognizes the language $\{a^n b^n | n \geq 0\} \cup \{a^{2n} b^n | n \geq 0\}$ in polynomial time and with one-sided error $2\epsilon - \epsilon^2 = \delta$. $\square$

4.2.4 Palindromes

Let $L_{pal} = \{x | x \in \{a, b\}^* \text{ and } x = x^R\}$ be the language of all palindromes over the alphabet $\{a, b\}$[11]. We now present a result of Ambainis and Watrous [2] stating that L_{pal} is recognizable by a 2QCFA in exponential time.

Theorem 4.4 *There is a 2QCFA* M *that recognizes the language* L_{pal} *with one-sided error* ϵ*,* $0 < \epsilon < \frac{1}{2}$*, and with expected number of steps exponential in the input size.*

Sketch of proof. The quantum states of M consist in a *qutrit*[12], with $|0\rangle$ being the initial state. Let also $U_a = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $U_b = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$ be unitary operators.

First M will scan the input string once, applying U_σ to the quantum state when the tape head is over symbol σ. Then, after returning to the left-end marker, it will scan the

[9] A CFG is unambiguous if every generated string has only one derivation tree. More details about derivation trees and ambiguity can be found in chapter 4 of Hopcroft and Ullman [9].

[10] By example 4.11 of Reghizzi [16] it can be seen that L is a non-deterministic unambiguous CFL.

[11] By example 4.12 of Reghizzi [16] it can be seen that L_{pal} is a non-deterministic unambiguous CFL.

[12] A *qutrit* is a quantum system with three possible states $\{|0\rangle, |1\rangle, |2\rangle\}$.

input a second time applying the operator U_σ^{-1} when tape head is over σ. After that, a measurement is applied.

It can be proved that if w is a palindrome, the probability of measuring $|0\rangle$ is 1. Otherwise there will be a non-zero probability of measuring $|1\rangle$ or $|2\rangle$. So, whenever the result of the measurement is not $|0\rangle$, the input can be rejected. Otherwise, a method based on coin flips, parametrized by the desired error probability ϵ, is performed. The end result will be the acceptance of the input string with a non-zero probability. If this method does not accept the string, the whole process is repeated.

It can be proved, based on the rejection and acceptance probabilities at each iteration, that the automata stops in expected exponential time.

For further details, see section 3 of Ambainis and Watrous [2]. □

4.2.5 2QCFAs and inherently ambiguous context-free languages

Inherently ambiguous CFLs are the ones that cannot be generated by any unambiguous CFG. Ambiguity causes problems when parsing, so inherently ambiguous CFLs are intrinsically not well suited for parsing.

In this subsection, we show a 2QCFA that recognizes with one-sided error the inherently ambiguous CFL $L_{ijk} = \{a^i b^j c^k | i = j \text{ or } j = k\}$[13]. Here, we present a simpler proof of Grilo and Moura [7].

Theorem 4.5 *There is a 2QCFA that recognizes the non-context-free language L_{ijk} with one-sided error δ in time $O(|w|^4)$, where w is the input string and $0 < \delta < \frac{1}{2}$.*

Sketch of proof. As described in Remark 4.2.1 the languages $L_1 = \{a^n b^n c^* | n \geq 0\}$ and $L_2 = \{a^* b^n c^n | n \geq 0\}$ can be recognized with one-sided error $\epsilon = 1 - \sqrt{1-\delta}$ by a 2QCFA in $O(|w|^4)$ steps. Since $L_{ijk} = L_1 \cup L_2$, using Theorem 3.1, we can also show that it is recognized by 2QCFA with one-sided error δ, in $O(|w|^4)$ expected time. □

4.3 Non-context-free languages

Non-context-free languages (non-CFLs) are the languages that cannot be recognized by PDAs. In this section, we present two non-CFLs recognized with one-sided error by 2QCFA.

4.3.1 A 2QCFA recognizing $a^n b^n c^n$

In this subsection, we show a 2QCFA that recognizes with one-sided error the non-CFL $L_n = \{a^n b^n c^n | n \geq 0\}$[14], based on proof by Grilo and Moura [7].

Theorem 4.6 *There is a 2QCFA that recognizes the non-context-free language L_n with one-sided error δ in time $O(|w|^4)$, where w is the input string and $0 < \delta < \frac{1}{2}$.*

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[13] It can be seen in Sipser [20] that L_{ijk} is an inherently ambiguous CFL.

[14] It can be seen in example 6.1 of Hopcroft and Ullman [9] that L_n is not a CFL.

Sketch of proof. As described in Remark 4.2.1 the languages $L_1 = \{a^n b^n c^* | n \geq 0\}$ and $L_2 = \{a^* b^n c^n | n \geq 0\}$ are recognized by 2QCFA with one-sided error $\epsilon = 1 - \sqrt{1-\delta}$ and in time $O(|w|^4)$. Since $L_n = L_1 \cap L_2$, using Theorem 3.2, we can show that L_n is recognized by 2QCFA with one-sided error δ and in $O(|w|^4)$ expected time. $\square$

4.3.2 A 2QCFA for wcw

In this subsection, we show a 2QCFA that recognizes with one-sided error the non-CFL $L_{twin} = \{wcw | w \in \{a, b\}^*\}$ in exponential time. The idea is to use a similar automata as the one described in Subsection 4.2.4.

Theorem 4.7 *There is a 2QCFA M that recognizes the language L_{twin} with one-sided error ϵ, $0 < \epsilon < \frac{1}{2}$, and with expected number of steps exponential in the input size.*

Sketch of proof. First, we will reject strings not in the form xcy, $x, y \in \{a, b\}^*$. Then, based on the 2QCFA for recognizing palindromes, shown in Theorem 4.4, we apply the transformation U_σ for each symbol σ of the prefix x, scanned forward, and U_σ^{-1} for each symbol σ of the suffix y, scanned backward. If $x = y$, then the final effect will be equivalent to the identity, otherwise there will be a non-zero probability of rejecting the input string.

Now, we can proceed as in Theorem 4.4 and get a method for accepting the input string whose expected halting time is exponential.

For further details, see Section 4 of Zheng *et al* [18]. $\square$

5 Languages not recognizable by 2QCFAs

One aspect of 2QCFA that is still very primitive is the characterization of languages that cannot be recognized by such models. For instance, there is no decidable language that was proved not to be recognizable by 2QCFA.

There are some languages, however, that we conjecture that cannot be recognized by 2QCFA with one-sided error, namely the languages $L_< = \{a^i b^j | i < j\}$ and derivations from it[15]. Another example is the Dyck language.

In this section we prove some partial results regarding the acceptance of $L_<$ by 1QCFA[16]. We will consider, in this section, that a 1QCFA recognizes a language L if for all $w \in L$, w is accepted with probability 1, and for all $w \notin L$, w is rejected with any non-zero probability. This is a weaker form of recognition, but we will show that even for it, there is no 1QCFA that accepts the language $L_<$.

First, we prove a lemma about unitary matrices.

Lemma 5.1 *For all unitary matrices U, there exists a value k for which U^k is within ϵ distance of the identity, for all $\epsilon > 0$.*

[15] If $L_<$ could be recognized, so would $L_> = \{a^i b^j | i > j\}$ and $L_{\neq} = \{a^i b^j | i \neq j\}$.
[16] See Remark 2 for the definition of 1QCFA model

Sketch of proof. Since U is unitary, it can be diagonalized, i.e., there exists matrices P and D, for which $U = PDP^{-1}$, where P is the matrix whose columns are the orthonormal eigenvectors of U, and D is a diagonal matrix whose diagonal value $D_{jj} = e^{\omega_j i}$ is the norm-1 eigenvalue of U associated to the j-th eigenvector of U.

Hence $U^s = PD^sP^{-1}$. We are looking for some k for which $|e^{k\omega_j i} - 1| < \epsilon$, so that U^k will be ϵ-close to I.

First, we partition the complex plane in $N = \lceil \frac{2\pi}{\epsilon} \rceil$ parts, by letting $w_j = \{e^{\frac{2\pi t i}{N}} | t \in \mathbb{R}, j \leq t < j+1\}$, $0 \leq j < N$. Since there is a finite number of partitions, there are some values k_1 and k_2, for which $e^{k_1 w_j i}$ and $e^{k_2 w_j i}$ are in the same partition w_{l_j} for all $0 \leq j < N$. Thus, $|e^{k_1 w_j i} - e^{k_1 w_j i}| < \frac{2\pi}{N} = \epsilon$, and so D^k, and therefore U^k, is at distance ϵ to the identity.

For further details, see Section 4 of Huschenbett [10]. $\square$

Now, we prove an auxiliary Lemma. It already gives the intuition for a more general proof.

Lemma 5.2 *If $\Theta(q_1, \sigma) = \Theta(q_2, \sigma) \in \mathcal{U}(\mathcal{Q})$, for all $\sigma \in \Sigma$ and all $q_1, q_2 \in Q$, then no 1QCFA will recognize the language $L_< = \{a^i b^j | i < j\}$.*

Sketch of proof. As Θ is independent of the classical states, let A and B be the unitary transformations when computing over a and b, respectively. If $w = a^i b^j$, the quantum state after computing over the input string will be $B^j A^i |0....0\rangle$.

We have, by Lemma 5.1, that for a chosen ϵ there exists a k for which B^k is ϵ close to I. Let us now look at the strings $w_i = a^l b^{l+ik}, i \geq 0$, $l \geq 0$. Since for $i > 0$, w_i should be accepted with probability 1, the operator B^k will induce a rotation over an hyper-plane that leads to acceptance. Thus, the string $w_0 = a^l b^l$ would also be accepted with probability 1, instead of being rejected with a non-zero probability. $\square$

Now we will see that, while computing over the input string, if the 1QCFA does not make any measurement, it will not be able to recognize $L_<$.

Lemma 5.3 *If $\Theta(q, \sigma) \in \mathcal{U}(\mathcal{Q})$, for all $\sigma \in \Sigma$ and all $q \in Q$, no 1QCFA can recognize the language $L_< = \{a^i b^j | i < j\}$.*

Sketch of proof. If there is no measurement until the one performed over the right-end marker, considering the finitely number of classical states, for an input string long enough, there will be a cycle of quantum operators of period r, after s initial steps. Using the same reasoning of Lemma 5.2, there is no possibility of the strings $w_i = \{a^s b^{s+ikr}$, $i \geq 0$ and some k associated to an error bound ϵ, be accepted or rejected with the correct probability. $\square$

Then, we will show that measurements do not help in the recognition of $L_<$.

Lemma 5.4 *If, at some point, the 1QCFA makes a measurement before processing the entire input, the 1QCFA cannot recognize the language $L_<$.*

Sketch of proof. If more than one measurement if applied, it is easy to see that the substring between the measurements is ignored in the computation, and it will not be possible to distinguish some string that is in $L_<$ from one that is not in $L_<$.

If just one measurment is applied, all computation after it is discarded, since it will act as a DFA. Using Lemma 5.3 and the fact that $L_<$ is not regular, we conclude that the 1QCFA will not be able to recognize it. □

Finally, we can state that no 1QCFA will recognize $L_<$.

Theorem 5.5 *No 1QCFA recognizes the language* $L_< = \{a^i b^j | i < j\}$.

Sketch of proof. Directly from Lemmas 5.2, 5.3 and 5.4. □

This is an important step for proving that $L_<$ is not recognized by 2QCFA, since it excludes the technique used in constructing the 2QCFA for recognizing $\{a^n b^n | n \geq 0\}$. However, there might still be another technique that moves the tape head in both directions during computation and that results in recognizing the language.

6 Future work

We now present some questions for future work.

1. **Is the set of languages recognized by 2QCFA, with arbitrary one-sided error, closed under complement, non-erasing homomorphism and general concatenation?**

   Qiu [15] showed that if we swap the acceptance and rejection states sets of a 2QCFA that recognizes the language L with one-sided error ϵ, there will be a 2QCFA that accept all strings not in L with probability $1 - \epsilon$, and all the strings in L with probability 0. However we are interested in a 2QCFA that accepts the strings not in L with probability 1 and the strings in L with probability ϵ.

   Macko [12] showed that the set of languages recognized by 2QCFA is closed under general homomorphism if and only if 2QCFAs are equivalent to Turing Machines (what we do not expect to be so). But the proof depends on homomorphism erasure (i.e., $h(\sigma)$ is the empty string for some σ). We would like to know what happens if $|h(\sigma)| \geq 1$.

   Also, we conjecture that the class of languages accepted by 2QCFAs is not closed under general concatenation, since if it was the case, the language $\{a^i b^j | i < j\}$ would be accepted.

2. **Is $\{a^i b^j | i < j\}$ recognized by a 2QCFA with arbitrary one-sided error?**

   We conjecture that the answer for this question is no. If it was not the case, the quantum states would have to be used, as the language is non-regular. However, using the quantum states seems not to be enough because for the family of strings $F_n = \{a^i b^n | i \geq 0\}$, for a fixed n, there is a threshold value for i that sepearates strings in the language from the ones that are not.

3. **Which problems are decidable for 2QCFAs?**

   Amano and Iwama [1] have shown that the emptiness problem is undecidable for 1.5QFA and their result is easily extended to 2QCFA. Blondel *et al* [3] have also studied the decidability of some problems involving other quantum automata models with fixed acceptance probability.

   We would like to know if it is possible to extend all of these results to 2QCFA and if other similar problems are also decidable for 2QCFAs, like language equality for 2QCFAs or whether a string w can be accepted by a 2QCFA with probability greater than a fixed threshold.

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