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# Minimal separators in extended $P_4$-laden graphs

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# Minimal separators in extended $P_4$-laden graphs*

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**Abstract**

We show that extended $P_4$-laden graphs have a linear number of minimal separators and present an algorithm to list them in linear time, extending an algorithm for $P_4$-sparse graphs given by Nikolopoulos and Palios. We also give bounds on the number and total size of all minimal separators of extended $P_4$-laden graphs and some of its subclasses, such as, $P_4$-tidy and $P_4$-lite graphs. Moreover, we show that these bounds are tight for all subclasses considered.

## 1 Introduction

A set $X$ of vertices is called an *ab-separator* of a graph $G$ if there is no path from $a$ to $b$ in $G[V(G) \setminus X]$, with $\{a, b\} \subseteq V(G) \setminus X$. A *minimal separator* of $G$ is a subset $X \subset V(G)$ such that $X$ is an $ab$-separator of $G$ for some $\{a, b\} \subset V(G)$ and there is no other $ab$-separator that is a proper subset of $X$.

Minimal separators have been extensivelly studied. Kloks and Kratsch [10] showed an $O(n^5 R)$ algorithm that lists all minimal separators of any graph, where $R$ is the number of minimal separators of the input graph. In 2000, Berry et al. [2] presented a faster algorithm for the same purpose, which takes $O(n^3 R)$ time. Moreover, it was conjectured [8, 9] that the existance of a polynomial-time algorithm for listing all minimal separators of any graph in a class of graphs imply the existance of polynomial-time algorithms for minimum fill-in and treewidth for graphs in the same class. These problems are NP-Complete for arbitrary graphs [1, 16].

Chordal, trapezoid, permutation, and $P_4$-sparse graphs have a polynomial number of minimal separators (see a review in [15]). Furthermore, there are linear-time algorithms to find the minimal separators of chordal [11], planar 3-connected [12], and $P_4$-sparse [15] graphs.

All graph classes considered in this article are known as classes with few $P_4$'s. These classes restrict the presence of induced $P_4$ on its graphs and all of them have nice characterizations in terms of special properties of the unique modular decomposition tree associated to each graph of the class. The modular decomposition tree of any graph can be computed in linear time [13, 14] and it is a powerful tool for obtaining linear-time algorithms, when restricted to some graph classes, for many problems that are hard in general.

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Among classes of graphs with few $P_4$'s, we highlight extended $P_4$-laden and $P_4$-laden graphs. They were introduced by Giakoumakis [4], who also showed that problems such as maximum cycle, vertex coloring, and maximum clique are solved in linear time for graphs in these classes. The extended $P_4$-laden graphs contain many well-studied classes, such as $P_4$-laden and $P_4$-tidy graphs, which, in their turn, contain $P_4$-lite, $P_4$-extendible, $P_4$-sparse, and cographs [5]. Each of these subclasses has specific applications. For example, scattering number and maximum matching are solved in linear time on $P_4$-tidy graphs [5, 3].

In this work we showed that extended $P_4$-laden graphs have a linear number of minimal separators and we extend the algorithm of Nikolopoulos and Palios to list all those separators in linear time.

In the following subsection, we give some definitions. In Section 2 we give some results about minimal separators which we use to construct an algorithm that lists all minimal separators of extended $P_4$-laden graphs. In Section 3, we show that the number and total size of all minimal separators of extended $P_4$-laden graphs are linearly bounded. Finally, in Section 4, we show that these bounds are tight.

## 1.1 Definitions

In this work, all graphs are finite, simple and undirected. A graph $G$ has vertex set $V(G)$ and edge set $E(G)$. Let $n_G = |V(G)|$ and $m_G = |E(G)|$. The set of vertices adjacent to $v \in V(G)$ is denoted by $N_G(v)$. The subgraph induced by $X \subseteq V(G)$ is denoted by $G[X]$. We denote by $P_n$ an induced path on $n$ vertices and by $C_n$ an induced cycle on $n$ vertices. The complement of a graph $G$ is denoted by $\overline{G}$. Let $\lambda(G)$ be the set of all minimal separators of $G$, $l_G = |\lambda(G)|$, and $s_G = \sum_{X \in \lambda(G)} |X|$.

Given two graphs $G_1$ and $G_2$ such that $V(G_1) \cap V(G_2) = \varnothing$, the disjoint union operation (denoted by $G_1 \cup G_2$) produces a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, and the join operation (denoted by $G_1 + G_2$) produces a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{\{v_1, v_2\} : v_1 \in V(G_1), v_2 \in V(G_2)\}$.

A clique is a set of pairwise adjacent vertices and a stable set is a set of pairwise non-adjacent vertices.

A *module* of a graph $G$ is a subset $M \subseteq V(G)$ such that, for every vertex $v \in (V(G) \setminus M)$, $v$ is adjacent to either every vertex in $M$ or to none of them. A module $M$ is *strong* if there is no other module $N$ of $G$ such that $M \nsubseteq N$, $M \nsupseteq N$, and $M \cap N \neq \varnothing$.

The *modular decomposition tree* [14] of $G$ is a tree where each node corresponds to a strong module of $G$ and the parent of a node $x$ is the one related to the smaller strong module that properly contains the module corresponding to $x$. The children of the root of a modular decomposition tree of a graph $G$ represent proper maximal strong modules of $G$ that are a partition of its vertices. If we contract each one of these modules to a single vertex (called representative vertex of the module) we get the *quotient graph* of $G$.

A *spider* is a graph whose vertex set can be partitioned into three sets $(K, S, R)$ such that:

- $|K| = |S| \geq 2$;
- $K$ is a clique and $S$ is a stable set;

- There is a bijection $b : K \to S$ such that, $\{u, v\} \in E(G)$, for $u \in S$ and $v \in K$, if, and only if, $b(v) = u$ (*thin spider*) or $b(v) \neq u$ (*thick spider*); and

- For every $r \in R$, $K \subseteq N_G(r)$ and $S \cap N_G(r) = \varnothing$.

A *split* operation produces a graph $G$ from another graph $G'$ by splitting a vertex $v \in V(G')$ into two vertices, $v'$ and $v''$, such that $N_G(v') = N_G(v'') = N_{G'}(v)$, and, optionally, adds the edge $\{v', v''\}$ to $E(G')$.

A *pseudo-spider* $G$ is a graph obtained from a spider $G'$ by a split operation applied on a single vertex $v \in K \cup S$. The vertices $v'$ and $v''$ are in the same part ($K$ or $S$) that contained the split vertex $v$.[1] Note that $\{v', v''\}$ is a module of $G$.

A *split* graph is a graph whose vertex set admits a partition into a clique $K$ and a stable set $S$. A split graph may have more than one such partitions. We say that a split graph has a split partition $(K, S)$.

Given a split graph $G$ and a partition $(K, S)$ of $V(G)$, we define three special sets, $S(G) = \{v \in S : N_G(v) \cap K \neq \varnothing \wedge K \not\subset N_G(v)\}$, $K(G) = \{v \in K : \exists u \in S(G) \wedge \{u, v\} \in E(G)\}$, and $R(G) = V(G) \setminus (K(G) \cup S(G))$.

The graphs $P_4$-lite, $P_4$-tidy, $P_4$-laden, and extended $P_4$-laden have a similar recursive characterization. A graph $G$ is *extended $P_4$-laden* or *$P_4$-tidy* if, and only if, it is:

- a disjoint union or join of graphs in the class;

- a spider or pseudo-spider with partition $(K, S, R)$ where $R$ is empty or $G[R]$ is a graph in the class;

- a $C_5$, $P_5$, its complement, or a single-vertex graph; or

- a graph whose quotient graph $Q$ is a split graph with a partition of $V(Q)$ in $S(Q)$, $K(Q)$, and $R(G)$, such that the module represented by $v \in V(Q)$ (only allowed for extended $P_4$-laden graphs):
  - is a stable set in $G$, if $v \in S(Q)$;
  - is a clique in $G$, if $v \in K(Q)$; and
  - induces a extended $P_4$-laden graph, if $v \in R(Q)$.

A graph is *$P_4$-lite* [7] if it is $P_4$-tidy graph with no induced $C_5$ and a graph is *$P_4$-laden* if it is a extended $P_4$-laden graph with no induced $C_5$.

## 2 Minimal separators of some graphs

In this section, we describe all minimal separators of some particular classes of graphs. Each of these classes is related to one of the cases in the definition of extended $P_4$-laden graphs given above. From these results, we construct an algorithm that lists all minimal separators of any extended $P_4$-laden graph.

The following lemma is due to Nikolopoulos and Palios [15].

[1]In our definition, a spider is not a pseudo-spider.

**Lemma 2.1.** *Let $G$ be any graph, then*

- *If $G$ is disconnected and $B_1, B_2, \ldots, B_p$, $p \geq 2$, are its connected components, then $\lambda(G) = \{\varnothing\} \cup \bigcup_{i=1}^{p} \lambda(B_i)$.*
- *If $\overline{G}$ is disconnected and $B_1, B_2, \ldots, B_p$, $p \geq 2$, are the complement of connected components of $\overline{G}$, then $T \in \lambda(G)$ if, and only if, $T = T_i \cup (V(G) \setminus V(B_i))$ and $T_i \in \lambda(B_i)$.*
- *If $G$ is a spider with partition $(K, S, R)$, then $T \in \lambda(G)$ if, and only if,*
    - *$T = K \cup T_H$ if $T_H \in \lambda(G[R])$; or*
    - *$T = \{v\}, v \in K$ if $G$ is a thin spider; or*
    - *$T = K \setminus \{v\}, v \in K$ if $G$ is a thick spider.*

Given a graph $H$, a set of pairwise disjoint modules of $H$, $\{M_1, \ldots, M_p\}$, and $v_i \in M_i$, for $1 \leq i \leq p$, we construct a contracted graph $G = H[V]$ such that $V = (V(H) \setminus (M_1 \cup M_2 \cup \ldots \cup M_p)) \cup \{v_1, v_2, \ldots, v_p\}$. We denote the contracted graph by $H|\{M_1, M_2, \ldots, M_p\}$ and we call $v_i$ the representative vertex of $M_i$.

The following lemma shows that the minimal separators of a graph $H$ can be computed from the minimal separators of the subgraph induced by the module $M$ of $H$ and from the minimal separators of the contracted graph $H|\{M\}$.

**Lemma 2.2.** *Let $H$ be a graph, $M$ be a module of $H$ with $|M| \geq 2$, $G = H|\{M\}$, and $v$ be the representative vertex of $M$. Then, $\lambda(H) = \{T : T \in \lambda(G) \wedge v \notin T\} \cup \{(T \setminus \{v\}) \cup M : T \in \lambda(G) \wedge v \in T\} \cup \{T \cup N_G(v) : T \in \lambda(H[M])\}$.*

*Proof.* For simplicity, in this proof we consider that all paths are *induced* and we introduce two notations. For a graph $G$, a set $T \subset V(G)$, non-adjacent vertices $a$ and $b$ in $V(G) \setminus T$, and $c \in T$: $L(G, T, a, b)$ is the set of all (induced) paths in $G$ from $a$ to $b$ that do not contain vertices in $T$ and $M(G, T, a, b, c)$ is the set of all (induced) paths in $G$ from $a$ to $b$ that contain vertex $c$ and no other vertex in $T$. Note that, $T$ is an $ab$-separator of $G$ if, and only if, $L(G, T, a, b) = \varnothing$ and $T$ is a minimal $ab$-separator of $G$ if, and only if, $L(G, T, a, b) = \varnothing$ and $M(G, T, a, b, c) \neq \varnothing$, for any $c \in T$.

First, we show that minimal separators of $H$ can be obtained from minimal separators of $G$ and $H[M]$, as described in this lemma. Note that any induced path from $a$ to $b$ in $H$ that contains one vertex in $V(H) \setminus M$, has at most one vertex of $M$, by construction of $H$.

Let $T$ be a minimal $ab$-separator of $H$.

- If $\{a, b\} \subseteq V(G)$ and $T \cap M = \varnothing$: We must show that $T \in \lambda(G)$. Since $T \in \lambda(H)$, then $L(H, T, a, b) = \varnothing$ and $M(H, T, a, b, c) \neq \varnothing$, for all $c \in T$.

  First, since $G$ is an induced subgraph of $H$, we have $L(G, T, a, b) = \emptyset$.

  Now, let $P \in M(H, T, a, b, c)$. If $P$ contains no vertex in $M$, then $P$ is an path of $G$ and $P \in M(G, T, a, b, c)$. Otherwise, $P$ contains one vertex $x \in M$. Then, we construct $P'$ changing $x$ by $v$ and we get $P' \in M(G, T, a, b, c)$. Therefore, we have $T \in \lambda(G)$.

- If $\{a,b\} \subseteq V(G)$ and $T \cap M \neq \varnothing$: First, we show that $M \subseteq T$. Let $y \in T \cap M$ and $P \in M(H,T,a,b,y)$. Note that $P$ contains no other vertex of $M$ besides $y$. Then, since $M$ is a module, we can change $y$ in $P$ by any other vertex of $M$, which implies $M \subseteq T$.

  Now, we show that $T' \in \lambda(G)$, if $T' = (T \setminus M) \cup \{v\}$. Suppose $T'$ is not an $ab$-separator of $G$, then there is an path $P \in L(G,T',a,b)$. But, since $P$ does not contain $v$, we would also have $P \in L(H,T,a,b)$, a contradiction.

  Let $x \in T'$. If $x \neq v$, then $x \in T$ and there is a path $P \in M(H,T,a,b,x)$. Since $P$ does not intercept $M$, $P \in M(G,T',a,b,x)$. For $x = v$, take one path in $M(H,T,a,b,y)$ for some vertex $y \in M$, and change $y$ by $v$ to get a path in $M(G,T',a,b,v)$.

- If $a \in V(G)$ and $b \in M$: Since any path $P$ from $a$ to $b$ in $H$ does not contain any vertex of $M$ besides $b$, we have $T \cap M = \varnothing$. Moreover, for each such path $P$, we construct an path $P'$ in $G$, changing $b$ by $v$. Thus, $T$ is a minimal $av$-separator of $G$.

- If $\{a,b\} \subseteq M$: First, we show that $T \cap V(G) = N_G(v)$. If there is some $x \in N_G(v)$ and $x \notin T$, the path $(a,x,b)$ would be in $L(H,T,a,b)$, a contradiction. So, $N_G(v) \subseteq T$. On the other hand, if $x \in T$ and $x \in V(G) \setminus N_G(v)$, there should be a path in $M(H,T,a,b,x)$. But, any such a path must contain some vertex in $N_G(v)$, a contradiction.

  Now, let $T' = T \setminus N_G(v)$. Clearly, $T' \subset M$. Since every path from $a$ to $b$ in $H[M]$ is a path from $a$ to $b$ in $H$, if $T'$ were not a minimal $ab$-separator of $H[M]$, $T$ would not be a minimal $ab$-separator of $H$.

Now, we show any minimal separator of $G$ or of $H[M]$ can be used with the rules on this lemma to construct a minimal separator of $H$. Consider the following cases:

- $T$ is a minimal $ab$-separator of $G$ and $v \notin T$. We show that $T \in \lambda(H)$. Consider $a \neq v$ and $b \neq v$ and suppose that $T$ is not an $ab$-separator of $H$. So, there is a path $P$ in $L(H,T,a,b)$. Note that $P$ can not contain more than one vertex of $M$. So, it can be transformed into a path of $G$ by changing the only vertex in $M$ by $v$, if it exists. This path would not contain vertices of $T$, so it would be in $L(G,T,a,b)$, a contradiction.

  Now, for each $x \in T$, select a path $P \in M(G,T,a,b,x)$ and construct $P'$ from $P$ by changing $v$ by any vertex of $M$ if $P$ contains $v$. Hence, $P' \in M(H,T,a,b,x)$.

  The previous arguments can also be applied (with few modifications) if $T$ is a minimal $av$-separator of $G$. In this case, the arguments show that $T$ is a minimal $ax$-separator of $H$, for any $x \in M$.

- $T$ is a minimal $ab$-separator of $G$ and $v \in T$. We show that $T' = (T \setminus \{v\}) \cup M$ is a minimal $ab$-separator of $H$. If there is a path $P \in L(H,T',a,b)$, then $P \in L(G,T,a,b)$, since $P$ contains only vertices in $V(G)$.

  Any path in $M(G,T,a,b,x)$, for $x \neq v$, is also a path in $H$ and it is in $M(H,T',a,b,x)$. We take a path in $M(G,T,a,b,v)$ and change $v$ by any vertex in $x \in M$ to get a path in $M(H,T',a,b,x)$.

- $T$ is a minimal $ab$-separator of $H[M]$. We show that $T' = T \cup N_G(v)$ is a minimal $ab$-separator of $H$. If there is some $P \in L(H, T', a, b)$, the vertices of $P$ must be in $M$, since to go from $M$ to $V(H) \setminus M$, the path must contain a vertex of $N_G(v)$. So, $P$ would be in $L(H[M], T, a, b)$, a contradiction.

  For any $x \in M \cap T'$, the path $P \in M(H[M], T, a, b, x)$ is also contained in $M(H, T', a, b, x)$. For $x \in N_G(v)$, the path $(a, x, b)$ is in $M(H, T', a, b, x)$. □

Note that the results given in Lemma 2.1 can be obtained from Lemma 2.2 and the minimal separators of prime spiders (spiders with $|R| \leq 1$).

From lemmas 2.1 and 2.2, we can characterize minimal separators of pseudo-spiders.

**Corollary 2.3.** *Let $G'$ be a spider and $G$ a pseudo-spider obtained from $G'$ by splitting vertex $v$ into $v'$ and $v''$. Then, $T \in \lambda(G)$ if, and only if,*

- *$T = T'$, where $T' \in \lambda(G')$ and $v \notin T'$; or*
- *$T = (T' \setminus \{v\}) \cup \{v', v''\}$, where $T' \in \lambda(G')$ and $v \in T'$; or*
- *$T = N_G(v') = N_G(v'')$, if $v \in K$ and $v'$ is not adjacent to $v''$.*

To characterize minimal separators of $P_4$-tidy graphs, we also need to analyze some base graphs, namely, $P_5$, $\overline{P_5}$, $C_5$, and $K_1$, that can be easily identified by inspection.

**Remark 2.4.** *Let $G$ be a graph. If*

- *$G \cong K_1$, there are no separator;*
- *$G \cong C_5$, each pair of non-adjacent vertices of $G$ is a minimal separator;*
- *$G \cong P_5$, every vertex with degree two is a minimal separator;*
- *$G \cong \overline{P_5}$, let $V(\overline{G}) = \{a, b, c, d, e\}$ and $E(\overline{G}) = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}\}$. Then, $\lambda(G) = \{\{a, b\}, \{a, e\}, \{d, e\}\}$.*

The last graphs we analyze are split graphs, for those we also characterize its minimal separators.

**Lemma 2.5.** *Let $G$ be a split graph and $(K, S)$ be a split partition of $G$ with maximal $K$. Then, $P \in \lambda(G)$ if, and only if, $P = N_G(v)$, $v \in S$.*

*Proof.* Let $G$ be a split graph with a split partition $(K, S)$ such that $K$ is a maximal clique.

If $v \in S$ and $P = N_G(v)$, then there is a $u \in K \setminus P$ since $K$ is maximal. Hence, $P$ is a $uv$-separator because $v$ has no neighbors in $G[V(G) \setminus P]$. Moreover, $P$ is a minimal $uv$-separator since for any $w \in P$, $w$ is adjacent to both $u$ and $v$.

Now, let $P$ be a minimal separator of $G$. Then, there is a $\{u, v\} \subset V(G)$ such that $P$ is a minimal $uv$-separator of $G$ and one of the following conditions must be satisfied:

- $u \in K$ and $v \in S$: Note that $P$ must contain all vertices in $N_G(v) \cap N_G(u)$, which is equal to $N_G(v)$ since $N_G(v) \subset K \subseteq N_G(u)$. Then, $P$ is a minimal $uv$-separator only if $P = N_G(v)$.

- $\{u, v\} \subseteq S$: Since $N_G(u) \subset K$ and $N_G(v) \subset K$, then $N_G(u) \subseteq P$ or $N_G(v) \subseteq P$. Otherwise, there are $x \in N_G(u) \setminus P$ and $y \in N_G(v) \setminus P$, which make a path $(u, x, y, v)$ in $G[V(G) \setminus P]$. Now, if $P$ is a minimal $uv$-separator, it must be $N_G(v)$ or $N_G(u)$ since these sets are $uv$-separators. □

### 2.1 An algorithm for extended $P_4$-laden graphs

Now, we describe an algorithm that lists the minimal separators of an extended $P_4$-laden graph $G$. First, it computes the modular decomposition tree of $G$, which contains all its strong modules. Then, it performs a depth-first search on that tree to compute $\lambda(G[X])$ for each strong module $X$ of $G$. At each node, it identifies if $G[X]$ is isomorphic to a graph considered in Lemma 2.1, Corollary 2.3, Remark 2.4, or Lemma 2.5 and apply the appropriated rules to find its minimal separators. Otherwise, $G[X]$ is a modified split graph, which is recognized testing if the associated quotient graph $Q$ is a split graph. In this case, we first expand any possible vertices in $S(Q)$ or $K(Q)$ that are non-trivial modules (modules with more than one vertex) of $G[X]$ to corresponding stable sets and cliques, producing a bigger split graph. Then, we use rules on Lemma 2.5 on $Q$ and apply Lemma 2.2 to vertices in $R(Q)$ that represent non-trivial modules of $G[X]$.

Note that the modular decomposition tree can be computed in linear time [13, 14], as well as the identification of the type of each strong module [4] and the computation of a maximal partition of each split graph [6]. So, our algorithm will be $O(n_G + m_G)$ if $s_G$ and $l_G$ are linearly bounded on the size of the graph $G$.

## 3 Bounds on minimal separators

In this section, we are interested in bounding $l_G$ and $s_G$ for extended $P_4$-laden graphs and some of its subclasses. We show $l_G$ and $s_G$ are linearly bounded in the size of $G$ for all of them.

First, we bound $l_G$ and $s_G$ for the graphs whose minimal separators were determined in Section 2. We aim to find the smaller value of $\alpha$ such that the following property is satisfied.

**Claim 3.1.** *For a graph $G$ and a real number $\alpha \geq 0$, the following conditions are satisfied.*

$$s_G \leq 2\alpha m_G \qquad\qquad l_G \leq \begin{cases} \alpha n_G - 1/3 & \text{if } G \text{ is disconnected;} \\ \alpha n_G - 2/3 & \text{if } G \text{ is connected.} \end{cases}$$

We analyze graphs that are disjoint unions and joins (complements of disjoint unions) of other graphs. We call them by union and join graphs, respectively.

**Lemma 3.2.** *Let $G = \bigcup_{i=1}^{p} G_i$, $p \geq 2$, be a union of connected graphs $G_i$. Then, $G$ satisfies Claim 3.1 for $\alpha$ if each $G_i$ satisfies Claim 3.1 for $\alpha$.*

*Proof.* Let $G = \bigcup_{i=1}^{p} G_i$, $p \geq 2$, be a union of connected graphs $G_i$. By Lemma 2.1,

$$l_G = 1 + \sum_{i=1}^{p} l_{G_i} \leq 1 + \sum_{i=1}^{p}(\alpha n_{G_i} - \frac{2}{3}) = \alpha n_G + \frac{3-2n}{3} \leq \alpha n_G - \frac{1}{3}$$

$$s_G = \sum_{i=1}^{p} s_{G_i} \leq \sum_{i=1}^{p} 2\alpha m_{G_i} = 2\alpha m_G.$$

□

**Lemma 3.3.** *Let $G = G_1 + G_2 + \ldots + G_p$, $p \geq 2$, be a join graph. Then, $G$ satisfies Claim 3.1 for $\alpha$ if each $G_i$ satisfies Claim 3.1 for $\alpha$.*

*Proof.* Let $G = G_1 + G_2 + \ldots + G_p$, $p \geq 2$, be a join graph. By Lemma 2.1,

$$l_G = \sum_{i=1}^{p} l_{G_i} \leq \sum_{i=1}^{p}(\alpha n_{G_i} - \frac{1}{3}) = \alpha n_G - \frac{p}{3} \leq \alpha n_G - \frac{2}{3}$$

$$s_G = \sum_{i=1}^{p}[(n_G - n_{G_i})l_{G_i} + s_{G_i}] \leq \sum_{i=1}^{p}[(n_G - n_{G_i})(\alpha n_{G_i} - \frac{1}{3}) + 2\alpha m_{G_i}]$$

Since $m_G = \sum_{i=1}^{p}[\frac{n_{G_i}}{2}(n_G - n_{G_i}) + m_{G_i}]$, to have $s_G \leq 2\alpha m_G$ we need

$$\sum_{i=1}^{p}[(n_G - n_{G_i})(\alpha n_{G_i} - \frac{1}{3}) + 2\alpha m_{G_i}] \leq 2\alpha \sum_{i=1}^{p}[\frac{n_{G_i}}{2}(n_G - n_{G_i}) + m_{G_i}]$$

which is satisfied since $\alpha n_{G_i} - \frac{1}{3} \leq \alpha n_{G_i}, \forall i, 1 \leq i \leq n.$

□

From these results we see that the minimum value of $\alpha$ that satisfies Claim 3.1 for a union or join graph $G$ is the greater among the minimum values of $\alpha$ that satisfy the same claim for each $G_i$, $1 \leq i \leq p$.

Now we analyze spiders and pseudo-spiders. First, we compute $l_G$ and $s_G$ of a spider $G$ with partition $(K, S, R)$. Let $k = |K|$ and $H = G[R]$. Then, $l_G = l_H + k$ and $n_G = n_H + 2k$. The values of $m_t$ and $s_t$ ($m_T$ and $s_T$) of a thin (thick) pseudo-spider $G$ are given in Table 1.

**Lemma 3.4.** *Let $G$ be a spider with partition $(K, S, R)$ and $H = G[R]$. If $R \neq \varnothing$ and $H$ satisfies Claim 3.1, then $G$ satisfies Claim 3.1 for* $\max\{\alpha, 2/3\}$. *If $R = \varnothing$, $G$ satisfies Claim 3.1 for $\alpha = 2/3$.*

| Type | Number of edges | Size of separators |
|---|---|---|
| Thin | $m_t = m_H + (k^2 + k)/2 + kn_H$ | $s_t = s_H + l_H k + k$ |
| Thick | $m_T = m_H + 3(k^2 - k)/2 + kn_H$ | $s_T = s_H + l_H k + k(k-1)$ |

Table 1: Parameters of spiders.

*Proof.* Let $G$ be a spider with partition $(K, S, R)$. Recall that $k \geq 2$. By Lemma 2.1 and once $R$ may be empty (but $l_H \geq 0$), then $l_G = k + l_H \leq k + \alpha n_H$. Since $G$ is connected, we want $l_G \leq \alpha n_G - 2/3$.

$$k + \alpha n_H \leq \alpha(n_H + 2k) - \frac{2}{3} \iff k \leq 2\alpha k - \frac{2}{3}$$
$$\alpha \geq \frac{k + \frac{2}{3}}{2k} \text{ which is true if } \alpha \geq \frac{2}{3}$$

Now, we want $s_G \leq 2\alpha m_G$. We analyze two cases, when $G$ is a thin spider and when it is a thick spider. In both cases, we build the desired relation from the hypothesis on $H$ and equations in Table 1.

If $G$ is a thin spider, we have:

$$\begin{aligned} s_G = s_H + l_H k + k &\leq 2\alpha m_H + \alpha n_H k + k \\ s_G \leq 2\alpha m_G \text{ if } 2\alpha m_H + \alpha n_H k + k &\leq 2\alpha(m_H + n_H k + \frac{k^2}{2} + \frac{k}{2}) \\ \alpha n_H k + k &\leq \alpha(2n_H k + k^2 + k) \\ \alpha n_H + 1 &\leq \alpha(2n_H + k + 1) \\ \alpha \geq \frac{1}{n_H + k + 1} &\text{ which is true if } \alpha \geq 1/3. \end{aligned}$$

Finally, if $G$ is a thick spider, we want:

$$\begin{aligned} s_G = s_H + l_H k + k^2 - k &\leq 2\alpha m_H + \alpha n_H k + k^2 - k \\ s_G \leq 2\alpha m_G \text{ if } 2\alpha m_H + \alpha n_H k + k^2 - k &\leq 2\alpha(m_H + n_H k + \frac{3k^2}{2} - \frac{3k}{2}) \\ \alpha n_H k + k^2 - k &\leq \alpha(2n_H k + 3k^2 - 3k) \\ \alpha n_H + k - 1 &\leq \alpha(2n_H + 3k - 3) \\ \text{which is true if } k - 1 &\leq \alpha(3k - 3) \\ \text{satisfied if } \alpha &\geq 1/3. \end{aligned}$$

□

Next, we compute $l_G$ and $s_G$ of a pseudo-spider $G$ with partition $(K, S, R)$. Let $k = \min\{|K|, |S|\}$ and $H = G[R]$. If $G$ is a pseudo-spider obtained from a spider by splitting a vertex $v \in K$ into two non-adjacent vertices $v'$ and $v''$, then $l_G = l_H + k + 1$. For all other pseudo-spiders, $l_G = l_H + k$. Note also that $n_G = n_H + 2k + 1$ for any pseudo-spider. Finally, in tables 2 and 3, we show the values of $m_G$ and $s_G$, based on the values of $s_t$, $m_t$, $s_T$, and $m_T$ for spiders, given in Table 1.

| Type | $\{v', v''\} \notin E(G)$ | | $\{v', v''\} \in E(G)$ | |
|---|---|---|---|---|
| | $v \in S$ | $v \in K$ | $v \in S$ | $v \in K$ |
| Thin | $m_t + 1$ | $m_t + k + n_H$ | $m_t + 2$ | $m_t + k + n_H + 1$ |
| Thick | $m_T + k - 1$ | $m_T + 2k + n_H - 2$ | $m_T + k$ | $m_T + 2k + n_H - 1$ |

Table 2: Number of edges in pseudo-spiders.

| Type | $v \in S$ | $v \in K$ | |
|---|---|---|---|
| | — | $\{v', v''\} \in E(G)$ | $\{v', v''\} \notin E(G)$ |
| Thin | $s_t$ | $s_t + l_H + 1$ | $s_t + l_H + k + n_R + 1$ |
| Thick | $s_T$ | $s_T + l_H + k - 1$ | $s_T + l_H + 3k + n_R - 3$ |

Table 3: Size of separators in pseudo-spiders.

**Lemma 3.5.** *Let $G$ be a pseudo-spider with partition $(K, S, R)$ and $H = G[R]$. If $R \neq \varnothing$ and $H$ satisfies Claim 3.1 for $\alpha$, then $G$ satisfies Claim 3.1 for* $\max\{\alpha, 11/15\}$. *If $R = \varnothing$, $G$ satisfies Claim 3.1 for $\alpha = 11/15$.*

*Proof.* Let $G$ be a pseudo-spider with partition $(K, S, R)$. Recall that $k \geq 2$. Then, by Corollary 2.3 and hypothesis on $H$, $l_G \leq k + l_H + 1 \leq k + \alpha n_H + 1$. So, to have $l_G \leq \alpha n_G - 2/3$, we need

$$
\begin{aligned}
k + \alpha n_H + 1 &\leq \alpha(n_H + 2k + 1) - \frac{2}{3} \\
k + 1 &\leq 2\alpha k + \alpha - \frac{2}{3} \\
\alpha &\geq \frac{k + \frac{5}{3}}{2k + 1}, \text{ which is true for } \alpha \geq \frac{11}{15}.
\end{aligned}
$$

By definition of pseudo-spiders, there are two vertices $v'$ and $v''$ in $K$ or in $S$ that we can contract to get a spider $G'$. Consider the following cases:

- $\{v', v''\} \subset S$: By Corollary 2.3, $s_G = s_{G'}$ and, by Lemma 3.4, $s_G \leq 2\alpha m_{G'} < 2\alpha m_G$, for $\alpha \geq 1/3$.

- $\{v', v''\} \subset K$ and $v'$ is non-adjacent to $v''$: In these case, we use equations given in tables 2 and 3.

  Let $G$ be a thin pseudo-spider with $R = \varnothing$. We have $s_G = 2k+1$ and $m_G = (k^2+3k)/2$. So, $s_G \leq 2\alpha m_G$ if

  $$\alpha \geq \frac{2k+1}{k^2+3k} \text{ which is true if } \alpha \geq 1/2.$$

  Now, if $R \neq \varnothing$, we have $s_G = s_H + l_H(k+1) + 2k + n_H + 1 \leq 2\alpha m_H + \alpha n_H(k+1) +$

$2k + n_H + 1$ and $m_G = m_H + n_H k + n_H + (k^2 + 3k)/2$. So, $s_G \leq 2\alpha m_G$ if

$$\alpha n_H(k+1) + 2k + n_H + 1 \leq \alpha(2n_H k + 2n_H + k^2 + 3k),$$
$$\text{satisfied if (a) } \alpha n_H(k+1) + n_H \leq 2\alpha n_H(k+1)$$
$$\alpha \geq \frac{1}{k+1}, \text{ true if } \alpha \geq \frac{1}{3}$$
$$\text{and if (b) } 2k+1 \leq \alpha(k^2+3k)$$
$$\alpha \geq \frac{2}{k+3} + \frac{1}{k^2+3k}, \text{ true if } \alpha \geq 1/2.$$

Otherwise, $G$ is a thick pseudo-spider. If $R = \varnothing$, then $s_G = k^2 + 2k - 3$ and $m_G = (3k^2 + k)/2 - 2$. So, $s_G \leq 2\alpha m_G$ if

$$\alpha \geq \frac{k^2 + 2k - 3}{3k^2 + k - 4}, \text{ which is true if } \alpha \geq 1/2.$$

Now, if $R \neq \varnothing$, we have $s_G = s_H + l_H(k+1) + k^2 + 2k + n_H - 3 \leq 2\alpha m_H + \alpha n_H(k+1) + k^2 + 2k + n_H - 3$ and $m_G = m_H + n_H k + n_H + (3k^2 + 3k)/2 - 2$. So, $s_G \leq 2\alpha m_G$ if

$$\alpha n_H(k+1) + k^2 + 2k + n_H - 3 \leq \alpha(2n_H k + 2n_H + 3k^2 + k - 2),$$
$$\text{satisfied if (a) } \alpha n_H(k+1) + n_H \leq 2\alpha n_H(k+1) \to \alpha \geq \frac{1}{3}$$
$$\text{and (b) } k^2 + 2k - 3 \leq \alpha(3k^2 + k - 4) \to \alpha \geq \frac{1}{2}$$

- $\{v', v''\} \subset K$ and $v'$ is adjacent to $v''$: Let $G^*$ the pseudo-spider obtained from $G$ removing the edge $\{v', v''\}$. Then, by Corollary 2.3 and the previous result, $s(G) < s(G^*) \leq 2\alpha m_{G^*} < 2\alpha m_G$ for $\alpha \geq \max\{\alpha, 1/2\}$.

□

Hence, the bounds that we obtain on the minimum $\alpha$ that satisfy Claim 3.1 for spiders and pseudo-spiders have a minimum that derives from the structure of these graphs and also depend on the graph $G[R]$.

Now, we analyze split graphs.

**Lemma 3.6.** *A split graph $G$ satisfies $l_G \leq n_G - 1$ and $s_G \leq m_G$. Moreover, if $G$ is connected and has 2 or more vertices, then $l_G \leq n_G - 2$.*

*Proof.* Let $G$ be a split graph with a split partition $(K, S)$, such that $K$ is maximal. The result follows directly from Lemma 2.5, which guarantees that $l_G \leq |S|$. Moreover, since $K$ is maximal, $|S| \leq n_G - 1$. Each minimal separator of $G$ is a set $P = N_G(v)$, $v \in S$. Hence, we can associate each vertex $u \in P$ to a distinct edge $\{u, v\}$ in $G$. If $G$ is connected and $n_G \geq 2$, $|K| \geq 2$ (since $K$ is maximal), so $l_G \leq n_G - 2$. □

As a direct consequence of Lemma 3.6, we know that split graphs satisfy Claim 3.1.

**Corollary 3.7.** *A split graph $G$ satisfies Claim 3.1 for any $\alpha \geq 1$.*

However, this bound is not tight on $s_G$. For this work, this is not a problem, since the graphs we analyze include join graphs of split graphs, a combination which makes bounds on Claim 3.1 tight. The following result is needed to consider modified split graphs that are $P_4$-laden graphs.

**Lemma 3.8.** *Let $G$ be a graph and let $M$ be a module of $G$, $2 \leq |M| < n_G$. Let $G' = G|\{M\}$ with $v$ the representative vertex of $M$. If $G'$ is a connected split graph and it has a split partition $(K', S')$ such that $v \in K'$, $N_G(v) \cap S' = \varnothing$, then $G$ satisfy Claim 3.1 for $\alpha$ if $G[M]$ does and $\alpha \geq 1$.*

*Proof.* Let $H = G[M]$. Note that, by construction of $G'$,

$$n_G = n_{G'} + n_H - 1 \qquad \text{and} \qquad m_G = m_{G'} + m_H + (|K'| - 1)(n_H - 1).$$

By hypothesis, $(K', S')$ is a split partition of $G'$ such that $K'$ is maximal. Then, by Lemma 2.5, we know that $v$ does not take part in any minimal separator of $G'$. So, by Lemma 2.2, we know that

$$l_G = l_{G'} + l_H \qquad \text{and} \qquad s_G = s_{G'} + s_H + l_H(|K'| - 1).$$

The graph $G'$ is connected and $n_{G'} \geq 2$. Then, by Lemma 3.6, $l_{G'} \leq n_{G'} - 2$ and $s_{G'} \leq m_{G'}$.

First, we prove the restriction on $l_G$, namely, $l_G \leq \alpha n_G - 2/3$.

$$\begin{aligned} l_{G'} + l_H &\leq \alpha(n_{G'} + n_H - 1) - 2/3 \\ \text{by hypothesis on } H,\ l_H &\leq \alpha n_H - 1/3 \\ \text{hence, } l_{G'} &\leq \alpha(n_{G'} - 1) - 1/3. \\ \text{Since } l_{G'} \leq n_{G'} - 2,\ 0 &\leq \alpha(n_{G'} - 1) - n_{G'} + 5/3. \\ \text{So, } \alpha &\geq \frac{n_{G'} - 5/3}{n_{G'} - 1} \text{ which is true if } \alpha \geq 1. \end{aligned}$$

Now, we show that $s_G \leq 2\alpha m_G$.

$$\begin{aligned} s_{G'} + s_H + l_H(|K'| - 1) &\leq 2\alpha(m_{G'} + m_H + (n_H - 1)(|K'| - 1)) \\ &\text{by hypothesis on } H,\ s_H \leq 2\alpha m_H \\ \text{so, } s_{G'} + l_H(|K'| - 1) &\leq 2\alpha(m_{G'} + (n_H - 1)(|K'| - 1)) \\ \text{again, by hypothesis on } H,\ l_H(|K'| - 1) &\leq (\alpha n_H - 1/3)(|K'| - 1) \\ \text{so, } s_{G'} &\leq \alpha(2m_{G'} + (n_H - 2)(|K'| - 1)) + |K'|/3 - 1/3 \\ \text{since } G' \text{ is split, } 0 &\leq \alpha(2m_{G'} + (n_H - 2)(|K'| - 1)) + |K'|/3 - 1/3 - m_{G'} \\ \text{so, } \alpha &\geq \frac{m_{G'} - (|K'| - 1)/3}{2m_{G'} + (n_H - 2)(|K'| - 1)} \\ &\text{then, since } m_{G'} \geq 1, \text{ it is enough that } \alpha \geq 1/2. \end{aligned}$$

(Note that $m_{G'} \geq 1$ because $n_{G'} \geq 2$ and $G'$ is a connected graph by hypothesis.) ∎

In order to bound the minimal separators of $P_4$-laden graphs, we consider the following modified split graphs, and we show that, for some set of modules, the contracted graph satisties the conditions of Lemma 3.8.

**Lemma 3.9.** *Let $H$ be a graph, $M_1, M_2, \ldots, M_p$ ($p \geq 1$) be a set of pairwise disjoint modules of $H$ satisfying $2 \leq |M_i| < |V(H)|$, for $1 \leq i \leq p$, $G = H|\{M_1, M_2, \ldots, M_p\}$, and let $v_i$ be the representative vertex of $M_i$, for $1 \leq i \leq p$. If $G$ is a connected split graph with partition $(K, S)$ such that for $1 \leq i \leq p$, $v_i \in R(G)$ and $H[M_i]$ satisfies Claim 3.1 for $\alpha$ and $\alpha \geq 1$, then $H$ also satisfies Claim for $\alpha$.*

*Proof.* Let $L = \{v_1, v_2, \ldots, v_p\}$ and $M = M_1 \cup M_2 \cup \ldots \cup M_p \cup (R(G) \setminus L)$. Note that $R(G)$ is a module of $G$ and $M$ is a module of $H$. Without loss of generality, let $M_1, M_2, \ldots, M_q$ be the modules such that $\{v_1, v_2, \ldots, v_q\} \subseteq K$ and $M_{q+1}, M_{q+2}, \ldots, M_p$ be the modules such that $\{v_{q+1}, v_{q+2}, \ldots, v_p\} \subseteq S$. Also, let $(R(G) \setminus L) \cap K = \{k_1, k_2, \ldots, k_t\}$ and $(R(G) \setminus L) \cap S = \{s_1, s_2, \ldots, s_u\}$. Hence, $H[M] = H[M_1] + H[M_2] + \ldots + H[M_q] + H[k_1] + H[k_2] + \ldots + H[k_t] + (H[M_{q+1}] \cup H[M_{q+2}] \cup \ldots \cup H[M_p] \cup H[s_1] \cup H[s_2] \cup \ldots \cup H[s_u])$. Since every $H[M_i]$ satisfy Claim 3.1 for $\alpha$ and every single-vertex graph satisfies the Claim for 2/3, then, by lemmas 3.2 and 3.3, $H[M]$ satisfy the Claim for $\alpha$, if $\alpha \geq 2/3$.

Now, if $M = V(H)$, we are done since $H = H[M]$. Otherwise, $G' = H|\{M\}$ is a split graph with 2 or more vertices. Let $v$ be the representative vertex of $M$. Note that $G' \cong G|\{R(G)\}$ and $N_{G'}(v) = K(G)$. So, $(K(G) \cup \{v\}, S(G))$ is a split partition of $G'$. Since $G'$ is connected (because $G$ is), by Lemma 3.8 $H$ satisfies Claim 3.1 for $\alpha$ if every $M_i$ does and $\alpha \geq 1$. □

## 3.1 Linear bounds

Combining the previous results with the minimum values of $\alpha$ that satisfy Claim 3.1 for a single-vertex graph, $P_5$, $\overline{P_5}$, and $C_5$, we are now able to state the main result of this section – the bounds of $l_G$ and $s_G$ for extended $P_4$-laden graphs and subclasses.

**Theorem 3.10.** *If $G$ is a $P_4$-lite graph, then $l_G \leq 11n_G/15$ and $s_G \leq 22m_G/15$. If $G$ is a $P_4$-laden graph, then $l_G \leq n_G$ and $s_G \leq 2m_G$. If $G$ is a $P_4$-tidy or extended $P_4$-laden graph, then $l_G \leq 17n_G/15$ and $s_G \leq 34m_G/15$.*

*Proof.* It is enough to show that Claim 3.1 is valid for $P_4$-lite graphs with $\alpha = 11/15$, for $P_4$-laden graphs with $\alpha = 1$, and for $P_4$-tidy and extended $P_4$-laden graphs with $\alpha = 17/15$.

First we analyze four special (connected) graphs, whose separators were identified in Remark 2.4, and compute the minimum $\alpha$ for each of these graphs to satisfy Claim 3.1. If $G \cong K_1$, then $\alpha \geq 2/3$ (since $G$ is a connected graph); if $G \cong C_5$, then $\alpha \geq 17/15$; if $G \cong P_5$ or $G \cong \overline{P_5}$, then $\alpha \geq 11/15$.

The desired minimum values of $\alpha$ for all these classes came from lemmas 3.2, 3.3, 3.4, 3.5, and 3.9, Corollary 3.7, and the previous observations and characterizations of the classes given in Section 1. □

## 4 Tightness of bounds

Finally, we are going to show that the bounds defined on Theorem 3.10 are tight for each class. The meaning of tight that we use here is that there is no smaller ratios of $l_G/n_G$ and $s_G/m_G$ valid for all graphs in each considered class than those given in the Theorem 3.10.

First, we define the following infinite sequence of graphs given a base graph $G_0$: $G_i = G_{i-1} \cup G_{i-1}$ for $i = 1 \pmod 2, i \geq 1$, and $G_i = G_{i-1} + G_{i-1}$, for $i = 0 \pmod 2, i \geq 2$. Let $l_i = l_{G_i}$, $s_i = s_{G_i}$, $n_i = n_{G_i}$, and $m_i = m_{G_i}$.

**Lemma 4.1.** *Given a base graph $G_0$ to produce the infinite sequence of graphs defined above, the ratios $l_i/n_i$ and $s_i/m_i$ are as close to $(l_0 + 2/3)/n_0$ and $2(l_0 + 2/3)/n_0$, respectively, as we want, using increasing values of $i$.*

*Proof.* As showed in Lemma 2.1, we may infer:

$$l_i = 2l_{i-1}, \text{ for } i = 0 \text{ (mod 2) and } l_i = 2l_{i-1} + 1, \text{ for } i = 1 \text{ (mod 2)},$$

which leads to, for $i = 0 \pmod 2, i \geq 0$:

$$\begin{aligned} l_i &= 2l_{i-1} = 4l_{i-2} + 2 = 4(4l_{i-4} + 2) + 2 = \dots \\ &= 2^i l_0 + 2 \sum_{j=0}^{i/2-1} 4^j = 2^i(l_0 + \frac{2}{3}) - \frac{2}{3}. \end{aligned}$$

Now, since $n_i = n_0 \cdot 2^i$, we state, for $i = 0 \pmod 2, i \geq 2$:

$$\lim_{i\to\infty} \frac{l_i}{n_i} = \lim_{i\to\infty} (\frac{l_0 + \frac{2}{3}}{n_0} - \frac{\frac{2}{3}}{2^i n_0}) = \frac{l_0 + \frac{2}{3}}{n_0}.$$

We now analyze the relation between $s_i$ and $m_i$. By Lemma 2.1:

$$s_i = 2s_{i-1}, \text{ for } i = 1 \text{ (mod 2) and } s_i = 2s_{i-1} + 2l_{i-1}n_{i-1}, \text{ for } i = 0 \text{ (mod 2)}.$$

For $i = 0 \pmod 2$, $i \geq 2$, $s_i$ may be written as

$$s_i = 2s_{i-1} + 2l_{i-1}n_{i-1} = 4s_{i-2} + 2\frac{l_i}{2}\frac{n_i}{2} = 4s_{i-2} + \frac{l_i n_i}{2}.$$

Then, follows, for $i = 0 \pmod 2$ and large enough

$$s_i = 4(4s_{i-4} + \frac{l_{i-2}n_{i-2}}{2}) + \frac{l_i n_i}{2} = 4^2 s_{i-4} + 4\frac{l_{i-2}n_{i-2}}{2} + \frac{l_i n_i}{2}.$$

$$\text{Generalizing, } s_i = 2^i s_0 + \sum_{j=1}^{i/2} \frac{4^{i/2-j} l_{2j} n_{2j}}{2} = 2^i s_0 + \frac{2^i}{2} \sum_{j=1}^{i/2} \frac{l_{2j} n_{2j}}{4^j}$$

$$\text{but, } \frac{l_{2j} n_{2j}}{4^j} = \frac{(4^j(l_0 + \frac{2}{3}) - \frac{2}{3})4^j n_0}{4^j} = 4^j n_0 (l_0 + \frac{2}{3}) - \frac{2n_0}{3}$$

$$\text{and } \sum_{j=1}^{i/2} \frac{l_{2j} n_{2j}}{4^j} = \sum_{j=1}^{i/2} 4^j n_0 (l_0 + \frac{2}{3}) - \sum_{j=1}^{i/2} \frac{2n_0}{3} = 4n_0(l_0 + \frac{2}{3})\frac{2^i - 1}{3} - \frac{i n_0}{3},$$

$$\text{which implies } s_i = 2^i(s_0 + 2n_0(l_0 + \frac{2}{3})\frac{2^i - 1}{3} - \frac{i n_0}{6}).$$

From Lemma 2.1 we also have

$$m_i = 2m_{i-1}, \text{ for } i = 1 \pmod 2 \text{ and } m_i = 2m_{i-1} + n_{i-1}^2, \text{ for } i = 0 \pmod 2.$$

Then, for $i = 0 \pmod 2), i \geq 2$, we have

$$\begin{aligned} m_i &= 2m_{i-1} + n_{i-1}^2 = 4m_{i-2} + n_{i-1}^2 = 4(m_{i-2} + n_{i-2}^2) \\ &= 4(4(m_{i-4} + n_{i-4}^2) + n_{i-2}^2) = 4^2 m_{i-4} + 4^2 n_{i-4}^2 + 4n_{i-2}^2 \\ &= 2^i m_0 + \sum_{j=0}^{i/2-1} 4^{i/2-j} n_{2j}^2 = 2^i m_0 + \sum_{j=0}^{i/2-1} 4^{i/2-j} (2^{2j} n_0)^2 \\ &= 2^i m_0 + n_0^2 \sum_{j=0}^{i/2-1} 4^{j+i/2} = 2^i m_0 + 2^i n_0^2 \frac{2^i - 1}{3}. \end{aligned}$$

Finally, we can establish the ratio $s_i/m_i$:

$$\begin{aligned} \frac{s_i}{m_i} = \frac{2^i(s_0 + 2n_0(l_0 + \frac{2}{3})\frac{2^i-1}{3} - \frac{in_0}{6})}{2^i m_0 + 2^i n_0^2 \frac{2^i-1}{3}} &= \frac{s_0 + 2n_0(l_0 + \frac{2}{3})\frac{2^i-1}{3} - \frac{in_0}{6}}{m_0 + n_0^2 \frac{2^i-1}{3}} \\ &= \frac{6s_0 + 4n_0(l_0 + \frac{2}{3})(2^i - 1) - in_0}{6m_0 + 2n_0^2(2^i - 1)}. \end{aligned}$$

Thus, we get

$$\lim_{i \to \infty} \frac{s_i}{m_i} = \frac{2(l_0 + \frac{2}{3})}{n_0}.$$

□

Theorem 3.10 gives a bound on the total count and size of minimal separators of any $P_4$-lite, $P_4$-tidy, $P_4$-laden, or extended $P_4$-laden graph. Now we show the bounds are tight.

**Theorem 4.2.** *The bounds set in Theorem 3.10 are tight for each considered class of graphs.*

*Proof.* Let $L_0$ be a pseudo-spider with partition $(K, S, R)$ obtained from a $P_4$ by splitting a vertex in $K$ into two non-adjacent vertices. If we make $G_0 = L_0$, every $G_i$ would be a $P_4$-lite graph, and if we make $G_0 = C_5$, every $G_i$ would be a $P_4$-tidy.

By Lemma 4.1, if we use $G_0 = L_0$, there are $P_4$-lite graphs whose ratios $l_i/n_i$ and $s_i/m_i$ are as close to 11/15 and 22/15, respectively, as we want. As well, we can produce $P_4$-tidy graphs using $G_0 = C_5$ for those, these ratios approach 17/15 and 34/15.

The only point left is to show a family of $P_4$-laden graphs for which $l_G/n_G$ approaches 1 as well as $s_G/m_G$ approaches 2. To do that, we define the following sequence of split graphs:

Let $H_j$ be a split graph defined as having the clique $K = \{k_1, k_2, \ldots, k_j\}$. Let $\mathcal{C} = \{Y \subset K : Y \neq \varnothing\}$ and for each $X \in \mathcal{C}$, create a vertex $v_X$ in $S$ such that $N_{H_j}(v_X) = X$. By construction and by Lemma 2.5, $\lambda(H_j) = \mathcal{C}$.

We note that $l_{H_j} = 2^j - 2$ and $s_{H_j} = j2^{j-1} - j$, whereas $n_{H_j} = 2^j + j - 2$ and $m_{H_j} = j2^{j-1} - j + j(j-1)/2$. This let us show

$$\lim_{j\to\infty}\frac{l_{H_j}}{n_{H_j}} = \lim_{j\to\infty}\frac{2^j-2}{2^j+j-2} = 1 \text{ and } \lim_{j\to\infty}\frac{s_{H_j}}{m_{H_j}} = \lim_{j\to\infty}\frac{j2^{j-1}-j}{j2^{j-1}+\frac{j(j-3)}{2}} = 1.$$

Now, if we use as $G_0$ a split graph $H_j$ in this sequence, we produce graphs $G_i$ such that $l_i/n_i$ tends to $(l_{H_j}+2/3)/n_{H_j}$ and $s_i/m_i$ tends to $2(l_{H_j}+2/3)/n_{H_j}$. Now, using increasing values of $j$ for $H_j$,

$$\lim_{j\to\infty}\frac{l_{H_j}+\frac{2}{3}}{n_{H_j}} = 1+\frac{\frac{2}{3}}{2^j+j-2} = 1 \text{ and } \lim_{j\to\infty} 2\frac{l_{H_j}+\frac{2}{3}}{n_{H_j}} = 2+\frac{\frac{4}{3}}{2^j+j-2} = 2.$$

□

## 5 Conclusion

We showed that extended $P_4$-laden graphs (which include $P_4$-tidy and $P_4$-sparse graphs) have the number and total size of minimal separators linearly bounded on the size of the graph. Hence, the algorithm that lists all minimal separators of extended $P_4$-laden graphs and its subclasses, described in Section 2, is $O(n_G + m_G)$.

We also give distinct bounds on number and total size of minimal separators of graphs in many subclasses of extended $P_4$-laden graphs and we proved that each bound is tight. Moreover, the bounds given for $P_4$-tidy are tight also for some of its subclasses, such as extended $P_4$-reducible, extended $P_4$-sparse, and $P_4$-extendible graphs. All these classes contain the graphs $G_i$, defined in Section 4, using $G_0 \cong C_5$, which were used to show the tightness of the bounds for $P_4$-tidy.

We expect that the proposed algorithm enables the use of minimal separators in other applications of extended $P_4$-laden graphs, since it is simple and efficient.

## References


[1] S. Arnborg, D. G. Corneil, and A. Proskurowski. Complexity of finding embeddings in a $k$-tree. *SIAM J. Algebraic and Discrete Methods*, 8(2):277–284, 1987.

[2] A. Berry, J.-P. Bordat, and O. Cogis. Generating all the minimal separators of a graph. *Internat. J. Foundations Comput. Sci.*, 11(3):397–403, 2000.

[3] J. L. Fouquet, I. Parfenoff, and H. Thuillier. An $O(n)$ time algorithm for maximum matching in $P_4$-tidy graphs. *Inf. Process. Lett.*, 62(6):281–287, 1997.

[4] V. Giakoumakis. $P_4$-laden graphs: A new class of brittle graphs. *Inf. Process. Lett.*, 80:29–36, 1996.

[5] V. Giakoumakis, F. Roussel, and H. Thuillier. On $P_4$-tidy graphs. *Discrete Math. and Theoret. Comput. Sci.*, 1(1):17–41, 1997.

[6] P. L. Hammer and B. Simeone. The splittance of a graph. *Combinatorica*, 1(3):275–284, 1981.

[7] B. Jamison and S. Olariu. A new class of brittle graphs. *Stud. Appl. Math.*, 81:89–92, 1989.

[8] T. Kloks, H. Bodlaender, H. Müller, and D. Kratsch. Computing treewidth and minimum fill-in: All you need are the minimal separators. *LNCS*, 726:260–271, 1993.

[9] T. Kloks, H. Bodlaender, H. Müller, and D. Kratsch. Erratum to the ESA'93 Proceedings. *LNCS*, 855:508, 1994.

[10] T. Kloks and D. Kratsch. Listing all minimal separators of a graph. *SIAM J. Comput.*, 27(3):605–613, 1998.

[11] P. S. Kumar and C. E. V. Madhavan. Minimal vertex separators of chordal graphs. *Discrete Appl. Math.*, 89:155–168, 1998.

[12] F. Mazoit. Listing all the minimal separators of a 3-connected planar graph. *Discrete Math.*, 306:372–380, 2006.

[13] R. M. McConnell and J. P. Spinrad. Linear-time modular decomposition and efficient transitive orientation of comparability graphs. In *SODA '94: Proceedings of the Fifth Annual ACM-SIAM Symp. on Discrete Algorithms*, pages 536–545, Philadelphia, PA, USA, 1994.

[14] R. M. McConnell and J. P. Spinrad. Modular decomposition and transitive orientation. *Discrete Math.*, 201(1-3):189–241, 1999.

[15] S. D. Nikolopoulos and L. Palios. Minimal separators in $P_4$-sparse graphs. *Discrete Math.*, 306(3):381–392, 2006.

[16] M. Yannakakis. Complexity the minimum fill-in is NP-Complete. *SIAM J. Algebraic and Discrete Methods*, 2(1):77–79, 1981.