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Abstract

We present the first verification methods that automatically generate bases of invariants expressed by *multivariate formal power series* and *transcendental functions*. We discuss the convergence of solutions generated over hybrid systems that exhibit non-linear models augmented with parameters. We reduce the invariant generation problem to linear algebraic matrix systems, from which one can provide effective methods for solving the original problem. We obtain very general sufficient conditions for the existence and the computation of formal power series invariants over multivariate polynomial continuous differential systems. The formal power series invariants generated are often composed by the expansion of some well-known transcendental functions like *log* or *exp* and have an analysable closed-form. This facilitates their use to verify safety properties. Moreover, we generate *inequality* and *equality* invariants. Our examples, dealing with non-linear continuous evolution similar to those present today in many critical hybrid embedded systems, show the strength of our results and prove that some of them are beyond the limits of other recent approaches.

1 Introduction

Hybrid systems [1], [2] exhibit discrete and continuous behaviors, as one often finds when modeling digital system embedded in analog environments. Moreover, most safety-critical systems, *e.g.* aircraft, automobiles, chemical plants and biological systems, operate as non-linear hybrid systems and can only be adequately modeled by means of non-linear arithmetic over the real numbers and involving multivariate polynomial, fractional or transcendental functions.

In this work, we will use hybrid automata as computational models for hybrid systems. A hybrid automaton can describe interactions between discrete transitions and continuous dynamics, the latter being governed by local differential equations. We look for invariants that strengthen what we wish to prove, and so allow us to establish the desired property. Also, they can provide precise over approximations of the set of reachable states in the

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continuous state space. Given that, they can be used to determine which discrete transitions are possible and can also be used to verify if a given property is fulfilled or not.

Some known verification approaches are based on inductive invariant generation [3] and abstract interpretation [4], [5], which can be extended to hybrid systems to verify safety-critical properties. Other recent approaches to invariant generation are constraint based [6], [7],[8]. In these cases, a template form, described by a polynomial with fixed degree and unknown parametric coefficients, is proposed as the target invariant to be generated. The conditions for invariance are then encoded, resulting in constraints over the unknown coefficients whose solutions yield the desired invariants. But they still require several computations of Gröbner bases [9], first-order quantifier elimination [10] or abstraction operations at several steps, and known algorithms for those problems are, at least, of double exponential time complexity. SAT modulo theory decision procedures and polynomial systems [11], [7], [12] also could eventually lead to decision procedures for treating linear theories and decidable systems. Such works strive to generate linear or polynomial invariants over hybrid systems that exhibit affine or polynomial systems as continuous evolution modes.

Despite tremendous progress over the past years [6], [13], [14], [7], [15], [16], [17], [18], [19], [20], [21], generating invariants for hybrid systems remains very challenging for non-linear discrete systems, as well as for non-linear differential systems with non-trivial local and initial conditions. In this work, we present new methods for the automatic generation of invariants in the form of assertions where continuous functions are expressed by multivariate formal power series. Such methods can then be applied to systems with continuous evolution modes described by multivariate polynomials or fractional differential rules. As far as we know, there are no other methods that deal with this type of systems or that can generate this type of invariants.

We develop the new methods [22], [23] by first extending our previous work on non-linear invariant generation for discrete models with nested loops and conditional statements that describe multivariate polynomial or fractional systems [24], [25]. Then, we generalize our previous work on non-linear invariant generation for hybrid systems [26],[27], [28], [29].

We summarize our contributions as follows:

- To the best of our knowledge, we present the first methods which generate bases of *formal power series and transcendental invariants*, while dealing with non-linear continuous models present in many critical hybrid and embedded systems. The problem of synthesizing power series invariants and the results are clearly novel. We consider very general forms of continuous modes, *i.e.*, they are non-linear and augmented with parameters. Moreover, we generate both *inequality* and *equality* invariants.
- We introduce a more general approximation of consecution, dealing with assertions expressed by multivariate formal power series. We show that the preconditions for discrete transitions and the Lie-derivatives for continuous evolution can be viewed as morphisms and suitably represented by matrices. In this way, we reduce the invariant generation problem to linear algebraic matrix manipulations. We present an analysis of these matrices.
- We also provide resolution and convergence analysis for techniques that generate non trivial bases of provable multivariate formal power series and generate transcendental invariants for each local continuous evolution rules.
- Mathematically, we develop very general sufficient conditions allowing the existence and

computation of solutions defined by formal power series for multivariate polynomial differential systems. In order to achieve this goal we develop new methods, in the spirit of Boularas [30].

• The contribution is significant as it provides invariants that can be used to prove safety properties which also exhibit formal power series expressions or transcendental functions. To reason symbolically about formal power series and transcendental functions, it is necessary to be able to generate formal power series invariants, since they provide a more precise reachability analysis. The formal power series invariants generated are often composed by expansions involving transcendental functions like *log* or *exp*, which have analyzable closed forms, and thus facilitates the use of these invariants to verify properties.

Example 1.1. (Motivational Example) *Consider the following non-linear continuous system with* 2 *variables* $x(t), y(t)$ *and* 2 *parameters* a, b*:*

$$(S_3) = \begin{cases} \dot{x}(t) = ax(t) \\ \dot{y}(t) = ay(t) + bx(t)y(t). \end{cases}$$

Note that for this kind of systems, one could prove that no invariant can be obtained via the standard constraint-based approaches based on constant, polynomial or fractional scaling methods. Our method exhibits the following basis for the vector space of invariants

$$\{x, e^{-bx/a}y\}.$$

Almost all elements in this space would provide transcendental invariants. More precisely, by considering

$$F^1(x, y) = x$$

and

$$F^2(x, y) = e^{-bx/a}y$$

we will be able to generate strong invariants expressed in a very simple form (e.g. $x = 0$*) and others expressed by multivariate formal power series and transcendental functions (e.g.* $e^{-bx/a}y = 0$*).*

For instance, given any initial condition $x(0) = x_0$*,* $y(0) = y_0$*, the following assertion*

$$x(e^{-bx_0/a})y_0 - x_0(e^{-bx/a})y = 0$$

is an inductive invariant whatever the initial conditions are. It depends smoothly on the initial value and it is convergent everywhere. We are also able to generate inequality invariants, e.g.*, if we initially have* $a \leq 0$ *and*

$$F^2(x_0, y_0) \leq 0$$

then we have the inequality invariant

$$e^{-bx_0/a}y_0 \leq e^{-bx/a}y \leq 0.$$

Such invariants are beyond the reach of current invariant generation techniques, even these in a simple forms. □

This article is organized as follows. In Section 2 we first recall the notion of algebraic hybrid systems, we introduce our notations and representations for multivariate formal series. In Section 3 we present new forms for approximating consecution with multivariate formal power series and we reduce the problem to triangular linear algebraic matrix systems. In Section 4 we provide very general sufficient conditions for the existence of invariants and, further, we show how to automatically compute such invariants. In Section 5 we show how we treat general triangularizable systems. In Section 6 we present a running example and a convergence analysis. We show the efficiency of our methods in Section 7 by generating closed-form invariants for systems that are intractable by other state-of-the-art formal methods and static analysis approaches. In Section 8 we show how we handle algebraic discrete transitions. Section 9 offers our conclusions.

2 Hybrid Systems, Inductive Assertions and Formal Power Series

We present our approaches within a framework for hybrid systems. Let $\mathbb{K}[X_1, .., X_n]$ be the ring of multivariate polynomials over the set of variables $\{X_1, .., X_n\}$. An ideal is any set $I \subseteq \mathbb{K}[X_1, .., X_n]$ which contains the null polynomial and is closed under addition and multiplication by any element in $\mathbb{K}[X_1, .., X_n]$. Let $E \subseteq \mathbb{K}[X_1, .., X_n]$ be a set of polynomials. The ideal generated by E is the set of finite sums $(E) = \{\sum_{i=1}^{k} P_i Q_i \mid P_i \in \mathbb{K}[X_1, \ldots, X_n], Q_i \in E, k \geq 1\}$. A set of polynomials E is said to be a *basis* of an ideal I if $I = (E)$. By the Hilbert basis theorem, we know that all ideals have a *finite basis*. Notationally, as is standard in static program analysis, a primed symbol x' refers to next state value of x after a transition is taken. We may also write $\dot{x}$ for the derivative $\frac{dx}{dt}$. We denote by $\mathbb{R}_d[X_1, .., X_n]$ the ring of multivariate polynomials of degree at most d over the set of real variables $\{X_1, .., X_n\}$.

We use the notion of hybrid automata as the computational models for hybrid systems.

Definition 2.1. *A* hybrid system *is described by a tuple $\langle V, V_t, L, \mathcal{T}, \mathcal{C}, \mathcal{S}, l_0, \Theta\rangle$, where $V = \{a_1, .., a_m\}$ is a set of parameters, $V_t = \{X_1(t), .., X_n(t)\}$ where $X_i(t)$ is a function of t, L is a set of locations and l_0 is the initial location. A* transition *$\tau \in \mathcal{T}$ is given by $\langle l_{pre}, l_{post}, \rho_\tau\rangle$, where l_{pre} and l_{post} name the pre- and post- locations of τ, and the* transition relation *ρ_τ is a first-order assertion over $V \cup V_t \cup V' \cup V_t'$. Also, Θ is the initial condition, given as a first-order assertion over $V \cup V_t$. And $\mathcal{C}$ maps each location $l \in L$ to a* local condition *$\mathcal{C}(l)$ denoting an assertion over $V \cup V_t$. Finally, $\mathcal{S}$ associates each location $l \in L$ to a* differential rule *$\mathcal{S}(l)$ corresponding to an assertion over $V \cup \{dX_i/dt | X_i \in V_t\}$. A* state *is any pair from $L \times \mathbb{R}^{|V \cup V_t|}$, that is a location and interpretation of the variables.* □

The evolution of variables and functions in an interval must satisfy the local conditions and differential rules.

Definition 2.2. *A run of a hybrid automaton is an infinite sequence $(l_0, \kappa_0) \to \cdots \to (l_i, \kappa_i) \to \cdots$ of states where l_0 is the* initial location *and $\kappa_0 \models \Theta$. For any two consecutive states $(l_i, \kappa_i) \to (l_{i+1}, \kappa_{i+1})$ in such a run, the condition describes a* discrete consecution *if* *there exists a transition* $\langle q, p, \rho_i \rangle \in \mathcal{T}$ *such that* $q = l_i$, $p = l_{i+1}$ *and* $\langle \kappa_i, \kappa_{i+1} \rangle \models \rho_i$ *where the primed symbols refer to* κ_{i+1}. *Otherwise, it is a* continuous consecution *condition and there is some* $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, *and a differentiable function* $\phi : [0, \varepsilon) \to \mathbb{R}^{|V \cup V_t|}$ *such that the following conditions hold: (i)* $l_i = l_{i+1} = q$; *(ii)* $\phi(0) = \kappa_i$, $\phi(\varepsilon) = \kappa_{i+1}$; *(iii) During the time interval* $[0, \varepsilon)$, ϕ *satisfies the local condition* $\mathcal{C}(q)$ *and the local differential rule* $\mathcal{S}(q)$ *such that for all* $t \in [0, \varepsilon)$ *we must have* $\phi(t) \models \mathcal{C}(q)$ *and* $\langle \phi(t), d\phi(t)/dt \rangle \models \mathcal{S}(q)$. *A state* (ℓ, κ) *is reachable if there is a run and some* $i \geq 0$ *such that* $(\ell, \kappa) = (\ell_i, \kappa_i)$. □

Definition 2.3. *Let* W *be a hybrid system. An assertion* φ *over* $V \cup V_t$ *is an* invariant *at* $l \in L$ *if* $\kappa \models \varphi$ *whenever* (l, κ) *is a reachable state of* W. □

Definition 2.4. *Let* W *be a hybrid system and let* $\mathbb{D}$ *be an assertion domain. An assertion map for* W *is a map* $\gamma : L \to \mathbb{D}$. *We say that* γ *is* inductive *if and only if the following conditions hold:*
1. ***Initiation***: $\Theta \models \gamma(l_0)$;
2. ***Discrete Consecution***: *for all* $\langle l_i, l_j, \rho_\tau \rangle \in \mathcal{T}$ *we have* $\gamma(l_i) \wedge \rho_\tau \models \gamma(l_j)'$;
3. ***Continuous Consecution***: *for all* $l \in L$, *and two consecutive states* (l, κ_i) *and* (l, κ_{i+1}) *in a possible run of* W *such that* κ_{i+1} *is obtained from* κ_i *according to the local differential rule* $\mathcal{S}(l)$, *if* $\kappa_i \models \gamma(l)$ *then* $\kappa_{i+1} \models \gamma(l)$. . □

Hence, if γ is an inductive assertion map then $\gamma(l)$ is an invariant at l for W. Note that, in a continuous consecution, if $\gamma(l) \equiv (P(X_1(t), .., X_n(t)) = 0)$, for all $t \in [0, \varepsilon)$, where P is a multivariate polynomial in $\mathbb{R}[X_1, .., X_n]$ such that it has null values on the trajectory $(X_1(t), .., X_n(t))$ during the time interval $[0, \varepsilon)$, which is not to say that P is the null polynomial, then $\mathcal{C}(l) \wedge (P(X_1(t), .., X_n(t)) = 0) \models (\frac{d(P(X_1(t),..,X_n(t))}{dt} = 0)$ during the local time interval.

Definition 2.5. *A formal power series in the indeterminates* $x_1, \ldots, x_n$ *is an expression of the following form:*

$$\sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n} f_{i_1,\ldots,i_n} x^{i_1} \ldots x^{i_n},$$

where the coefficients $f_{i_1,\ldots,i_n}$ *belong to* $\mathbb{R}$. □

Definition 2.6. *Whenever* $i = (i_1, ..., i_n) \in \mathbb{N}^n$, *we denote the sum* $i_1 + \cdots + i_n$ *by* $|i|$. *We say that an order* $<$ *is a lexicographical total ordering in* $\mathbb{N}^n$ *if for any two elements* $i = (i_1, ..., i_n)$ *and* $j = (j_1, ..., j_n)$ *in* $\mathbb{N}^n$ *we have that* $(j_1, ..., j_n) < (i_1, ..., i_n)$ *holds if and only if one of the following condition holds: (i)* $|j| < |i|$; *or (ii)* $|j| = |i|$, *and the first non null component of* $i - j$ *is positive.* □

With $|i| = k$, where $i = (i_1, .., i_n)$, the monomials $x_1^{i_1} \cdots x_n^{i_n}$, form an ordered basis for the vector space of homogeneous polynomials of total degree k. This means that any homogeneous polynomial of total degree k can be written in the following ordered form: $\sum_{|i|=k} f_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}$. As a consequence, since a formal power series $F(x_1, .., x_n)$ is the direct sum of its homogeneous components, it can be written in the following ordered form:

$$F(x_1, .., x_n) = \sum_{k\geq 1} \sum_{|i|=k} f_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}.$$

We will denote the coefficients of homogeneous polynomials of degree k by [30]

$$F_k = \begin{bmatrix} f_{k,0,0,\ldots,0} & f_{k-1,1,0,\ldots,0} & f_{k-1,0,1,\ldots,0} & \cdots & f_{0,0,0,\ldots,k} \end{bmatrix}^\top$$

and the basis of homogeneous monomials of degree k will be denoted by the following vector

$$X^k = [\ {x_1}^k\ {x_1}^{k-1}x_2\ {x_1}^{k-1}x_3\ \ldots\ {x_n}^k\ ]^\top,$$

with the coordinates ordered with respect to the lexicographical total ordering given in Definition 2.6. We may now write the formal power series $F(x_1,..,x_n)$ as

$$\sum_{k\geq 1} F_k \cdot X^k = F_1 \cdot X^1 + ... + F_k \cdot X^k + ...,$$

where $F_k \cdot X^k$ denotes the scalar product $\langle F_k, X^k\rangle$. The polynomial $P_i(x_1,\ldots,x_n)$ can thus be written in the form $P_i(x_1,\ldots,x_n) = P^i_1 \cdot X^1 + ... + P^i_m \cdot X^m$, where m is the maximal degree among all polynomials P_i, and the P^i_j are the coefficient vectors of P_i. Denote by $x(t)$ the vector $(x_1(t),...,x_n(t))^\top$. Then S can be written as

$$\dot{x} = A_1 \cdot X^1(t) + ... + A_m \cdot X^m(t),$$

where $A_j = \begin{bmatrix} P^1_j & \ldots & P^n_j \end{bmatrix}^\top$. In particular, A_1 is the $n \times n$ matrix equal to the *Jacobian* matrix of the polynomial system given by the P_i's at zero.

From now on, let us describe the continuous evolution rules by a polynomial differential system S of the form:

$$S = \begin{bmatrix} \dot{x}_1(t) = P_1(x_1(t),...,x_n(t)) \\ \dot{x}_2(t) = P_2(x_1(t),...,x_n(t)) \\ \ldots \\ \dot{x}_n(t) = P_n(x_1(t),...,x_n(t)) \end{bmatrix} \tag{1}$$

3 Reduction to linear algebra

Now, we encode differential continuous consecution conditions. Let S be a polynomial differential system as in Eq.(1).

Definition 3.1. *A function F from $\mathbb{R}^n$ to $\mathbb{R}$ is said to be a λ-invariant for a system S if $\frac{d}{dt}F(x_1(t),..,x_n(t)) = \lambda F(x_1(t),..,x_n(t))$, for any solution $x(t) = (x_1(t),..,x_n(t))$ of S.* □

In Definiton 3.1, the numerical value of the Lie derivative of F is given by λ times its numerical value throughout the time interval $[0, \varepsilon)$. Without loss of generality we will assume that λ is a constant. It is worth noticing, however, that our methods will also work when λ is a multivariate fractional or multivariate polynomial, as is the case for multivariate polynomial invariants generation [26], [27], [28]. Next, we establish sufficient conditions over S for it to admit λ-invariants which are formal power series. Note that a formal power series $F(x) = F_1 \cdot X^1 + ... + F_k \cdot X^k + ...$ is a λ-invariant if the following conditions holds: $\sum_{i=0}^{n} \frac{\partial F(x)}{\partial x_i} P_i(x) = \lambda F(x)$. Using our notation, we obtain:

$$\sum_{i=0}^{n} \frac{\partial(F_1 \cdot X^1 + ... + F_k \cdot X^k + ...)}{\partial x_i}(P_1^i \cdot X^1 + ... + P_m^i \cdot X^m) \\ -\lambda(F_1 \cdot X^1 + ... + F_k \cdot X^k + ...) = 0.$$

By directly expanding the left side of the equation described just above and collecting terms corresponding to increasing degrees, we have:

$$\begin{array}{ll}
(1): & \sum_{j=1}^{n} \frac{\partial(F_1 X^1)}{\partial x_j} P_1^j X^1 - \lambda F_1 X^1 = 0 \\
(2): & \sum_{j=1}^{n} [\frac{\partial(F_1 X^1)}{\partial x_j} P_2^j X^2 + \frac{\partial(F_2 X^2)}{\partial x_j} P_1^j X^1] - \lambda F_2 X^2 = 0 \\
(3): & \sum_{j=1}^{n} [\frac{\partial(F_1 X^1)}{\partial x_j} P_3^j X^3 + \frac{\partial(F_2 X^2)}{\partial x_j} P_2^j X^2 + \frac{\partial(F_3 X^3)}{\partial x_j} P_1^j X^1] \\
 & -\lambda F_3 X^3 = 0 \\
\dots & \dots \\
(m): & \sum_{j=1}^{n} [\frac{\partial(F_1 X^1)}{\partial x_j} P_m^j X^m + \frac{\partial(F_2 X^2)}{\partial x_j} P_{m-1}^j X^{m-1} + \\
 & + \dots + \frac{\partial(F_m X^m)}{\partial x_j} P_1^j X^1] - \lambda F_m X^m = 0 \\
(m+1): & \sum_{j=1}^{n} [\frac{\partial(F_2 X^2)}{\partial x_j} P_m^j X^m + \frac{\partial(F_3 X^3)}{\partial x_j} P_{m-1}^j X^{m-1} + \\
 & + \dots + \frac{\partial(F_{m+1} X^{m+1})}{\partial x_j} P_1^j X^1] - \lambda F_{m+1} X^{m+1} = 0 \\
\dots & \dots
\end{array}$$

The equation corresponding to degree k is:

$$\begin{array}{l}
\sum_{j=1}^{n} [\ \frac{\partial(F_{k-min(k,m)+1} X^{k-min(k,m)+1})}{\partial x_j} P_{min(k,m)}^j X^{min(k,m)} \\
+\frac{\partial(F_{k-min(k,m)+2} X^{k-min(k,m)+2})}{\partial x_j} P_{min(k,m)-1}^j X^{min(k,m)-1} \\
+ \dots + \frac{\partial(F_k X^k)}{\partial x_j} P_1^j X^1\ ] - \lambda F_k X^k = 0.
\end{array}$$

With a different notion of consecution, we can treat more general systems than those that appeared in the determinant analysis of integrability of differential systems in Boularas [30]. Take the linear morphism $D_{p-k,p}$ from $\mathbb{R}_{p-k}[x_1, \dots, x_n]$ to $\mathbb{R}_p[x_1, \dots, x_n]$, given by

$$\begin{array}{rl}
\mathbb{R}_{p-k}[x_1, \dots, x_n] & \mapsto \mathbb{R}_p[x_1, \dots, x_n] \\
P(X = x_1, \dots, x_n) & \mapsto \sum_{j=1,\dots,n} (\partial_j P(X)) P_{k+1}^j . X^{k+1}.
\end{array}$$

which is the matrix $M_{p-k,p}$ in the ordered canonical basis of $\mathbb{R}_{p-k}[x_1, \dots, x_n]$ and $\mathbb{R}_p[x_1, \dots, x_n]$, respectively. Its l-th column represents the decomposition of the polynomial

$$\sum_{j=1,\dots,n} (\partial_j P(X)) P_{k+1}^j . X^{k+1},$$

where $P(X)$ is the l-th monomial in the ordered basis

$$\{x_1^p, x_1^{p-1} x_2, x_1^{p-1} x_3, \dots, x_n^p\}.$$

We can reduce the infinite system, described just above, to the following linear algebraic system:

$$\begin{cases} (M_{1,1}-\lambda I_2)F_1 = 0 \\ M_{1,2}F_1 + (M_{2,2}-\lambda I_2)F_2 = 0 \\ M_{1,3}F_1 + M_{2,3}F_2 + (M_{3,3}-\lambda I_4)F_3 = 0 \\ \dots \\ M_{k-min(k,m)+1,k}F_{k-min(k,m)+1} + \\ +M_{k-min(k,m)+2,k}F_{k-min(k,m)+2} + \\ +\cdots + (M_{k,k}-\lambda I_{k+1})F_k = 0 \\ \dots \end{cases}$$

By using the definitions of $D_{p-k,p}$ and $D_{p,p}$, we will see that we can symbolically compute all the matrices that will appear during the resolution of the mentioned linear algebraic system. We will use the following result

Lemma 3.1. *Assume that matrix $A = M_{1,1}$ is triangular,i.e.* $A = \begin{bmatrix} \lambda_1 & & & & \\ \star & \lambda_2 & & & \\ \star & \star & \ddots & & \\ \star & \star & \star & \lambda_{n-1} & \\ \star & \star & \star & \star & \lambda_n \end{bmatrix}$.
Then $M_{p,p}$ is also triangular with diagonal terms $i_1\lambda_1+\cdots+i_n\lambda_n$, where $i_1+\cdots+i_n = p$. □

Proof. We get $P_1^j.X^1 = \lambda_j x_j + a_{j,j+1}x_{j+1} + \cdots + a_{j,n}x_n$. Now consider the monomial basis $P(X) = x_1^{i_1}\dots x_n^{i_n}$, where $i_1+\cdots+i_n = p$. One has

$$\begin{aligned} D_{p,p}(X) = {} & i_1 x_1^{i_1-1}\dots x_n^{i_n}(\lambda_1 x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n) \\ & + i_2 x_1^{i_1} x_2^{i_2-1}\dots x_n^{i_n}(\lambda_2 x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n) \\ & + \cdots + i_n x_1^{i_1}\dots x_n^{i_n-1}(\lambda_n x_n) \\ = {} & (i_1\lambda_1 + \cdots + i_n\lambda_n)x_1^{i_1}\dots x_n^{i_n} + \Omega, \end{aligned}$$

where Ω is a sum of higher term monomials that come after $x_1^{i_1}\dots x_n^{i_n}$ in the ordered basis of $R_p[x_1,\dots,x_n]$.

Then, the matrix $M_{p,p}$ corresponding to $D_{p,p}$ in the canonical ordered basis of $\mathbb{R}_p[x_1,\dots,x_n]$, is:

$$\begin{bmatrix} p\lambda_1 & & & & & & \\ \star & (p-1)\lambda_1+\lambda_2 & & & & & \\ \star & \star & \ddots & & & & \\ \star & \star & \star & \sum_{k=1}^{n} i_k\lambda_k & & & \\ \star & \star & \star & \star & \ddots & & \\ \star & \star & \star & \star & \star & \lambda_{n-1}+(p-1)\lambda_n & \\ \star & \star & \star & \star & \star & \star & p\lambda_n \end{bmatrix}$$

Thus, it is also triangular with diagonal $i_1\lambda_1+\cdots+i_n\lambda_n$, where $i_1+\cdots+i_n = p$. □

4 Sufficient existence conditions

First, we show what happens when a λ-invariant converges. Next, we examine the computation of λ-invariants.

Theorem 4.1. (Soundness) *Let F be a λ-invariant for a system S. Let U be an open subset of $\mathbb{R}^n$, where F is defined by a normally convergent power series. If there is an initial condition $x_1(0), ..., x_n(0)$ in U such that $F(x_1(0), ..., x_n(0)) = 0$, then $F(x_1(t), ..., x_n(t)) = 0$ for all t such that $x_1(t), ..., x_n(t)$ remain in U, i.e., F is an invariant of S for the initial condition $(x_1(0), ..., x_n(0))$.* □

Proof. As the power series defining F converges normally on U, so does any of its derivatives. Thus,

$$\begin{array}{rcl} \dot{F}(x_1(t), ..., x_n(t)) & = & \sum_{i=1}^{n} \partial_i F(x_1(t), ..., x_n(t)) \dot{x}_i(t) \\ & = & \lambda F(x_1(t), ..., x_n(t)) \end{array},$$

given the λ-invariant property. So, $F(x_1(t), ..., x_n(t))$ must be equal to $t \mapsto ke^{\lambda t}$ for some constant k. Thus k is zero since $F(x_1(0), ..., x_n(0)) = 0$. Hence, so is $F(x_1(t), ..., x_n(t))$, for any t such that $(x_1(t), \ldots, x_n(t)) \in U$. □

Now we can state the following results, which allow for the computation of invariants which are inequality assertions.

Corollary 4.1. (Inequality invariants) *Given the system S and an λ-invariant F for S. Let U be an open subset of $\mathbb{R}^n$ on which F is defined by a normally convergent power series. For any initial value $x_1(0), ..., x_n(0)$ in U denote $F(x_1(0), .., x_n(0))$ by $F(x(0))$ and denote $F(x_1(t), ..., x_n(t))$ by $F(x(t))$. The following holds:*

(i) If $\lambda \geq 0$ and $F(x(0)) \geq 0$, then for all $t \geq 0$ we have the invariant $F(x(t)) \geq F(x(0))$;

(ii) If $\lambda \geq 0$ and $F(x(0)) \leq 0$, then for all $t \geq 0$ we have the inequality invariant $F(x(t)) \leq F(x(0))$;

(iii) If $\lambda < 0$ and $F(x(0)) \geq 0$, then for all $t \geq 0$ we have the invariant $0 \leq F(x(t)) \leq F(x(0))$;

(iv) If $\lambda < 0$ and $F(x(0)) \leq 0$, then for all $t \geq 0$ we have the invariant $F(x(0)) \leq F(x(t)) \leq 0$. □

Proof. Let F be a λ-invariant system S. By definition, we have $\frac{d}{dt}F(x(t)) = \lambda F(x(t))$ for solutions $(x_1(t), .., x_n(t))$ of S. So, $F(x(t))$ must be equal to $t \mapsto ce^{\lambda t}$ with $c = F(x_1(0), .., x_n(0))$. (i)For all $t \geq 0$ we know that $e^{\lambda t} \geq 1$ (as the function *exp* increases). So, we have $F(x(t)) = F(x(0))e^{\lambda t} \geq F(x(0))$ for all $t \geq 0$. (ii) We still have $e^{\lambda t} \geq 1$ but $F(x(t)) = F(x(0))e^{\lambda t} \leq F(x(0))$. (iii) We have $0 < e^{\lambda t} \leq 1$ and $0 \leq F(x(t)) = F(x(0))e^{\lambda t} \leq F(x(0))$ for all $t \geq 0$. (iv) We still have $0 < e^{\lambda t} \leq 1$ but $0 \geq F(x(t)) = F(x(0))e^{\lambda t} \geq F(x(0))$ for all $t \geq 0$. □

4.1 Sufficient general existence conditions

We have the following main results on the existence of formal power series invariants for systems described by Eq. (1).

Theorem 4.2. *Let A be the Jacobian matrix at zero for the polynomial $P = (P_1, ..., P_n)$ defining system S. Its expression is: $(\partial_i P_j(0, ..., 0), i, j \in [1, n]^2)$. Let $P_k(0, .., 0) = 0$. If A is triangularizable with eigenvalues $\lambda_1 \leq ... \leq \lambda_n$, then there exists a λ-invariant formal power series for S when all eigenvalues are positive, or are all negative, with $\lambda = \lambda_1$.* □

Proof. Up to a linear change of variables, we can assume that matrix A is triangular with diagonal terms $\lambda_1 \leq ... \leq \lambda_n$. We know that matrix $M_{k,k}$ has the form described in Lemma 3.1 and its proof. As A is triangular, so is $M_{k,k}$, and its diagonal terms are the real numbers $i_1\lambda_1 + \cdots + i_n\lambda_n$, where $i_1 + \cdots + i_n = k$. Hence, the diagonal terms of $M_{k,k} - \lambda I_{k+1}$ are $0 \leq \lambda_2 - \lambda ... \leq \lambda_n - \lambda$ when $k = 1$. Also, it has a nontrivial kernel, and so we can chose a nonzero F_1 such that $(M_{1,1} - \lambda I_2)F_1 = 0$. For $k \geq 2$ and $i_1 + \cdots + i_n = k$, the diagonal terms $i_1\lambda_1 + \cdots + i_n\lambda_n - \lambda$ of the triangular matrix $M_{k,k} - \lambda I_{k+1}$ are greater than $i_1\lambda_1 + \cdots + i_n\lambda_n - \lambda = k\lambda - \lambda > \lambda > 0$. So, $M_{k,k} - \lambda I_{k+1}$ is invertible.

Hence, we can choose:

- $F_2 = -(M_{2,2} - \lambda I_3)^{-1} M_{1,2} F_1$,
- $F_3 = -(M_{3,3} - \lambda I_4)^{-1}(M_{1,3}F_1 + M_{2,3}F_2)$,

  ...
- $F_k = -(M_{k,k} - \lambda I_{k+1})^{-1}(M_{\ell,k}F_{\ell,k} + \cdots + M_{k-1,k}F_{k-1})$,

where $\ell = k - \min\{k, m\} + 1$. Then, $(F_1, F_2, \ldots, F_k, \ldots)$ is a nonzero solution of the system and the formal power series $\sum_i F_i X^i$ is a λ-invariant. □

The proof also describes a method for the resolution of the triangular matrix system. We can, then, generate nonzero formal power series $\sum_i F_i X^i$ which are λ-invariants associated to the nonzero solution $(F_1, F_2, \ldots)$. In the examples that follow, we used Maple to compute the matrix products necessary to obtain F_k in its symbolic form. We treat the case when all eigenvalues are negative in a similar way. That is, with $\lambda = \lambda_n$, λ will be the eigenvalue with the minimum absolute value. Also, we recall that triangularizable matrices of $M_n(\mathbb{R})$ with eigenvalues of the same sign form a positive measure in the set of all matrices.

4.2 Inductive invariants and initial conditions

We state the following important result.

Theorem 4.3. *Let A be the Jacobian matrix at zero of the polynomial $P = (P_1, ..., P_n)$ defining a system S, as in Eq. (1), and whose expression is $(\partial_i P_j(0, ..., 0),\ i, j \in [1, n]^2)$. Assume, further, that $P_k(0, .., 0) = 0$. Suppose that A is triangularizable with eigenvalues $\lambda_1 \leq ... \leq \lambda_n$. Denote λ_1 by λ and assume that the eigenspace associated with λ is of dimension at least 2. Let F_1 and F_2 be two independent λ-invariants. Then, for any initial value*

$(x_{1,0},\dots,x_{n,0})$, the power series $F_2(x_{1,0},\dots,x_{n,0})F_1 - F_1(x_{1,0},\dots,x_{n,0})F_2$ defines an inductive invariant on U for the solution of S with initial conditions $x_1(0) = x_{1,0},\dots,x_n(0) = x_{n,0}$. □

Proof. We can see that both F_1 and F_2 converge to a solution $(x_1(t),...,x_n(t))$ with initial values $(x_{1,0},...,x_{n,0})$. Moreover, since F_1 and F_2 are independent,

$$F = F_2(x_{1,0},\dots,x_{n,0})F_1 - F_1(x_{1,0},\dots,x_{n,0})F_2$$

is a nonzero λ-invariant which vanishes at $(x_{1,0},..,x_{n,0})$. So, according to Theorem 4.1, F is an inductive invariant. □

5 Triangularizable systems

Now we show how to treat the following general system with parameters a, b, c, $a_{1,1}$, $a_{1,2}$, $a_{2,2}$, $b_{1,1}$,$b_{1,2}$, $b_{2,2}$ in V, and variables x, y in V_t:

$$\begin{aligned}\dot{x}(t) &= ax(t) + by(t) + a_{1,1}x^2(t) + a_{1,2}x(t)y(t) + a_{2,2}y^2(t)\\ \dot{y}(t) &= cy(t) + b_{1,1}x^2(t) + b_{1,2}x(t)y(t) + b_{2,2}y^2(t).\end{aligned}$$

The Jacobian matrix at zero of the polynomials defining the system is $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$. From Theorem 4.2, we already know how to find a formal power series F which is an a-invariant. Looking more closely at the coefficients of such a series we will show that it must converge in some appropriate neighborhood of 0. In this section it is not necessary to assume that $a > c$ or $a < c$, since we will choose $\lambda = min\{a, c\}$.

In subsection 5.1 we compute symbolically the matrices $M_{p-k,p}$ as they naturally appear in the resolution of the linear system. In subsection 5.2 we show how to reduce and solve the latter in order to generate λ-invariants by applying Theorem 4.2. In subsection 5.3 we present our convergence analysis methods for the discovered λ-invariants. We take $a = c$ only from subsection 5.4 onwards, in which case Theorem 4.3 applies and we obtain inductive invariants that hold for any initial conditions.

5.1 The matrices $M_{p-k,p}$

Using our notation, we have $P_i^1 = 0$ and $P_i^2 = 0$ for all $i > 2$. Then $M_{p-k,p}$ is the matrix whose l-th column is the vector corresponding to the decomposition of the polynomial

$$\begin{aligned}&\partial_1[(0,...,0,\underbrace{1}_{l-th\ position},0,...,0)X^{p-k}]P_{k+1}^1X^{k+1}\\ &+\partial_2[(0,...,0,\underbrace{1}_{l-th\ position},0,...,0)X^{p-k}]P_{k+1}^2X^{k+1}\end{aligned}$$

in the ordered canonical basis of $\mathbb{R}_p[x,y]$. Here, the polynomial $(0,..,0,\underbrace{1}_{l-th\ position},0,..,0)X^{p-k}$ is the l-th monomial of the canonical basis of $\mathbb{R}_{p-k}[x,y]$. Therefore, the matrices $M_{p-k,p}$

$$\begin{pmatrix} (p-1)a_{1,1} & b_{1,1} & & & & \\ (p-1)a_{1,2} & (p-2)a_{1,1}+b_{1,2} & 2b_{1,1} & & & \\ (p-1)a_{2,2} & (p-2)a_{1,2}+b_{2,2} & (p-3)a_{1,1}+2b_{1,2} & 3b_{1,1} & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 3a_{2,2} & 2a_{1,2}+(p-3)b_{1,2} & a_{1,1}+(p-2)b_{1,2} & (p-1)b_{1,1} \\ & & & 2a_{2,2} & a_{1,2}+(p-2)b_{2,2} & (p-1)b_{1,2} \\ & & & & a_{2,2} & (p-1)b_{2,2} \end{pmatrix}.$$

Figure 1: The matrix $M_{p-1,p}$

are zero unless $k=0$ or $k=1$. When $k=0$, the general form of $M_{p,p}$ is given in Section 3 and, in our particular case, it is

$$\begin{pmatrix} pa & & & & & \\ p.b & (p-1)a+c & & & & \\ & (p-1)b & (p-2)a+2c & & & \\ & & \ddots & \ddots & & \\ & & & 2b & a+(p-1)c & \\ & & & & b & pc \end{pmatrix}.$$

Note that $p+1$ is actually the dimension of $\mathbb{R}_p[x,y]$ and so $M_{p-1,p}$ is rectangular with $p+1$ rows and p columns. Here, the l-th monomial in the basis of $\mathbb{R}_{p-1}[x,y]$ is $x^{p-l-1}y^l$. Also, the polynomial $P_2^1X^2$ is $a_{1,1}x^2+a_{1,2}xy+a_{2,2}y^2$ and the polynomial $P_2^2X^2$ is $b_{1,1}x^2+b_{1,2}xy+b_{2,2}y^2$. Hence, matrix $M_{p-1,p}$ can be written as depicted in Figure (1):

5.2 Resolution of the infinite system

We are looking for λ-scale invariants and we know that we can choose $\lambda = min\{a,c\}$. Then, the system to solve is

$$\begin{aligned} (M_{1,1}-\lambda I_2)F_1 &= 0 \\ M_{1,2}F_1 + (M_{2,2}-\lambda I_3)F_2 &= 0 \\ M_{2,3}F_2 + (M_{3,3}-\lambda I_4)F_3 &= 0 \\ &\cdots \\ M_{k-1,k}F_{k-1} + (M_{k,k}-\lambda I_{k+1})F_k &= 0 \\ &\cdots . \end{aligned}$$

This linear algebraic system can be written as:

$$\begin{aligned}
(M_{1,1} - \lambda I_2)F_1 &= 0 \\
F_2 &= -(M_{2,2} - \lambda I_3)^{-1} M_{1,2} F_1 \\
F_3 &= -(M_{3,3} - \lambda I_4)^{-1} M_{2,3} F_2 \\
&\cdots \\
F_k &= -(M_{k,k} - \lambda I_{k+1})^{-1} M_{k-1,k} F_{k-1} \\
&\cdots .
\end{aligned}$$

One can choose any F_1, and then let $F_k = (-1)^{k+1} U_k(F_1)$, where U_k is the matrix with $k+1$ rows and 2 columns:

$$\begin{aligned}
&[\ (M_{k,k} - \lambda I_{k+1})^{-1} M_{k-1,k}] \cdot [(M_{k-1,k-1} - \lambda I_k)^{-1} M_{k-2,k-1}] \cdot \\
&\cdots\cdot [(M_{3,3} - \lambda I_4)^{-1} M_{2,3}] \cdot [(M_{2,2} - \lambda I_3)^{-1} M_{1,2}\ ].
\end{aligned}$$

We know that matrix $M_{k,k}$ has the form described in Section 3, Lemma 3.1 and its proof. Then, $M_{k,k} - \lambda I_{k+1}$ is

$$\begin{pmatrix}
ka-\lambda & & & & \\
k.b & (k-1)a+c-\lambda & & & \\
 & (k-1)b & (k-2)a+2c-\lambda & & \\
 & \ddots & \ddots & & \\
 & 2b & a+(k-1)c-\lambda & & \\
 & & b & kc-\lambda &
\end{pmatrix}$$

which can be decomposed as the product DT:

$$\begin{pmatrix}
d_1 & & & & & \\
 & d_2 & & & & \\
 & & d_3 & & & \\
 & & & \ddots & & \\
 & & & & d_k & \\
 & & & & & d_{k+1}
\end{pmatrix}
\begin{pmatrix}
1 & & & & & \\
t_2 & 1 & & & & \\
 & t_3 & 1 & & & \\
 & & \ddots & \ddots & & \\
 & & & t_k & 1 & \\
 & & & & t_{k+1} & 1
\end{pmatrix},$$

where $d_i = (k+1-i)a+(i-1)c-\lambda$ and $t_j = (k+2-j)b/d_j$. So, $(M_{k,k} - \lambda I_{k+1})^{-1} = T^{-1}D^{-1}$, where D^{-1} has the obvious form and T^{-1} is

$$\begin{pmatrix}
1 & & & & & \\
-t_2 & 1 & & & & \\
t_2t_3 & -t_3 & 1 & & & \\
-t_2t_3t_4 & t_3t_4 & -t_4 & 1 & & \\
\star & \star & \star & \star & \star & \\
(-1)^k t_2 \dots t_{k+1} & (-1)^{k-1} t_3 \dots t_{k+1} & \dots & t_k t_{k+1} & -t_{k+1} & 1
\end{pmatrix}.$$

We also know that $M_{k-j,k}$ has the form described in Section 5.1. Finally, all the matrices appearing in the product U_k are defined and F_k can be symbolically computed.

5.3 Convergence of the λ-invariant

We want to show that if $\lambda > 2b$, the coefficients of the F_i vectors decrease quickly enough so that the invariant F converges in a neighborhood of zero. Let us first recall some basic properties of norms in finite dimension real vector spaces, as well as the associated matrix norms. If v, with coordinates v_i, belongs to $\mathbb{R}^n$, we denote by $|v|_\infty$ the value $\max_{i=1,\dots,n} |v_i|$. If A is a matrix with m rows and n columns, representing a morphism from $(\mathbb{R}^n, |.|_\infty)$ to $(\mathbb{R}^m, |.|_\infty)$ in the canonical basis, it is well-known that associated with the norm $|.|_\infty$ is the matricial norm $||.||$ on $M_{m,n}(\mathbb{R})$, where $||A|| = \max_{i=1,\dots,m}(\sum_{j=1}^n |A_{i,j}|)$. Moreover, using this norm, if $v \in \mathbb{R}^n$ then one has $|Av|_\infty \leq ||A||.|v|_\infty$. This implies that if A and B are two matrices belonging, respectively, to $M_{m,n}(\mathbb{R})$ and $M_{n,p}(\mathbb{R})$, then we get

$$||AB|| \leq ||A|| \cdot ||B||.$$

In particular,

$$||U_k|| \leq ||M_{k,k} - \lambda I_{k+1}| \cdot |||M_{k-1,k}|| \dots ||M_{2,2} - aI_3|| \cdot ||M_{1,2}||.$$

But, from the expressions for the $M_{k-1,k}$ matrices, we have that $||M_{k-1,k}|| \leq f(k-1)$, where $f = 4 \cdot \max(|a_{i,j}|, |b_{i',j'}|)$. From the preceding paragraph again, we deduce that

$$||(M_{k,k} - \lambda I_{k+1})^{-1}|| \leq ||D^{-1}|| \cdot ||T^{-1}||.$$

But $||D^{-1}|| = \max_i(d_i^{-1}) \leq [(k-1)\lambda]^{-1}$, because $\lambda = \min(a, c)$, and so

$$||T^{-1}|| = \max_i(1 + t_i + t_{i-1}t_i + \cdots + t_2t_3 \dots t_{i-1}t_i).$$

But each t_j is less than

$$(k + 2 - j)b/d_j \leq kb/[(k-1)\lambda] \leq 2b/\lambda.$$

Suppose that $\lambda > 2b$. Then

$$||T^{-1}|| \leq 1 + 2b/\lambda + \cdots + (2b/\lambda)^k \leq 1/(1 - 2b/\lambda).$$

By letting e be the constant $1/(1 - 2b/\lambda)$, we can write

$$||(M_{k,k} - \lambda I_{k+1})^{-1}|| \leq e/(k-1)\lambda.$$

Finally, $||U_k||$ is less than $(ef/\lambda)^{k-2} = r^{k-2}$. Eventually,

$$|F_k|_\infty = |U_k(F_1)|_\infty \leq ||U_k||.|F_1|_\infty \leq r^{k-2}|F_1|_\infty.$$

Let t be $\max\{|x|, |y|\}$. Then,

$$\begin{aligned} |F(x,y)| &\leq |F_1X^1| + |F_2X^2| \cdots + |F_kX^k| + \dots \\ &\leq 2|F_1|_\infty t + 3|F_2|_\infty t^2 + \cdots + (k+1)|F_k|_\infty t^k + \dots. \end{aligned}$$

The right part of the inequality is itself inferior to

$$1/r^2|F_1|_\infty[2(rt) + 3(rt)^2 + \cdots + (k+1)(rt)^k + \dots],$$

which, from the classical theory of one variable power series, is convergent in the open disk centered at zero and of radius $1/r$. Hence, we have proved the following.

Proposition 5.1. *Consider the system described at the beginning of Section 5 with a and c positive and greater than $2b$. Let λ be the minimum between a and c. Then there exists a λ-invariant, obtained as described in Theorem 4.2, and which always converges in a neighborhood of zero.* □

5.4 The case of eigenspaces with dimension 2

Now, suppose that the eigenspace corresponding to λ has multiplicity 2, *i.e.* $a = c = \lambda > 0$ and $b = 0$. We know, from the previous subsection, that any λ-invariant will converge in a ball of radius $1/r$ and centered at zero. Moreover, according to Theorem 4.3, this will give an inductive invariant for the system for any initial solutions within this ball. More precisely, by letting $F_1^1 = (1,0)^\top$ and $F_1^2 = (0,1)^\top$, we get a basis $F^1(x,y)$ and $F^2(x,y)$ of λ-invariants that converge in the open $|.|_\infty$-disk of radius $1/r$ and centered at zero.

Note that the monomial of degree one in the Taylor series of F^1 is x, and it is y in the Taylor series of F^2. In other words, if we take the first coefficient of F as $(1,0)^\top$, we obtain a λ-invariant

$$F = F^1(x,y)$$

and, similarly, if we take the second coefficient of F as $(0,1)^\top$, we obtain another λ-invariant

$$F = F^2(x,y).$$

Moreover, these two invariants form a basis for invariants that converge in the open $|.|_\infty$-disk of radius $1/r$ and centered at zero. Assume now that we are given initial values, $x(0) = x_0$ and $y(0) = y_0$, as solutions in this open disk. Then, there will always exist two real numbers, λ and μ, such that

$$\lambda(x_0,y_0)F^1(x_0,y_0) + \mu(x_0,y_0)F^2(x_0,y_0) = 0,$$

where

$$\lambda(x_0,y_0) = F^2(x_0,y_0)$$

and

$$\mu(x_0,y_0) = -F^1(x_0,y_0).$$

Then,

$$(\lambda(x_0,y_0)F^1 + \mu(x_0,y_0)F^2 = 0)$$

is an inductive invariant for the solution corresponding to the initial condition (x_0,y_0). So, given (x_0,y_0) in the $|.|_\infty$-disk of radius $1/r$ and centered at zero, the invariant depends smoothly on the initial condition.

6 A Running example

In this section, we discuss a running example and explain how the sufficient conditions for invariance are used, and how a basis for invariant ideals is automatically obtained. We treat sub-classes of non linear differential rules that we often find in local continuous modes in hybrid systems. More specifically, we show how our method applies to systems:

$$\dot{x}(t) = ax(t) + bx(t)y(t)$$
$$\dot{y}(t) = ay(t) + dx(t)y(t).$$

Next, we show how to generate invariant ideals. The Jacobian matrix at zero is $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. From Theorem 4.2, we can find a formal power series F which is an a-invariant. We will show that it must converge in some neighborhood of 0.

Step 1: Computation of the matrices $M_{p-k,p}$.

The coefficient vectors P_i are zero, for $i \geq 2$. So, $M_{p-k,p}$ is the matrix whose l-th column is the vector corresponding to the decomposition of the polynomial

$$\partial_1[(0,\dots,0,\underbrace{1}_{l-th\ position},0,\dots,0)X^{p-k}]P^1_{k+1}X^{k+1}$$
$$+\partial_2[(0,\dots,0,\underbrace{1}_{l-th\ position},0,\dots,0)X^{p-k}]P^2_{k+1}X^{k+1}$$

in the ordered canonical basis of $\mathbb{R}_p[x,y]$. Hence, in this case, the matrices $M_{p-k,p}$ are zero unless $k = 0$ or $k = 1$. When $k = 0$, the general form of $M_{p,p}$, as detailed in Section 3, is given by paI_{p+1}. But $p+1$ is actually the dimension of $\mathbb{R}_p[x,y]$ and so $M_{p-1,p}$ is rectangular with $p+1$ rows, and p columns. Then the l-th monomial of the basis of $\mathbb{R}_{p-1}[x,y]$ is $x^{p-l-1}y^l$. Then, the polynomial $P^1_2X^2$ is bxy, and the polynomial $P^2_2X^2$ is dxy. Hence, $\partial_1[(0,..,0,\underbrace{1}_{l-th\ position},0,..,0)X^{p-1}]P^1_2X^2+$

$$\partial_2[(0,..,0,\underbrace{1}_{l-th\ position},0,..,0)X^{p-1}]P^2_2X^2$$

reduces to $b(p-l-1)x^{p-l-1}y^{l+1} + dlx^{p-l}y^l$. Eventually, it can be seen that the matrix can be written as:

$$M_{p-1,p} = \begin{pmatrix} 0 & & & & & \\ (p-1)b & d & & & & \\ & (p-2)b & 2d & & & \\ & & \ddots & \ddots & & \\ & & & 2b & (p-2)d & \\ & & & & b & (p-1)d \\ & & & & & 0 \end{pmatrix}.$$

Step 2: Resolution of the linear system.

When looking for λ-scale invariants, we already know that we must choose $\lambda = a$. Then, we need to solve

$$\begin{aligned} (M_{1,1} - aI_2)F_1 &= 0 \\ M_{1,2}F_1 + (M_{2,2} - aI_2)F_2 &= 0 \\ M_{2,3}F_2 + (M_{3,3} - aI_3)F_3 &= 0 \\ &\dots \\ M_{k-1,k}F_{k-1} + (M_{k,k} - aI_k)F_k &= 0 \\ &\dots . \end{aligned}$$

Since $M_{k,k}$ is equal to kaI_{k+1}, the system becomes:

$$\begin{aligned} 0 \cdot F_1 &= 0 \\ F_2 &= -a^{-1}M_{1,2}F_1 \\ F_3 &= -(2a)^{-1}M_{2,3}F_2 \\ &\dots \\ F_k &= -[(k-1)a]^{-1}M_{k-1,k}F_{k-1} \\ &\dots \end{aligned}$$

This means that one can choose any F_1, and then choose F_k as $(-1)^{k+1}a^{-k+1}U_k(F_1)$, where U_k is the matrix with $k+1$ rows and 2 columns given by the product

$$[1/(k-1)M_{k-1,k}] \cdot [1/(k-2)M_{k-2,k-1}] \dots [1/2M_{2,3}]M_{1,2}.$$

Step 3: Convergence of the invariant.

Now we show that the invariant F converges in a neighborhood of zero. In particular, the norm $||U_k||$ is less than or equal to the product $\frac{1}{(k-1)!}||M_{k-1,k}|| \dots ||M_{1,2}||$. Then from $M_{k-1,k}$ we get $||M_{k-1,k}|| \leq ck$, where $c = \max\{|b|, |d|\}$. Hence, $||U_k|| \leq ck!/(k-1)! = ck$. Eventually, we get

$$\begin{aligned} |F(x,y)| &\leq |F_1X^1| + |F_2X^2| \dots + |F_kX^k| + \dots \\ &\leq 2|F_1|_\infty t + 3|F_2|_\infty t^2 + \dots + (k+1)|F_k|_\infty t^k + \dots \end{aligned}$$

The right side of the inequality is inferior to

$$ac|F_1|_\infty[2(\frac{t}{a}) + 3.2(\frac{t}{a})^2 + \dots + (k+1)k(\frac{t}{a})^k + \dots].$$

From the theory of one variable power series, it must converge in the open disk of radius a and centered at zero.

More precisely, taking F^1 and F^2, respectively, as $(1,0)^\top$ and $(0,1)^\top$, we get a basis $F^1(x,y)$ and $F^2(x,y)$ for a-invariants which converge in the open $|.|_\infty$-disk of radius a and centered at zero. Assume now that we are given initial values with parameters $x_0 = x(0)$ and $y_0 = y(0)$ for solutions of the system within this open disk. Then,

$$F^2(x_0,y_0)F^1(x,y) - F^1(x_0,y_0)F^2(x,y) = 0$$

is an inductive invariant. Again, for (x_0,y_0) in the $|.|_\infty$-disk of radius a and centered at zero it depends smoothly on the initial condition.

7 Experimental results

We give some examples in dimension 2 where x, y are variables in V_t and a, b are parameters in V. All the following systems will have a Jacobian matrix $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. We will see that we already get interesting invariants which are polynomial in the first case, rational in the second case, transcendental in the last case.

First, as we already saw in the previous sections, for all $k \geq 1$, only $M_{k,k}$ and $M_{k-1,k}$ are nonzero. Moreover $M_{k,k}$ is equal to $Diag(ka, \dots, ka)$ in $\mathcal{M}_{k+1,k+1}(\mathbb{R})$. And we saw in Section 5.2 that

$$F(X) = \sum_i F_i X^i$$

is an a-invariant if and only if

$$F_k = (-1)^{k-1} U_k(F_1)$$

with $k \geq 2$ and where

$$U_k = [(M_{k,k} - aI_{k+1})^{-1} M_{k-1,k}]. \dots .[(M_{2,2} - aI_3)^{-1} M_{1,2}]$$

is in $\mathcal{M}_{k+1,2}(\mathbb{R})$. Because of the form of $M_{k,k}$, we get

$$U_k = \frac{1}{a^{k-1}(k-1)!}.M_{k-1,k}. \dots .M_{1,2}.$$

From the results of Section 5, in all these cases the vector space of a-invariants will be of dimension 2 and we can find precise invariants convergent near zero . Also, the transcendental invariants obtained in the forthcoming Example (7.3) converge everywhere for any initial conditions.

Example 7.1. *Consider S_1 as*

$$\dot{x}(t) = ax(t)$$
$$\dot{y}(t) = ay(t) + bx^2(t).$$

Here $P_1(x, y) = ax$ and $P_2(x, y) = ay + bx^2$ and we have

$$M_{k-1,k} = \begin{pmatrix} 0 & b & & & \\ 0 & 0 & 2b & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 0 & (k-1)b \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} \in \mathcal{M}_{k+1,k}(\mathbb{R}).$$

Thus $U_2 = \begin{pmatrix} 0 & b/a \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_{3,2}(\mathbb{R})$, *and U_k is zero for $k \geq 3$.*

Taking first $F_1 = (1, 0)^\top$, one has $F_k = (0, 0, \dots, 0)^\top$ for $k \geq 2$ and then we get $F^1(x, y) = x$ as the first basis-vector of the space of a-invariants.

Now, with

$$F_1 = (0,1)^\top,$$

one has

$$F_2 = (-b/a)(1,0,\ldots,0)^\top,$$

and F_k is zero for $k \geq 3$. So, we get

$$F^2(x,y) = y - bx^2/a$$

as the second basis-vector of the space of a-invariants.
Hence, $F^1(x,y) = x$ and $F^2(x,y) = y - bx^2/a$ form a basis of the vector space of a-invariants:

```
Lambda=a
U[2]=[[0,0,0],[-b/a,0,0]]
F[1]=[[1,0]]; F[k]=[[0,..,0]]
F[1]=[[0,1]]; F[k]=[[-b/a,0,..,0]]
Basis of Vector Space Lambda Invariants:
{x, y- b*x^2/a}
```

Finally for an initial condition (x_0, y_0),

$$F(x,y) = x_0(y - bx_0^2/a) - x(y_0 - bx^2/a) = 0$$

is an inductive polynomial invariant *whatever are the initial conditions, i.e. for all x_0 and y_0. Applying Corollary 4.1, we can observe several box or inequality invariants. If $a > 0$ and $F^2(x_0,y_0) \geq 0$ then we have the following inequality invariants, that hold for any solution $x(t), y(t)$ of S_1 with initial conditions (x_0, y_0):*

$$y - bx^2/a \geq y_0 - b{x_0}^2/a \geq 0.$$

If we still have $a > 0$ and $F^1(x_0,y_0) \geq 0$ then we have the following inequality invariants

$$x \geq x_0 \geq 0.$$

Also, if $a < 0$ and

$$F^2(x_0,y_0) \geq 0$$

then we have the following (box) inequality invariants:

$$0 \leq y - bx^2/a \leq y_0 - b{x_0}^2/a.$$

□

Example 7.2. *Let S_2 be*

$$\dot{x}(t) = ax(t) + bx^2(t)$$
$$\dot{y}(t) = ay(t).$$

Here $P_1(x,y) = ax + bx^2$ *and* $P_2(x,y) = ay$ *and*

$$M_{k-1,k} = \begin{pmatrix} (k-1)b & 0 & & & & \\ 0 & (k-2) & 0 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 0 & b & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} \in \mathcal{M}_{k+1,k}(\mathbb{R}).$$

Thus $U_k = \begin{pmatrix} (-b/a)^{k-1} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_{k+1,2}(\mathbb{R}).$

Taking

$$F_1 = (1,0)^\top,$$

one has

$$F_k = (-b/a)^{k-1}(1,0,\ldots,0)^\top.$$

Then, we obtain the rational function

$$F^1(x,y) = \sum_{k\geq 1} (-b/a)^{k-1} x^k = x/(1+bx/a)$$

as the first basis-vector of the space of a-invariants, which is convergent for $|x| < |a/b|$.

Now, if we take

$$F_1 = (0,1)^\top,$$

one has

$$F_k = (0,0,\ldots,0)^\top$$

for $k \geq 2$, *and we get*

$$F^2(x,y) = y$$

as the second basis-vector of the space of a-invariants:

```
Lambda=a
U[k]=[[(-b/a)^(k-1),0,..,0],[0,..,0]]
F[1]=[[1,0]]; F[k]=[[(-b/a)^(k-1),..,0]]
F[1]=[[0,1]]; F[k]=[[0,..,0]]
Basis of Vector Space Lambda Invariants:
{x/(1+b*x/a), y}
```

For an initial condition (x_0, y_0) *with* $|x_0| < |a/b|$, *we obtain, for instance, the following inductive* rational invariants*:*

$$F(x,y) = x_0 y/(1 + bx_0/a) - xy_0/(1 + bx/a) = 0.$$

Using Corollary 4.1, we can also identify several inequality invariants. For instance, if we initially have $a > 0$ *and* $F^1(x_0, y_0) \geq 0$ *then we get the inequality invariant* $x/(1 + bx/a) \geq$

$x_0/(1+bx_0/a) \geq 0$. *Also, if we initially have* $a < 0$ *and* $F^1(x_0, y_0) \leq 0$ *the have the inequality invariant*

$$x_0/(1+bx_0/a) \leq x/(1+bx/a) \leq 0.$$

□

Example 7.3. *Here is an example where our method exhibits a transcendental invariant. Most importantly, note that this kind of results can not be obtained via the classical constant, polynomial or fractional scale methods. Moreover, the invariant obtained converge everywhere.*

The formal power series invariant generated are often composed by expansion of some well-known transcendental function and hence has an analyzable closed form. Being able to compute closed forms for the invariants allows us to reason symbolically about formal power series. This facilitates the use of the invariants to verify properties. Consider the system S_3 *given by*

$$\dot{x(t)} = ax(t)$$
$$\dot{y(t)} = ay(t) + bx(t)y(t).$$

Here $P_1(x,y) = ax$ *and* $P_2(x,y) = ay + bxy$ *and we get*

$$M_{k-1,k} = \begin{pmatrix} 0 & 0 & & & \\ 0 & b & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & (k-2)b & 0 \\ & & & 0 & (k-1)b \\ & & & & 0 \end{pmatrix} \in \mathcal{M}_{k+1,k}(\mathbb{R}),$$

thus U_k *is equal to* $\frac{(-b)^{k-1}}{a^{k-1}(k-1)!}\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_{k+1,2}(\mathbb{R})$.

Taking $F_1 = (1,0)^\top$, *one has* $F_k = (0,0,\ldots,0)^\top$ *for* $k \geq 2$, *so that we get* $F^1(x,y) = x$ *as the first basis-vector of the space of* a*-invariants.*

With

$$F_1 = (0,1)^\top,$$

one has

$$F_k = (0, \frac{(-b)^{k-1}}{a^{k-1}(k-1)!}, \ldots, 0)^\top$$

for $k \geq 2$. *Then we obtain the transcendental function*

$$F^2(x,y) = \sum_{k\geq 1} \frac{(-bx)^{k-1}y}{a^{k-1}(k-1)!} = e^{-bx/a}y$$

as the second basis-vector of the space of a*-invariants.*

```
Lambda=a
U[k]=[[0,..,0],[((-b)^(k-1))/(a^(k-1)*(k-1)!),0,..0]]
F[1]=[[1,0]]; F[k]=[[0,..,0]]
F[1]=[[0,1]]; F[k]=[[0,((-b)^(k-1))/(a^(k-1)*(k-1)!),0,..,0]]
Basis of Vector Space Lambda Invariants:
{x, exp(-b*x/a)*y}
```

Finally, for any given initial condition (x_0, y_0), the following assertion

$$F(x,y) = x(e^{-bx_0/a})y_0 - x_0(e^{-bx/a})y = 0$$

is an inductive invariant whatever are the initial conditions, i.e. for all x_0 and y_0. Clearly it depends smoothly on the initial value and is convergent everywhere. By applying Corollary 4.1, we can also identify several inequality invariants. For instance, if we initially have $a > 0$ and $F^1(x_0, y_0) \leq 0$ then we get the inequality invariant

$$e^{-bx/a}y \leq e^{-bx_0/a}y_0.$$

Also, if we initially have $a \leq 0$ and $F^1(x_0, y_0) \leq 0$ we get the inequality invariant

$$e^{-bx_0/a}y_0 \leq e^{-bx/a}y \leq 0.$$

□

In Table 1 we summarizes some experimental results. The second column gives the closed-form type of the basis generators. We emphasize the fact that the issue of finding invariants absolutely does not reduce only to the computation of such a radius. The main issue is the computations of the coefficients U_k, which is equivalent to the knowledge of the invariant. As a consequence, the fact that we are able to find closed forms is a nice observation, but should not be considered as the most important one. The invariant is really given by its coefficients.

8 Handling discrete transitions

The methods presented so far automatically generate bases of non trivial multivariate formal power series invariants for each differential rule associated to locations in the hybrid automaton. The basis of vector space invariants provided by our techniques, generate very precise invariants that could be used as a primitive in any static reachability analysis and verification framework for hybrid systems with non-linear continuous modes.

In order to handle the discrete transition relations, we can adapt and extend the methods proposed in our previous works on static analysis of hybrid systems [26], [27], [28] and discrete programs [26], [24]. Such techniques are completely orthogonal and different from those presented here. Those provide methods to handle discrete algebraic transitions that can be integrated in order to develop a full technique for hybrid systems. In fact, many other methods from different approaches could be seen as complementary techniques.

Table 1: Examples and experimental results: generation of basis of transcendental invariants

Diff. Syst.	Closed-form	Time/s
Ex.(7.1)	Polynomial	6.1
Ex.(7.2)	Rational	7.6
Ex.(7.3)	Transcendental	18.9
$\dot{x} = 7x$, $\dot{y} = 1/2x^2 + 7y$.	Polynomial	.7
$\dot{x} = 3x$, $\dot{y} = 9x * y + 3y$.	Transcendental	1.1
$\dot{x} = 8x^2 + 5x$, $\dot{y} = 5y$.	Rational	.8
$\dot{x} = b * x$, $\dot{y} = b * y * (1 + x)$.	Transcendental	11.3
$\dot{x} = b * x * (1 + x)$, $\dot{y} = b * y$.	Rational	6.5
$\dot{x} = b * x$, $\dot{y} = b * (x^2 + y)$.	Polynomial	4.5
$\dot{x} = x - x * y$, $\dot{y} = y - x * y$.	Transcendental	2.1
$\dot{x} = x * y$, $\dot{y} = x * y$.	Transcendental	1.6
$\dot{x} = x + a * x^2$, $\dot{y} = y$.	Rational	4.1
[26] Ex.(2)	Transcendental	1.8
[29] Eq.(3)	Transcendental	1.4
[29] Eq.(4)	Transcendental	.4
[28]	Polynomial	2.4

9 Conclusions

Invariant generation problems for continuous time evolution is the most challenging step in static analysis and verification of hybrid systems. Computationally, hybrid systems were an inspiration and motivation for this research. Once these invariants are generated, we are able to use and compose several techniques from static analysis for discrete state jumps. In order to verify safety properties expressed with transcendental functions and to reason symbolically about formal power series, it is necessary to be able to generate formal power series invariants. We presented methods which generate bases of *multivariate formal power series and transcendental invariants* for hybrid systems with non-linear behavior. Also, our methods generate *inequality* and *equality* invariants.

The problem of generating power series invariants and the results are clearly novel. Importantly, there is no other known methods that generate this type of invariants. We can prove that some of the examples that were dealt with do not have"finite" polynomial invariants. Hence, they are beyond the limits of other recent approaches. As for efficiency, we used linear algebra methods which do not require several Gröbner basis computations or quantifier eliminations. We also provide very general sufficient conditions allowing for the existence and computation of invariants defined by convergent formal power series for multivariate polynomial differential systems. Those conditions could also be used directly as primitives in any static analysis and verification framework for hybrid systems.

References

[1] T. Henzinger, The theory of hybrid automata, in: Proc. of the 11th Annual IEEE Symposium on Logic in Computer Science (LICS '96), 1996, pp. 278–292.

[2] R. Alur, C. Courcoubetis, N. Halbwachs, T. A. Henzinger, P. h. Ho, X. Nicollin, A. Olivero, J. Sifakis, S. Yovine, The algorithmic analysis of hybrid systems, Theoretical Computer Science 138 (1995) 3–34.

[3] Z. Manna, Mathematical Theory of Computation, McGrw-Hill, 1974.

[4] P. Cousot, R. Cousot, Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints, in: Conf. Record of the 4th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, ACM Press, NY, Los Angeles, California, 1977, pp. 238–252.

[5] P. Cousot, R. Cousot, Abstract interpretation and application to logic programs, Journal of Logic Programming 13 (2–3) (1992) 103–179.

[6] S. Sankaranarayanan, H. Sipma, Z. Manna, Constructing invariants for hybrid system, in: Hybrid Systems: Computation and Control HSCC, Vol. 2993 of LNCS, 2004, pp. 539–554.

[7] S. Gulwani, A. Tiwari, Constraint-based approach for analysis of hybrid systems, in: Proc. of the 14th Int. Conf. on Computer Aided Verification CAV, 2008.

[8] S. Prajna, A. Jadbabaie, Safety verification of hybrid systems using barrier certificates (2004).
URL `citeseer.ist.psu.edu/prajna04safety.html`

[9] B. Buchberger, Symbolic computation: Computer algebra and logic, in: Frontiers of Combining Systems: Proc. of the 1st Int. Workshop, 1996, pp. 193–220.
URL `citeseer.ist.psu.edu/buchberger96symbolic.html`

[10] V. Weispfenning, Quantifier elimination for real algebra - the quadratic case and beyond, Applicable Algebra in Engineering, Communication and Computing 8 (2) (1997) 85–101.
URL `citeseer.ist.psu.edu/weispfenning93quantifier.html`

[11] M. Franzle, C. Herde, T. Teige, S. Ratschan, T. Schubert, Efficient solving of large non-linear arithmetic constraint systems with complex boolean structure., JSAT 1 (3-4) (2007) 209–236.

[12] C. Borralleras, S. Lucas, R. Navarro-Marset, E. Rodriguez-Carbonell, A. Rubio, Solving non-linear polynomial arithmetic via sat modulo linear arithmetic, CADE (2009) 294–305.

[13] A. Tiwari, G. Khanna, Nonlinear systems: Approximating reach sets, in: Hybrid Systems: Computation and Control HSCC, Vol. 2993 of LNCS, 2004, pp. 600–614.

[14] E. Rodriguez-Carbonell, A. Tiwari, Generating polynomial invariants for hybrid systems, in: Hybrid Systems: Computation and Control, HSCC 2005, Vol. 3414 of LNCS, 2005, pp. 590–605.

[15] B. Akbarpour, L. C. Paulson, Applications of metitarski in the verification of control and hybrid systems, in: HSCC '09: Proc. of the 12th Inter. Conf. on Hybrid Systems: Computation and Control, 2009.

[16] A. Platzer, E. M. Clarke, Computing differential invariants of hybrid systems as fixedpoints, in: Computer-Aided Verification, CAV'08, LNCS, 2008.

[17] E. Asarin, T. Dang, A. Girard, Hybridization methods for the analysis of nonlinear systems, Acta Inf. 43 (7) (2007) 451–476.

[18] T. Sturm, A. Tiwari, Verification and synthesis using real quantifier elimination, in: Proceedings of the 36th international symposium on Symbolic and algebraic computation, ISSAC '11, ACM, New York, NY, USA, 2011, pp. 329–336. doi:http://doi.acm.org/10.1145/1993886.1993935.
URL `http://doi.acm.org/10.1145/1993886.1993935`

[19] S. Sankaranarayanan, Automatic abstraction of non-linear systems using change of variables transformations, in: Inter. Conf. on Hybrid Systems: Computation and Control (HSCC), 2011.

[20] S. Sankaranarayanan, Automatic invariant generation for hybrid systems using ideal fixed points, in: Proc. of the 13th ACM Inter. Conf. on Hybrid systems: computation and control, HSCC '10, 2010.

[21] T. Dang, D. Salinas, Image computation for polynomial dynamical systems using the bernstein expansion, in: CAV, 2009, pp. 219–232.

[22] R. Rebiha, N. Matringe, A. V. Moura, Transcendental inductive invariants generation for non-linear differential and hybrid systems, in: 15th Int. Conf. Hybrid Systems: Computation and Control (part of CPS Week 2012), HSCC2012, Beijing, China, April 17-19, 2012, ACM, 2012, pp. 25–34.

[23] R. Rebiha, A. V. Moura, N. Matringe, Multivariate formal power series invariants generation for non linear hybrid systems, Tech. Rep. IC-09-48, Institute of Computing, University of Campinas (December 2009).

[24] R. Rebiha, A. V. Moura, N. Matringe, Endomorphisms for non-trivial non-linear loop invariant generation, in: 5th Int. Conf. Theoretical Aspects of Computing, LNCS, 2008, pp. 425–439.

[25] R. Rebiha, A. V. Moura, N. Matringe, Endomorphism for non-trivial semi-algebraic loop invariant generation, Tech. Rep. TR-IC-08-31, Institute of Computing, University of Campinas (November 2008).

[26] R. Rebiha, A. V. Moura, N. Matringe, Generatin invariants for non-linear hybrid systems by linear algebraic methods, in: 17th Int. Static Analysis Symposium, SAS2010, LNCS, 2010.

[27] R. Rebiha, A. V. Moura, N. Matringe, Morphisms for non-trivial non-linear invariant generation for algebraic hybrid systems, in: 12th Int. Conf. Hybrid Systems: Computation and Control (HSCC2009), LNCS, 2009.

[28] N. Matringe, A. V. Moura, R. Rebiha, Morphisms for analysis of hybrid systems, in: ACM/IEEE Cyber-Physical Systems CPSWeek'09, 2nd Inter. Conf. on Numerical Software Verification.(NSV2009) Verification of Cyber-Physical Software Systems, 2009.

[29] R. Rebiha, A. V. Moura, N. Matringe, Morphisms for non-trivial non-linear invariant generation for algebraic hybrid systems, Tech. Rep. TR-IC-08-32, Institute of Computing, University of Campinas (November 2008).

[30] D. Boularas, A. Chouikrat, Equations d'amorcage d'integrales premieres formelles, in: Linear and Multilinear Algebra, Vol. 54, 2006, pp. 219–233.